INTRODUCTION TO SYMPLECTIC GEOMETRY

FOR NOVEMBER 4

1. On the coadjoint orbit

Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be the dual vector space of $\mathfrak{g}$. The adjoint representation of G on $\mathfrak{g}$ is defined by

$$\mathrm{Ad}_g(Y) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(g\, e^{tY}\, g^{-1}) \ . \tag{1.1}$$

The coadjoint action of G on $\mathfrak{g}^*$ is characterized by

$$\langle \mathrm{Ad}^*_g(\xi), Y\rangle = \langle \xi, \mathrm{Ad}_{g^{-1}}(Y)\rangle \tag{1.2}$$

for any $Y \in \mathfrak{g}$. Here, $\langle\, ,\, \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the dual pairing. The purpose of this note is to explain that there is a *canonical symplectic form* on the *orbits* of the coadjoint action. Part of the material here is taken from [CdS1, Homework 17].

1.1. **The vector fields associated to the adjoint and coadjoint action.**

For any $X \in \mathfrak{g}$, it induces a vector field on $\mathfrak{g}$ by the adjoint action:

$$\begin{aligned}\text{The vector field at } Y &= \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\, \mathrm{Ad}_{e^{tX}}(Y) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} e^{tX} e^{sY} e^{-tX} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(e^{-tX})_*(Y|_{e^{tX}}) = [X, Y] \ .\end{aligned} \tag{1.3}$$

The computation relies on the following facts:

- If we think Y as a *left* invariant vector field, $Y|_g = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} g e^{sY}$.
- On G, the flow generated by X for time t is tantamount to the *right* multiplication by e^{tX}. This is basically the same as the previous fact.
- The equality (1.3) is an equivalent definition for the Lie derivative.

For any $X \in \mathfrak{g}^*$, denote by $X^\sharp$ the vector field on $\mathfrak{g}^*$ induced by the coadjoint action. Its value at ξ is characterized by

$$\begin{aligned}\langle X^\sharp|_\xi, Y\rangle &= \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\langle \mathrm{Ad}^*_{e^{tX}}\, \xi, Y\rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}\langle \xi, \mathrm{Ad}_{e^{-tX}}\, Y\rangle = \langle \xi, [Y, X]\rangle \ .\end{aligned} \tag{1.4}$$

1.2. **The skew-symmetric bilinear form.** For any $\xi \in \mathfrak{g}^*$, define a skew-symmetric bilinear form on $\mathfrak{g}$ by

$$\omega_\xi(X, Y) = \langle \xi, [X, Y] \rangle \ . \tag{1.5}$$

We now examine the kernel of this skew-symmetric bilinear form. If $X \in \ker(\omega_\xi)$, we apply (1.4) to find that

$$0 = \omega_\xi(X, Y) = -\langle X^\sharp|_\xi, Y \rangle$$

for any $Y \in \mathfrak{g}$. Therefore, $X^\sharp$ vanishes at ξ. It follows that $\ker(\omega_\xi)$ is the Lie algebra of the stabilizer of the action (1.2) at ξ.

We denote the stabilizer by G_ξ, and its Lie algebra by $\mathfrak{g}_\xi$. Since $\ker(\omega_\xi) = \mathfrak{g}_\xi$, ω_ξ induces a non-degenerate skew-symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_\xi$. This quotient space is identified with the *tangent space* of the *orbit* of the coadjoint action at ξ.

Hence, ω induces a non-degenerate 2-form on the orbit of the coadjoint action. It is defined by

$$\omega_\xi(X^\sharp|_\xi, Y^\sharp|_\xi) = \langle \xi, [X, Y] \rangle \ . \tag{1.6}$$

Notice that (1.5) is *not* a 2-form on $\mathfrak{g}^*$. The inputs are elements of $\mathfrak{g}$, but not $\mathfrak{g}^*$.

1.3. **Exterior derivative of ω.** We calculate the exterior derivative of (1.6). Due to the above comment, it only makes sense to compute the exterior derivative on the orbit.

$$\begin{aligned}(\mathrm{d}\omega)(X^\sharp, Y^\sharp, Z^\sharp) = {} & X^\sharp(\omega(Y^\sharp, Z^\sharp)) + Z^\sharp(\omega(X^\sharp, Y^\sharp)) + Y^\sharp(\omega(Z^\sharp, X^\sharp)) \\ & - \omega([Y^\sharp, Z^\sharp], X^\sharp) - \omega([X^\sharp, Y^\sharp], Z^\sharp) - \omega([Z^\sharp, X^\sharp], Y^\sharp) \ .\end{aligned}$$

For the terms of the first line,

$$\begin{aligned} X^\sharp(\omega(Y^\sharp, Z^\sharp)) &= X^\sharp(\langle \xi, [Y, Z] \rangle) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle \mathrm{Ad}^*_{e^{tX}} \xi, [Y, Z] \rangle \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \langle \xi, \mathrm{Ad}_{e^{-tX}} [Y, Z] \rangle \\ &= -\langle \xi, [X, [Y, Z]] \rangle \ . \end{aligned}$$

For the terms of the second line,

$$\begin{aligned} \omega([Y^\sharp, Z^\sharp], X^\sharp) &= \omega(([Y, Z])^\sharp, X^\sharp) \\ &= \langle \xi, [[Y, Z], X] \rangle \ . \end{aligned}$$

By the Jacobi identity, $\mathrm{d}\omega = 0$. Thus, ω defines a symplectic form on the orbit of the coadjoint action. It is also known as the *Kostant–Kirillov symplectic structure.*

1.4. **The moment map.** Surely the Lie group G acts on the coadjoint orbit. It turns out that this group action is Hamiltonian, and the moment map is very simple.

For any $X \in \mathfrak{g}$, we claim that $\iota_{X^\sharp}\omega$ is d-exact. By (1.6) and (1.4),

$$\begin{aligned} \iota_{Y^\sharp}\iota_{X^\sharp}\omega &= \langle \xi, [X,Y] \rangle \\ &= \langle Y^\sharp|_\xi, X \rangle = Y^\sharp(\langle \xi, X \rangle) \end{aligned}$$

for any $Y \in \mathfrak{g}$. It follow that $\iota_{X^\sharp}\omega = \mathrm{d}(\langle \xi, X \rangle)$.

Hence, we can just take the moment map to be ξ, and it is automatically equivariant with respect to the coadjoint action. In other words, the moment map of the coadjoint orbit is the *inclusion map* to $\mathfrak{g}^*$.

1.5. **Remarks.**

- In fact, we can apply this framework for $G = \mathrm{SO}(3)$. Then we will get the answer for (6) of the Midterm.
- For Item (1.e) of Homework 8, the symplectic form should be

  $$\omega_\xi(X^\sharp, Y^\sharp) = i\,\mathrm{trace}([X,Y]\xi) \ .$$

  The correct statement of the first fact should be

  $$\begin{aligned} (\mathrm{d}\omega)(X^\sharp, Y^\sharp, Z^\sharp) &= X^\sharp(\omega(Y^\sharp, Z^\sharp)) + Z^\sharp(\omega(X^\sharp, Y^\sharp)) + Y^\sharp(\omega(Z^\sharp, X^\sharp)) \\ &\quad - \omega([Y^\sharp, Z^\sharp], X^\sharp) - \omega([X^\sharp, Y^\sharp], Z^\sharp) - \omega([Z^\sharp, X^\sharp], Y^\sharp) \ . \end{aligned}$$

  For the second fact, we consider the derivative of

  $$\xi \mapsto \omega_\xi(X^\sharp, Y^\sharp) = i\,\mathrm{trace}([X,Y]\xi)$$

  along $Z^\sharp$, which is equal to

  $$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} i\,\mathrm{trace}([X,Y](e^{tZ}\xi e^{-tZ})) \ .$$

  As mentioned above, the whole business of the symplectic form computation is done on the coadjoint orbit.