NOTE ON THE SUBBUNDLE

DIFFERENTIAL GEOMETRY I

1. Rank of a subbundle

Let M be an n-dimensional, connected manifold, and $\pi : E \to M$ be a rank k vector bundle. We define a subbundle E' to be a *submanifold* of E such that $E'_p = E' \cap E_p$ is a vector subspace of E_p for any $p \in M$. The main purpose of this section is to explain that the dimension of E'_p is independent of p.

Since the zero section is contained in E', $\dim E' \geq n$ and assume it to be $n + k'$. Consider a local trivialization of E: $E|_U \cong U \times \mathbb{R}^k$ over some open set $U \subset M$. Regard the open set U as an open subset of $\mathbb{R}^n$.

Consider the projection map

$$\begin{array}{rccc} \pi : & U \times \mathbb{R}^k & \to & U \subset \mathbb{R}^n \\ & (\mathbf{x}, \mathbf{v}) & \mapsto & \mathbf{x} \end{array} , \tag{$\clubsuit$}$$

and let ψ be the restriction of π on $E'|_U = E' \cap E|_U$. Fix a point $p \in U$. Since the zero sections is contained in E', $\psi_* : T_{(p,0)}E' \to \mathbb{R}^n$ must be surjective. It follows that there exists an open neighborhood W of $(p, 0)$ in $E'|_U$ such that ψ_* is surjective at every point of this neighborhood. The implicit function theorem says that $\psi^{-1}(p) \cap W$ is a submanifold of dimension k'. On the other hand, it is not hard to see that $\psi^{-1}(p) \cap W$ is the same as the intersection of E'_p with some open set of E. Therefore, $\dim E'_p = k'$.

The above argument is called the transverse intersection. You can see [T, §5.7].

2. Tangent bundle of a vector bundle

2.1. **Short exact sequence of vector spaces.** Before the discussion of the tangent bundle of a vector bundle, recall a *short exact sequence* of vector spaces means that there are three vector spaces V_1, V_2 and V_3 with an injective homomorphism $f : V_1 \to V_2$ and a surjective homomorphism $g : V_2 \to V_3$ such that $\ker(g) = \mathrm{im}(f)$. It is usually denoted by

$$0 \longrightarrow V_1 \xrightarrow{f} V_2 \xrightarrow{g} V_3 \longrightarrow 0 \ .$$

In other words, V_1 can be viewed as a subspace of V_2 via f, and V_3 can be viewed as the quotient space $V_2/f(V_1)$ via g. Note that unless there is an additional structure on V_2 (such as an inner product), there is no canonically defined left inverse of f, and no canonically defined right inverse of g. When V_2 carries an inner product, $V_2 \to f(V_1)$ can be defined to be the orthogonal projection, and V_3 can be identified as the orthogonal complement of $f(V_1)$ in V_2.

2.2. **Change of basis.** Let us focus on the local trivialization $E|_U \cong U \times \mathbb{R}^k$, and assume that U is an open subset of $\mathbb{R}^n$ as before. Such a local trivialization over U is definitely not unique. Any trivialization is equivalent to a smooth map $g(\mathbf{x}) : U \to \mathrm{Gl}(k;\mathbb{R})$. We are going to study the bundle transition map for the *tangent bundle* of $E|_U$ for the following change of coordinate

$$\begin{array}{rccc} \varphi : & U \times \mathbb{R}^k & \to & U \times \mathbb{R}^k \\ & (\mathbf{x},\mathbf{v}) & \mapsto & (\mathbf{x}, \mathbf{w} = g(\mathbf{x})\mathbf{v}) \end{array} .$$

The tangent space of $U \times \mathbb{R}^k$ at a point $(\mathbf{x},\mathbf{v})$ is $\mathbb{R}^n \oplus \mathbb{R}^k$, whose basis is $\{\frac{\partial}{\partial x^i}\}_{i=1}^n \cup \{\frac{\partial}{\partial v^\alpha}\}_{\alpha=1}^k$. Take the similar basis for the $(\mathbf{x},\mathbf{w})$ side. The bundle transition map is given by the partial derivatives of φ. At $(\mathbf{x},\mathbf{v})$, it looks like

$$\begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n\times k} \\ \frac{\partial g(\mathbf{x})}{\partial \mathbf{x}}\mathbf{v} & g(\mathbf{x}) \end{bmatrix} .$$

In other words,

$$\varphi_*(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^k \big(\sum_{\beta=1}^k \frac{\partial g_{\alpha\beta}(\mathbf{x})}{\partial x^i} v^\beta\big) \frac{\partial}{\partial w^\alpha} , \tag{$\diamondsuit$}$$

$$\varphi_*(\frac{\partial}{\partial v^\beta}) = \sum_{\alpha=1}^k \frac{\partial w^\alpha(\mathbf{x},\mathbf{v})}{\partial v^\beta} \frac{\partial}{\partial w^\alpha} = \sum_{\beta=1}^k g_{\alpha\beta}(\mathbf{x}) \frac{\partial}{\partial w^\alpha} . \tag{$\heartsuit$}$$

The above computation ($\heartsuit$) suggests that there exists a *subbundle* of TE whose bundle transition is exactly the same as that of E. However, ($\diamondsuit$) says that the tangent bundle of M (or of U) does *not* sit inside TE *canonically*, except at $\mathbf{v} = \mathbf{0} = \mathbf{w}$.

More precisely, it means that

$$0 \longrightarrow \pi^*E \longrightarrow TE \longrightarrow \pi^*TM \longrightarrow 0 . \tag{$\spadesuit$}$$

Namely, the pull-back of E and TM by $\pi : E \to M$ give two vector bundles over E. The bundle π^*E is a subbundle of TE, and their quotient bundle is isomorphic to π^*TM. But there is no *canonical* isomorphism between TE and $\pi^*E \oplus \pi^*TM$, except over the zero section.

2.2.1. *The push-forward of the projection.* There is a slightly different way to describe the subbundle $\pi^*E \subset TE$. Consider the push-forward of the projection $\pi_*|_e : T_eE \to T_{\pi(e)}M$. By using the local trivialization ($\clubsuit$), it is not hard to see that π_* is surjective, and the kernel of π_* is exactly π^*E. In fact, π_* is the second map in ($\spadesuit$). With this understood, $\ker(\pi_*) \cong \pi^*E$ is usually referred as the *vertical bundle* over E.

2.2.2. *How to differentiate a section?* Suppose that there is a section $s : M \to E$ and a vector field u on M. A naturally question is that can we *differentiate* s along u such that the output is *still a section* of E? If one follows the recipe of the differential of a map, it produces a section of TE over $s(M)$, or s^*TE over M. However, there is no canonical map from TE to π^*E in ($\spadesuit$).

The answer to this question is the notion of a *connection*, which we shall discuss later. It is in [T, §11]. There is also a very similar short exact sequence ([T, §11.4.3]).