DIFFERENTIAL GEOMETRY I
HOMEWORK 1

DUE: WEDNESDAY, SEPTEMBER 24

(1) Let (x, y, z) be coordinates on $\mathbb{R}^3$. Let Y_r be the set of points in $\mathbb{R}^3$ at distance[1] $r > 0$ from the circle
$$C = \big\{(x, y, z) \in \mathbb{R}^3 \;\big|\; x^2 + y^2 = 1 \;, z = 0\big\} \;.$$
  (a) Let $A = \{r \in (0, \infty) \mid Y_r \text{ is a smooth submanifold of } \mathbb{R}^3\}$. Find out A.
  (b) Let $\mathbf{S}^1$ be the circle, and let $\mathbf{S}^1 \times \mathbf{S}^1$ be the product manifold. Prove that Y_r is diffeomorphic to $\mathbf{S}^1 \times \mathbf{S}^1$ for any $r \in A$.

(2) Let $F : \mathbb{R}^3 \to \mathbb{R}^4$ be given by
$$F(x, y, z) = (x^2 - y^2, xy, zx, yz) \;.$$
Let $\mathbf{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Observe that $f = F|_{\mathbf{S}^2}$ satisfies $f(x, y, z) = f(-x, -y, -z)$, so that it descends to a map
$$\tilde{f} : \mathbb{RP}^2 = \mathbf{S}^2/\{\pm 1\} \to \mathbb{R}^4 \;.$$
Prove that $\tilde{f}$ is a smooth embedding. [*Hint*: A bijective continuous map from a compact topological space to a Hausdorff topological space is a homeomorphism.]

(3) The main purpose of this exercise is to get you working on the Grassmannians (see [T, §1.6]) hands-on. You are asked to work out $\mathbf{Gr}(4; 2)$, the space of 2-planes in $\mathbb{R}^4$. As a manifold, $\mathbf{Gr}(4; 2)$ has dimension 4.

Endow $\mathbb{R}^4$ with the standard Euclidean inner product. Let $V \subset \mathbb{R}^4$ be a 2-plane, and let $\pi_V : \mathbb{R}^4 \to V$ be the orthogonal projection. It has an open neighborhood $\mathcal{O}_V \subset \mathbf{Gr}(4, 2)$ consisting of 2-planes $V' \subset \mathbb{R}^4$ such that $\pi_V|_{V'} : V' \to V$ is an isomorphism.

Denote the orthogonal complement of V by $V^\perp$. As explained in [T, §1.6], $\mathcal{O}_V$ is identified with $\mathrm{Hom}(V, V^\perp)$, which is isomorphic to $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2) \cong \mathbb{R}^4$ in the current setting.
  (a) Consider $V = \{(\xi, \eta, 0, 0) \mid \xi, \eta \in \mathbb{R}\}$. Work out the map explicitly
$$\varphi_V^{-1} : \mathbb{R}^4 \cong \mathrm{Hom}(V, V^\perp) \to \mathcal{O}_V \subset \mathbf{Gr}(4; 2) \;.$$
[*Hint*: Write $\mathbb{R}^4$ as $V \oplus V^\perp$. For any element in $\mathrm{Hom}(V, V^\perp)$, the corresponding 2-plane is it graph. Note that any 2-plane can be described by two linearly independent vectors. After fixing a basis of $\mathbb{R}^4$ (or of V and $V^\perp$), elements of $\mathbf{Gr}(4; 2)$ can be described by 2×4 matrices. To sum up, you shall be able to realize φ_V^{-1} as a map from $\mathbb{R}^4$ to the space of 2×4 matrices, which is isomorphic to $\mathbb{R}^8$.]

[1]The distance from (u, v, w) to C is defined by $\inf_{(x,y,z)\in C} \sqrt{(u-x)^2 + (v-y)^2 + (w-z)^2}$. Note that the infimum is actually minimum in the case.

(b) Let $U = \{(0, \mu, 0, \nu) \mid \mu, \nu \in \mathbb{R}\}$. The open neighborhood $\mathcal{O}_U$ and the coordinate map $\varphi_U : \mathbb{R}^4 \to \mathcal{O}_U \subset \mathbf{Gr}(4; 2)$ can be defined in the same way. Find out the open set $\varphi_V(\mathcal{O}_U \cap \mathcal{O}_V) \subset \mathbb{R}^4 \cong \mathrm{Hom}(V, V^\perp)$.

(c) Compute the transition function

$$\varphi_U \circ \varphi_V^{-1} : \varphi_V(\mathcal{O}_U \cap \mathcal{O}_V) \subset \mathbb{R}^4 \to \varphi_U(\mathcal{O}_U \cap \mathcal{O}_V) \subset \mathbb{R}^4 \ ,$$

and check that it is a diffeomorphism between open subsets of $\mathbb{R}^4$.