INTRODUCTION TO SYMPLECTIC GEOMETRY

FOR SEPTEMBER 12

1. On the determinant of symplectic matrices

We adopt the notation of Differential Geometry. Let $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^n}\}$ be the basis for $\mathbb{R}^{2n}$. And let $\{\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^n, \mathrm{d}y^1, \mathrm{d}y^2, \ldots, \mathrm{d}y^n\}$ be the dual basis for $(\mathbb{R}^{2n})^*$. The standard symplectic bilinear map is

$$\omega_0 = \sum_{j=1}^{n} \mathrm{d}x^j \wedge \mathrm{d}y^j \ .$$

Let $\mathcal{B}$ be a linear change of basis:

$$\begin{array}{rccl} \mathcal{B}: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & \dfrac{\partial}{\partial x^j} & \mapsto & \displaystyle\sum_{k=1}^{n} \big(B_j^k \frac{\partial}{\partial x^k} + B_j^{n+k} \frac{\partial}{\partial y_k}\big) \\ & \dfrac{\partial}{\partial y^j} & \mapsto & \displaystyle\sum_{k=1}^{n} \big(B_{n+j}^k \frac{\partial}{\partial x^k} + B_{n+j}^{n+k} \frac{\partial}{\partial y_k}\big) \end{array}$$

The induce map on the dual space goes another direction (the *pull-back* map):

$$\begin{array}{rcll} (\mathbb{R}^n)^* & \leftarrow & (\mathbb{R}^n)^* & : \mathcal{B}^* \\ \sum_{k=1}^{n} \big(B_k^j \mathrm{d}x^k + B_{n+k}^j \mathrm{d}y^k\big) & \leftarrow\!\shortmid & \mathrm{d}x^j & \\ \sum_{k=1}^{n} \big(B_k^{n+j} \mathrm{d}x^k + B_{n+k}^{n+j} \mathrm{d}y^k\big) & \leftarrow\!\shortmid & \mathrm{d}y^j & \end{array} \tag{1.1}$$

Preserving the symplectic form. Suppose that $B^*\omega_0 = \omega_0$. That is to say,

$$\omega_0(\mathcal{B}(\frac{\partial}{\partial x^j}), \mathcal{B}(\frac{\partial}{\partial x^\ell})) = \omega_0(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^\ell}) \quad \Rightarrow \quad \sum_{k=1}^{n} (B_j^k B_\ell^{n+k} - B_j^{n+k} B_\ell^k) = 0 \ , \tag{1.2}$$

$$\omega_0(\mathcal{B}(\frac{\partial}{\partial y^j}), \mathcal{B}(\frac{\partial}{\partial y^\ell})) = \omega_0(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^\ell}) \quad \Rightarrow \quad \sum_{k=1}^{n} (B_{n+j}^k B_{n+\ell}^{n+k} - B_{n+j}^{n+k} B_{n+\ell}^k) = 0 \ , \tag{1.3}$$

$$\omega_0(\mathcal{B}(\frac{\partial}{\partial x^j}), \mathcal{B}(\frac{\partial}{\partial y^\ell})) = \omega_0(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial y^\ell}) \quad \Rightarrow \quad \sum_{k=1}^{n} (B_j^k B_{n+\ell}^{n+k} - B_j^{n+k} B_{n+\ell}^k) = \delta_{j\ell} \ . \tag{1.4}$$

Let B be the $2n \times 2n$ matrix whose j-th row is $[B_1^j \ B_2^j \ \cdots \ B_{2n}^j]$.

Claim. The conditions (1.2), (1.3) and (1.4) are equivalent to that $B^T J_n B = J_n$ where J_n is the following $2n \times 2n$ matrix

$$J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \ .$$

Preserving the symplectic volume. Note that

$$\frac{1}{n!}\omega_0^n = \mathrm{d}x^1 \wedge \mathrm{d}y^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^n$$

is a volume form. If $\mathcal{B}^*\omega_0 = \omega_0$, then $\mathcal{B}^*(\frac{1}{n!}\omega_0^n) = \frac{1}{n!}\omega_0^n$.

Claim. It follows from (1.1) that

$$\begin{aligned} &\mathcal{B}^*(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^n) \\ = \; &\det(B^T)(\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^n \wedge \mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \cdots \wedge \mathrm{d}y^n) \end{aligned}$$

As a result, $\mathrm{Sp}(n) \subseteq \mathrm{SL}(2n;\mathbb{R})$. I will not assign [CdS1, Homework 2]. You can read that homework, and compare it with this section.

2. On the space of all Lagrangians in $(\mathbb{R}^{2n}, \omega_0)$

2.1. Quick review of orthogonal group and unitary group.

(i) The general linear group $\mathrm{GL}(n;\mathbb{R})$ consists of all $n \times n$ invertible matrices (with real entries). The group multiplication is the matrix multiplication. The group $\mathrm{GL}(n;\mathbb{C})$ consists of all $n \times n$ invertible matrices with complex entries.

(ii) The special linear group $\mathrm{SL}(n;\mathbb{R})$ consists of all $n \times n$ matrices with determinant 1. It is a subgroup of $\mathrm{GL}(n;\mathbb{R})$.

(iii) The orthogonal group is defined to be

$$\mathrm{O}(n) = \left\{ A \in \mathrm{GL}(n;\mathbb{R}) \;\middle|\; A^T A = I_n \right\}$$

where T means transpose, and I_n is the $n \times n$ identity matrix. The following statements are equivalent to each other:

  (a) $A \in \mathrm{O}(n)$;
  (b) the column vectors of A form an orthonormal basis for $\mathbb{R}^n$;
  (c) the row vectors of A form an orthonormal basis for $\mathbb{R}^n$.

(iv) The unitary group is defined to be

$$\mathrm{U}(n) = \left\{ T \in \mathrm{GL}(n;\mathbb{C}) \;\middle|\; T^* T = I_n \right\}$$

where $*$ means conjugate-transpose. The following statements are equivalent to each other:

  (a) $T \in \mathrm{U}(n)$;
  (b) the column vectors of T form an unitary basis for $\mathbb{C}^n$;
  (c) the row vectors of T form an unitary basis for $\mathbb{C}^n$.

(v) Note that $\mathrm{O}(n)$ is a subgroup of $\mathrm{U}(n)$. We can form a homogeneous space $\mathrm{U}(n)/\mathrm{O}(n)$ where $\mathrm{O}(n)$ acts by right multiplication. Namely, it is

$$\mathrm{U}(n)/\{T \sim TA \text{ where } A \in \mathrm{O}(n)\} \; .$$

You can find a brief introduction to homogeneous spaces in most textbook of Differential Geometry.

(vi) Let W^k be a subspace of $\mathbb{R}^m$. We endow $\mathbb{R}^m$ with the standard inner product. Fix an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ for W^k. Then $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ is an orthonormal basis for W^k if and only if

$$\underset{(m\times k)}{\begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_k \\ | & | & & | \end{bmatrix}} = \underset{(m\times k)}{\begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix}} \underset{(k\times k)}{A}$$

for some $A \in \mathrm{O}(k)$. This is an exercise in linear algebra.

2.2. **Lagrangians in** $(\mathbb{R}^{2n}, \omega_0)$**.** The coordinate for $\mathbb{R}^{2n}$ is $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)$, and $\omega_0 = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$.

Let L be a Lagrangian n-plane in $(\mathbb{R}^{2n}, \omega_0)$. Choose an *orthonormal*[1] basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ for L. Each $\mathbf{v}_j$ is an $2n$-column vector. Denote its upper n-components by $\mathbf{x}_j$, and lower n-components by $\mathbf{y}_j$. We may regard $\mathbf{x}_j$ and $\mathbf{y}_j$ as vectors of $\mathbb{R}^n$. The orthonormal condition is equivalent to

$$\langle \mathbf{x}_j, \mathbf{x}_k \rangle + \langle \mathbf{y}_j, \mathbf{y}_k \rangle = \delta_{jk} \ . \tag{2.1}$$

Here, $\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{b}^* \, \mathbf{a}$ is the standard Euclidean (Hermitian if vectors are complex) inner product.

Claim. The Lagrangian condition is equivalent to

$$\omega_0(\mathbf{v}_j, \mathbf{v}_k) = 0 \qquad \Rightarrow \qquad \langle \mathbf{x}_j, \mathbf{y}_k \rangle - \langle \mathbf{x}_k, \mathbf{y}_j \rangle = 0 \ . \tag{2.2}$$

Set $\mathbf{u}_j$ to be $\mathbf{x}_j + i\mathbf{y}_j \in \mathbb{C}^n$

Claim. It follows from (2.1) and (2.2) that

$$\langle \mathbf{u}_j, \mathbf{u}_k \rangle = \delta_{jk} \ .$$

By (iv),

$$\begin{bmatrix} | & | & & | \\ \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \\ | & | & & | \end{bmatrix}$$

is a unitary $n \times n$ matrix. Also, reversing the procedure produces a Lagrangian n-plane from a unitary matrix.

According to (vi), the freedom of choice of orthonormal basis for L is $\mathrm{O}(n)$. Therefore, all the Lagrangian n-planes in $\mathbb{R}^{2n}$ is $\mathrm{U}(n)/\,\mathrm{O}(n)$, the space defined in (v).

[1]We cheat here by using the metric.