DIFFERENTIAL GEOMETRY I
HOMEWORK 2

DUE: WEDNESDAY, OCTOBER 1

(1) Consider the determinant function on $\mathrm{Gl}(n;\mathbb{R})$. Show that

$$\mathrm{d}(\det)|_{\mathfrak{m}} = \det(\mathfrak{m}) \sum_{i,j=1}^{n} (\mathfrak{m}^{-1})_{ji} \mathrm{d}\mathfrak{m}_{ij} \ .$$

If we denote by $\mathrm{d}\mathfrak{m}$ the $n \times n$ matrix whose (i,j)-element is the coordinate differential $\mathrm{d}\mathfrak{m}_{ij}$, we can write the above formula as

$$\mathrm{d}(\det)|_{\mathfrak{m}} = \det(\mathfrak{m})\,\mathrm{tr}(\mathfrak{m}^{-1}\mathrm{d}\mathfrak{m}) \ .$$

[*Hint*: This is a direct consequence of the Leibniz rule and the cofactor matrix construction of inverse matrices.]

(2) The main purpose of this exercise is to show that $\mathrm{SO}(3)$ is diffeomorphic to $\mathbb{RP}^3$. Part of this exercise is similar to #2 of Homework 1.

Let $F : \mathbb{R}^4 \to \mathbb{M}(3;\mathbb{R}) \cong \mathbb{R}^9$ be given by

$$F(x,y,z,w) = \begin{bmatrix} x^2+y^2-z^2-w^2 & 2(-xw+yz) & 2(xz+yw) \\ 2(xw+yz) & x^2-y^2+z^2-w^2 & 2(-xy+zw) \\ 2(-xz+yw) & 2(xy+zw) & x^2-y^2-z^2+w^2 \end{bmatrix}$$

Let $\mathbf{S}^3 = \{(x,y,z,w) \in \mathbb{R}^4 \mid x^2+y^2+z^2+w^2 = 1\}$. Observe that $f = F|_{\mathbf{S}^3}$ satisfies $f(x,y,z,w) = f(-x,-y,-z,-w)$, so that it descends to a map

$$\tilde{f} : \mathbb{RP}^3 = \mathbf{S}^3/\{\pm 1\} \to \mathbb{M}(3;\mathbb{R}) \ .$$

(a) Check that the image of $\tilde{f}$ belongs to $\mathrm{SO}(3)$.

(b) Prove that $\tilde{f}$ is injective. [*Hint*: x^2, y^2, z^2 and w^2 can be solved from the diagonal elements.]

(c) Show that $\tilde{f}$ is a smooth embedding. [*Hint*: A bijective continuous map from a compact topological space to a Hausdorff topological space is a homeomorphism.]

To sum up, the image of $\tilde{f}$ is a smooth submanifold of $\mathrm{SO}(3)$. Since both $\mathbb{RP}^3$ and $\mathrm{SO}(3)$ are of dimension three, the image of $\tilde{f}$ must be open in $\mathrm{SO}(3)$. Since $\mathbb{RP}^3$ is compact, the image of $\tilde{f}$ is closed in $\mathrm{SO}(3)$. By the *connectedness* (think about this fact) of $\mathrm{SO}(3)$, $\tilde{f}$ is a diffeomorphism from $\mathbb{RP}^3$ to $\mathrm{SO}(3)$. Or, you can use Item (b) to prove surjectivity directly.

(3) Reading Assignment: p.75–85 of *Calculus on Manifolds* by Michael Spivak.