NOTE ON THE PARTITION OF UNITY

DIFFERENTIAL GEOMETRY I

Let M be an n-dimensional, connected smooth manifold[1]. Suppose that $\mathcal{U} = \{(U, \varphi)\}$ is a coordinate atlas for M. The main purpose of this note is to show that there exist a *partition of unity* subordinating to $\mathcal{U}$. Namely, there exists smooth functions $\{\theta_j\}_{j\in J}$ on M such that

- $0 \le \theta_j \le 0$ for any $j \in J$;
- for any $j \in J$, there exists some $(U, \varphi) \in \mathcal{U}$ such that $\operatorname{supp}\theta_j \subset U$;
- for any $p \in M$, there exists a neighborhood O_p such that $\theta_j|_{O_p} \equiv 0$ except for finitely many j's;
- $\sum_{j\in J} \theta_j \equiv 1$.

step 1: locally finite atlas. For simplicity, assume that the image of each φ is exactly $\mathbb{R}^n$. For any $p \in M$, there exists some $(U, \varphi) \in \mathcal{U}$ such that $p \in U$. Let $V_p = \varphi^{-1}(B(\varphi(p); 1))$. It is clear that $\{V_p\}_{p\in M}$ is an open cover of M. Note that any closed subset of $\overline{V_p}$ is compact.

Since M is paracompact, there exists a locally finite refinement. Let $\mathcal{W} = \{W_j\}_{j\in J}$ be a locally finite refinement of $\{V_p\}_{p\in M}$.

step 2: J is countable. Fix an element of $\mathcal{W}$, call it W_1. Consider the following function

$$\begin{aligned} h : J &\longrightarrow \mathbb{N} \\ j &\mapsto \min \big\{ n \in \mathbb{N} \;\big|\; \text{there exists } \{S_\ell\}_{\ell=1}^n \subset \mathcal{W} \text{ such that } S_1 = W_1, S_n = W_j, \\ &\qquad\qquad \text{and } S_\ell \cap S_{\ell+1} \neq \varnothing \text{ for } \ell = 1, \ldots, n-1 \big\} . \end{aligned}$$

In other words, $h(j)$ is the minimal number of elements of $\mathcal{W}$ that is needed to connect W_1 with W_j.

The claim is that $h^{-1}(n)$ is finite for any $n \in \mathbb{N}$. Let

$$J_{\le n} = \big\{ j \in J \;\big|\; h(j) \le n \big\} .$$

Note that $J_{\le 1}$ has only one elements, which is from W_1. Suppose that $J_{\le n}$ has only finitely many elements. Then $K_n = \overline{\cup_{h(j)\le n} W_j} = \cup_{h(j)\le n}\overline{W_j}$ is compact. For any $p \in K_n$, there exists a neighborhood O_p such that O_p intersects only finitely many elements of $\mathcal{W}$. Since K_n is compact, $\{O_p \mid p \in K_n\}$ has a finite subcover $\{O_{p_\ell}\}_{\ell=1}^{N_n}$. It is not hard to show that

$$J_{\le n+1} \subset \big\{ j \in J \;\big|\; W_j \cap O_{p_\ell} \neq \varnothing \text{ for some } \ell \in \{1, \ldots, N_n\} \big\} .$$

It follows that $J_{\le n+1}$ is a finite set.

[1]Our manifolds are assumed to be Hausdorff and paracompact.

step 3: shrink W_j. From now on, write $\mathcal{W}$ as[2] $\{W_j\}_{j=1}^{\infty}$. The claim is that there exists a sequence of open sets $\{W_j'\}_{j=1}^{\infty}$ such that

$$M = \cup_{j=1}^{\infty} W_j' \quad \text{and} \quad \overline{W_j'} \subset W_j \text{ for any } j \in \mathbb{N} \, .$$

We only explain how to construct W_1'. Choose an open set O_1 such that $\overline{O_1} \subset W_1$. For any $q \in \partial W_1$, there exists a neighborhood O_q such that

$$\overline{O_q} \cap \overline{O_1} = \varnothing \quad \text{and} \quad \overline{O_q} \subset W_j \text{ for some } j \in \mathbb{N} \, .$$

Since ∂W_1 is compact, $\{O_q \mid q \in \partial W_1\}$ has a finite subcover $\{O_{q_\ell}\}_\ell$. Let $W_1' = W_1 \backslash \{\cup_\ell \overline{O_{q_\ell}}\}$. It is clear that $W_1' \supset O_1$, $\overline{W_1'} \subset W_1$ and $\{W_j\}_{j=2}^{\infty} \cup W_1'$ covers M.

Other W_j''s can be obtained by the same construction.

step 4: construct θ_j. If there exist a sequence of non-negative smooth functions $\{\chi_j\}_{j=1}^{\infty}$ with

$$\operatorname{supp} \chi_j \subset W_j \quad \text{and} \quad \chi_j|_{W_j'} > 0$$

for any $j \in \mathbb{N}$, then

$$\theta_j = \frac{\chi_j}{\sum_{j=1}^{\infty} \chi_j}$$

would be a partition of unity subordinating to $\mathcal{U}$. Note that the summation in the denominator is locally a finite sum due to the locally finiteness of $\mathcal{W}$. The function χ_j can be constructed using the bump function together with the compactness of $\overline{W_j'} \subset W_j$.

[2] If $\mathcal{W}$ has only finitely many elements, the argument is simpler.