INTRODUCTION TO SYMPLECTIC GEOMETRY

FOR SEPTEMBER 30

On the homotopy map in the proof of the Poincaré–Birkhoff theorem

In the proof of the Poincaré–Birkhoff theorem, we construct a curve $\gamma : [0, \infty) \to \mathbb{R}^2$ such that

- $\gamma(t)$ belongs to the strip $S = \mathbb{R} \times [-1, 1]$ when $t \in [0, T]$;
- $\gamma(0) \in \mathbb{R} \times \{-1\}$ and $\gamma(T) \in \mathbb{R} \times \{1\}$;
- $\gamma(T + 1)$ lies outside the strip; more precisely, its y-component is greater than 1;
- γ has no self-intersection, namely, $\gamma(s) \neq \gamma(t)$ for any $s \neq t$.

For this curve γ, we associate a map $\rho(t) : [0, T] \to \mathbf{S}^1$:

$$\rho(t) = \frac{\varphi_\epsilon(\gamma(t)) - \gamma(t)}{|\varphi_\epsilon(\gamma(t)) - \gamma(t)|} = \frac{\gamma(t + 1) - \gamma(t)}{|\gamma(t + 1) - \gamma(t)|} \ . \tag{0.1}$$

Note that

$$\rho(0) = \frac{\gamma(1) - \gamma(0)}{|\gamma(1) - \gamma(0)|} \qquad \text{and} \qquad \rho(T) = \frac{\gamma(T + 1) - \gamma(T)}{|\gamma(T + 1) - \gamma(T)|} \ . \tag{0.2}$$

Now, we take a continuous angle function $\theta(t) : [0, T] \to \mathbb{R}$ such that $\rho(t) = e^{i\theta(t)}$, and consider $\operatorname{sign}(\rho) = \operatorname{sign}(\theta(T) - \theta(0)) \in \{\pm 1\}$.

Goal. Deform $\rho(t)$ to another map $\rho_0(t) : [0, T] \to \mathbf{S}^1$ such that

(i) during the deformation process, the boundary condition (0.2) is preserved;
(ii) the image of $\rho_0(t)$ belongs to the upper semi-circle.

If this can be done, Item (i) implies that $\operatorname{sign}(\rho) = \operatorname{sign}(\rho_0)$, and Item (ii) implies that $\operatorname{sign}(\rho_0) = -1$.

Idealistically (also from the picture), we can always deform $\rho(t)$ to be $\frac{\gamma(T+1)-\gamma(0)}{|\gamma(T+1)-\gamma(0)|}$. However, this constant map does not meet Item (i). A good candidate of $\rho_0(t)$ is the following:

- when $t = \frac{T}{2}$, $\rho_0(\frac{T}{2}) = \dfrac{\gamma(T + 1) - \gamma(0)}{|\gamma(T + 1) - \gamma(0)|}$;
- when $t \leq \frac{T}{2}$, $\rho_0(t) = \dfrac{\gamma(f_1(t)) - \gamma(0)}{|\gamma(f_1(t)) - \gamma(0)|}$;
- when $t \geq \frac{T}{2}$, $\rho_0(t) = \dfrac{\gamma(T + 1) - \gamma(f_2(t))}{|\gamma(T + 1) - \gamma(f_2(t))|}$.

It is not hard to see that $f_1(t) = 2t + 1$ and $f_2(t) = 2t - T$ would make ρ_0 continuous.

Now, we can construct the (boundary fixing) homotopy between $\rho(t)$ and $\rho_0(t)$ directly:

$$H(t,s) = \begin{cases} \dfrac{\gamma((1-s)(t+1)+s(2t+1)) - \gamma((1-s)t)}{|\gamma((1-s)(t+1)+s(2t+1)) - \gamma((1-s)t)|} & \text{when } 0 \le t \le \frac{T}{2} \\ \dfrac{\gamma((1-s)(t+1)+s(T+1)) - \gamma((1-s)t + s(2t-T))}{|\gamma((1-s)(t+1)+s(T+1)) - \gamma((1-s)t + s(2t-T))|} & \text{when } \frac{T}{2} \le t \le T \end{cases} \tag{0.3}$$

where $s \in [0,1]$ is the deformation parameter. It follows from the non-self intersection of γ that $H(t,s)$ is well-defined. We can check that

$$H(t,0) = \rho(t) \ , \qquad H(t,1) = \rho_0(t) \ , \tag{0.4}$$

$$H(0,s) = \frac{\gamma(1) - \gamma(0)}{|\gamma(1) - \gamma(0)|} \ , \qquad H(T,s) = \frac{\gamma(T+1) - \gamma(T)}{|\gamma(T+1) - \gamma(T)|} \ . \tag{0.5}$$

We can always find a continuous angle function $\Theta(t,s)$ such that $H(t,s) = \exp(i\Theta(t,s))$. It follows from (0.5) that $\Theta(T,s)$ and $\Theta(0,s)$ must be constants. Thus,

$$\text{sign}(\rho) = \text{sign}(\rho_0) = \text{sign}(\Theta(T,s) - \Theta(0,s)) \ . \tag{0.6}$$

From the geometric construction of γ, we know that ρ_0 lies on the upper semi-circle. Therefore, we may take $\Theta(T,1) \in (0, \frac{\pi}{2})$ and $\Theta(0,1) \in (\frac{\pi}{2}, \pi)$. It follows that

$$\text{sign}(\rho) = \text{sign}(\rho_0) = -1 \ . \tag{0.7}$$