INTRODUCTION TO SYMPLECTIC GEOMETRY
SOLUTION FOR THE MIDTERM

THURSDAY, NOVEMBER 7, 2013

(1) True/False questions, no justications needed. (1.5points each item)

(a) F Consider $\mathbb{R}^6$ with the standard symplectic form ω_0. Any 4-dimensional subspace $U \subset \mathbb{R}^6$ is coisotropic, i.e. $U^{\omega_0} \subset U$.
For instance, take $U = \operatorname{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\}$. Then $U^{\omega_0} = \operatorname{span}\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial y_3}\} \not\subset U$.

(b) T Any manifold X^n can always be realized as a Lagrangian submanifold of some symplectic manifold (M^{2n}, ω).
X is always Lagrangian as the zero section in $(T^*X, \omega_{\text{can}})$.

(c) F There exists a symplectic matrix $A \in \operatorname{Sp}(n)$ with $\det(A) = -1$.
Symplectic form induces an orientation. It follows that $\operatorname{Sp}(n) \subseteq \operatorname{SL}(2n; \mathbb{R})$.

(d) F Let (M, ω) be a compact symplectic manifold. Suppose that there is an $\mathbf{S}^1$-action on M which preserves the symplectic form. Then, there exists a moment map $\mu : M \to \mathbb{R}$ of this $\mathbf{S}^1$-action.
Consider $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Consider the $\mathbf{S}^1$-action as rotating the x-component. It is only symplectic, but not Hamiltonian.

(e) F The Klein bottle admits a symplectic structure. The Klein bottle is constructed by identifying the boundary of $[0, 1] \times [0, 1] \subset \mathbb{R}^2$ by the relations $(0, y) \sim (1, y)$ and $(x, 0) \sim (1 - x, 1)$.
The Klein bottle is not orientable.

(f) F $\mathbf{S}^3 \times \mathbb{CP}^3$ admits a symplectic structure.
Since $\dim \mathbf{S}^3 \times \mathbb{CP}^3 = 9$, it cannot be symplectic.

(g) T $\mathbf{T}^2 \times \mathbf{S}^2$ admits a symplectic structure.
The product of symplectic manifolds is still symplectic.

(h) F $\mathbf{S}^6$ admits a symplectic structure.
Since $\mathbf{S}^6$ is compact and $\mathrm{H}^2_{\mathrm{dR}}(\mathbf{S}^6) = 0$, it cannot be symplectic.

(i) T Let (M, ω) be a compact symplectic manifold, and f, g be two smooth functions on M. Suppose that they Poisson commute, i.e. $\{f, g\} = 0$. Then, $\rho_t^* g = g$ where $\rho_t = \exp(tX_f)$.
$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t^* g = \rho_t^* \mathcal{L}_{X_f} g = \rho_t^* \omega(X_g, X_f) = 0.$

(j) $\boxed{\text{F}}$ There exists a smooth, strictly convex function F on $\mathbb{R}^2$ such that the image of its Legendre transform is $S_F = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,|\, r > 0 \text{ and } 0 < \theta < \frac{3\pi}{2}\}$.
The region is not convex.

(k) $\boxed{\text{T}}$ Consider $\mathbb{R}^4$ with the standard symplectic form $\omega_0 = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2$. The flow ρ_t generated by the vector field $V = \sin(y_2)\frac{\partial}{\partial x_1} + y_1 \cos(y_2)\frac{\partial}{\partial x_2}$ preserves the symplectic form ω_0.
$\mathcal{L}_V\omega_0 = \mathrm{d}(\iota_V\omega) + \iota_V(\mathrm{d}\omega) = \mathrm{d}(\sin(y_2)\mathrm{d}y_1 + y_1\cos(y_2)\mathrm{d}y_2) = \mathrm{d}^2(y_1\sin(y_2)) = 0.$

(l) $\boxed{\text{F}}$ Let (M^6, ω) be a compact, 6-dimensional symplectic manifold. Suppose that $\{\gamma_j : \mathbf{S}^3 \hookrightarrow M\}_{j\in\mathbb{N}}$ and $\gamma : \mathbf{S}^3 \hookrightarrow M$ are Lagrangian embeddings of the three sphere into M. It is possible that $\lim_{j\to\infty}\gamma_j = \gamma$ in the $\mathcal{C}^1$-topology, and $\#\{\gamma_j(\mathbf{S}^3) \cap \gamma(\mathbf{S}^3)\} = 1$ for all j. (Here we simply count the total number of intersection points without considering the sign of intersection.)
Apply the Weinstein tubular neighborhood theorem on $\gamma(\mathbf{S}^3)$. Since $\lim_{j\to\infty}\gamma_j = \gamma$ in the $\mathcal{C}^1$-topology, $\gamma_j(\mathbf{S}^3)$ must be given by the graph of closed 1-forms when j sufficiently large. Since $\mathrm{H}^1_{\mathrm{dR}}(\mathbf{S}^3) = 0$, closed 1-forms are the exterior derivative of functions. A smooth function on $\mathbf{S}^3$ has at least two critical points.

(2) The question concerns about basic symplectic-geometric properties of cotangent bundles.

(a) Let X be a manifold. Write down the local expression for the tautological 1-form and the canonical symplectic form of T^*X.
Let $\{x_j\}_{j=1}^n$ be a local coordinate on X, and $\{\xi_j\}_{j=1}^n$ be the corresponding coordinate for the fibers of T^*X. The tautological 1-form is $\alpha = \sum_{j=1}^n \xi_j \mathrm{d}x_j$. The canonical symplectic form is $\omega_{\mathrm{can}} = -\mathrm{d}\alpha = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}\xi_j$.

(b) Think $\mathbf{S}^3$ as the unit sphere in $\mathbb{R}^4$.

(i) Let $f : \mathbb{R}^4 \to \mathbb{R}$ be a smooth function. Construct a Lagrangian submanifold L_f of $(T^*\mathbf{S}^3, \omega_{\mathrm{can}})$ using f.
Take L_f to be the grapf of $\mathrm{d}f$.

(ii) Let $H = \{(\cos t, \sin t, 0, 0) \,|\, t \in \mathbb{R}/2\pi\mathbb{Z}\} \cup \{(0, 0, \cos t, \sin t) \,|\, t \in \mathbb{R}/2\pi\mathbb{Z}\}$ be the Hopf link in $\mathbf{S}^3$. Construct a Lagrangian submanifold L_H of $(T^*\mathbf{S}^3, \omega_{\mathrm{can}})$ such that $L_H \cap \iota_0(\mathbf{S}^3) = H$ where $\iota_0(\mathbf{S}^3)$ is the zero section in $T^*\mathbf{S}^3$.
Take L_H to be the conormal bundle of H. It follows from the construction of the conormal bundle that $L_H \cap \iota_0(\mathbf{S}^3) = H$.

$(2 + 2 + 2 = 6\text{points})$

the Hopf link

(3) Let ω_0 and ω_1 be two symplectic forms on M. Let $i : N \hookrightarrow M$ be a compact submanifold. Suppose that $\omega_0|_p = \omega_1|_p$ for any $p \in N$. Then, the *local Moser thoerem* asserts that there exists neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$ of N in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}_1$ such that

- $\varphi^*\omega_1 = \omega_0$;
- $\varphi \circ i = i$; namely, φ is the identity map on N.

Now,

(a) State the Darboux theorem.

Let (M, ω) be a symplectic manifold. Then, for any $p \in M$, there exists a coordinate neighborhood $p \in U \subset M$ with coordinates $\{x_j, y_j\}_{j=1}^n$ such that $\omega = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$.

(b) Prove the Darboux theorem. You are allowed to use the local Moser theorem. By symplectic linear algebra, $\omega|_p = \sum_{j=1}^n e_j^* \wedge f_j^*$ for a basis $\{e_j^*, f_j^*\}_{j=1}^n$ for T_p^*M. Consider the dual basis $\{e_j, f_j\}_{j=1}^n$ for T_pM. Choose a coordinate neighborhood $p \in U \subset M$ such that $\frac{\partial}{\partial x_j}|_p = e_j$ and $\frac{\partial}{\partial y_j}|p = f_j$ for all j; this can always be done by a linear transform. Endow U with the symplectic form $\omega_0 = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$. On the other hand, $\omega|_U$ is another symplectic form on U. By the construction of ω_0, these two symplectic form meets the requirement of the local Moser theorem, with the manifold to be U and the compact submanifold to be p. Therefore, there exists neighborhoods $\mathcal{U}_0$ and $\mathcal{U}_1$, and a diffeomorphism $\varphi : \mathcal{U}_0 \subset U \to \mathcal{U}_1 \subset U$ such that $\varphi^*\omega = \omega_0$. Then $\varphi : \mathcal{U}_0 \to \mathcal{U}_1 \subset M$ is a Darboux chart.

$(2 + 4 = 6$points$)$

(4) Suppose that τ is a symplectomorphism of (M, ω). Its graph $\{(p, \tau(p)) \,|\, p \in M\} \subset M \times M$ is a Lagrangian submanifold with respect to $\mathrm{pr}_1^*\,\omega - \mathrm{pr}_2^*\,\omega$. This relates a symplectormophism to a Lagrangian submanifold.

Let X be a manifold. Given a smooth function $f : X \times X \to \mathbb{R}$, we can apply the method of generating functions to construction a symplectomorphism of $(T^*X, \omega_{\text{can}})$ to itself.

(a) Describe the method of generating functions.

The graph of $\mathrm{d}f$, $Y_f = \{(x, \partial_x f, y, \partial_y f) \subset T^*X \times T^*X\}$, is Lagrangian with respect to $\mathrm{pr}_1^*\,\omega_{\text{can}} + \mathrm{pr}_2^*\,\omega_{\text{can}}$. By flipping the second fiber, $Y_f^\sigma = \{(x, \partial_x f, y, -\partial_y f) \subset T^*X \times T^*X\}$ is Lagrangian with respect to $\mathrm{pr}_1^*\,\omega_{\text{can}} - \mathrm{pr}_2^*\,\omega_{\text{can}}$. We may construct a symplectomorphism by solving $y = y(x, \xi)$ from $\partial_x f(x, y) = \xi$.

(b) What is the necessary condition on f to guarantee that we can locally construct a symplectomorphism?

From the Jacobian computation, also for the implicit function theorem, the condition is that $[\frac{\partial^2 f}{\partial x_j \partial y_k}]_{j,k}$ is non-degenerate.

(c) Consider the case when $X = \mathbb{R}$ and

$$\begin{array}{rccl} f: & X \times X & \to & \mathbb{R} \\ & (x, y) & \mapsto & (x^3 + 3x^2 + 4x)y + y^2 - 4 \ . \end{array}$$

Use the method of generating functions to construct the symplectomorphism *explicitly.*

$\partial_x f = (3x^2 + 6x + 4)y = (3(x+1)^2 + 1)y \quad \Rightarrow \quad y = (3x^2 + 6x + 4)^{-1}\xi.$

$-\partial_y f = -(x^3 + 3x^2 + 4x) - 2y \quad \Rightarrow \quad \eta = -(x^3 + 3x^2 + 4x) - 2(3x^2 + 6x + 4)^{-1}\xi.$

$(2 + 1 + 3 = 6$points)

(5) Let Σ_g be a compact, oriented surface[1] of genus g. Let ω be an area form on Σ_g. Suppose that τ is an orientation-preserving self-diffeomorphism of Σ. It follows that $\tau^*\omega$ is also an area form. Prove that ω and $\tau^*\omega$ are *strongly isotopic*. That is to say, there is an isotopy $\rho_t : M \to M$ such that $\rho_1^*(\tau^*\omega) = \omega$.

Since $\dim \Sigma = 2$, a symplectic form (with the same orientation) is the same as an area form. Since convex combinations of area forms are still area forms, $\omega_t = (1-t)\omega + t\tau^*\omega$ is a symplectic form for any $t \in [0, 1]$.

Since $\int_\Sigma \omega = \int_\Sigma \tau^*\omega$, ω and $\tau^*\omega$ define the same class in $\mathrm{H}^2_{\mathrm{dR}}(\Sigma)$. Thus, $\tau^*\omega - \omega = \mathrm{d}\eta$ for some 1-form η.

Consider the one-parameter family of vector field $v_t = -\omega_t^{-1}(\eta)$. Let ρ_t be the one-parameter family of diffeomorphism generated by v_t. We compute

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\rho_t^*\omega_t &= \rho_t^*\big(\mathcal{L}_{v_t}\omega_t + \frac{\mathrm{d}\omega_t}{\mathrm{d}t}\big) \\ &= \rho_t^*\big(\mathrm{d}(\iota_{v_t}\omega_t) + \iota_{v_t}(\mathrm{d}\omega_t) - \omega + \tau^*\omega\big) \\ &= \rho_t^*(-\mathrm{d}\eta + \mathrm{d}\eta) = 0 \ . \end{aligned}$$

It follows that $\rho_1^*(\tau^*\omega) = \omega$.

(7points)

(6) Regard $\mathbf{S}^2$ as the unit sphere in $\mathbb{R}^3$. For any $p \in \mathbf{S}^2$, $T_p\mathbf{S}^2$ consists of all the vectors orthogonal to p. Define the symplectic form ω by $\omega_p(u, v) = \langle p, u \times v\rangle$ where $\times$ is the usual cross product. The standard action of SO(3) on $\mathbb{R}^3$ maps $\mathbf{S}^2$ onto itself, and thus induces an action on $\mathbf{S}^2$. The purpose of this exercise is to compute the moment map of this SO(3)-action on $(\mathbf{S}^2, \omega)$.

The Lie algebra $\mathfrak{so}(3)$ consists of skew-symmetric matrices. It can be identified with $\mathbb{R}^3$ by

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \mapsto (a_1, a_2, a_3) \ . \tag{0.1}$$

[1] A surface is a manifold of real dimension 2.

(a) The adjoint action for matrix groups are given by the usual matric multiplication, $\operatorname{Ad}_g A = gAg^{-1}$. What is the adjoint action in terms of (0.1)? The dual space $\mathfrak{so}^*(3)$ can also be identified with $\mathbb{R}^3$ via the usual inner product. What is the coadjoint action?

It follows from a straightforward (but tedious) computation that $\operatorname{Ad}_g A = g\vec{a}$ as the matrix multiplication. For the coadjoint action,

$$\begin{aligned}\langle \operatorname{Ad}_g^* \vec{\xi}, \vec{a}\rangle &= \langle \vec{\xi}, \operatorname{Ad}_{g^{-1}} \vec{a}\rangle \\ &= \langle \vec{\xi}, g^{-1}\vec{a}\rangle \\ &= \langle g\vec{\xi}, g(g^{-1}\vec{a})\rangle = \langle g\vec{\xi}, \vec{a}\rangle\end{aligned}$$

for any $\vec{a} \in \mathbb{R}^3$. Hence, $\operatorname{Ad}_g^* \vec{\xi} = g\vec{\xi}$.

(b) Each component of the cross product is a 2-form on $\mathbb{R}^3$. What are they?

Let x, y, z be the coordinate for $\mathbb{R}^3$. The cross product is $(\mathrm{d}y\wedge \mathrm{d}z, \mathrm{d}z\wedge \mathrm{d}x, \mathrm{d}x\wedge \mathrm{d}y)$.

(c) With Item (b), construct a 2-form $\tilde{\omega}$ on $\mathbb{R}^3$ whose restriction on $\mathbf{S}^2$ is the symplectic form. The 2-form $\tilde{\omega}$ needs not to be d-closed on $\mathbb{R}^3$.

From $\omega_p(u, v) = \langle p, u \times v\rangle$, we can take $\tilde{\omega}$ to be $x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$.

(d) Compute $\iota_{A^\sharp}\tilde{\omega}$ on $\mathbb{R}^3$ for any $A \in \mathfrak{so}(3)$.

Let $g = e^{tA}$. It follows from a straighforward computation that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{tA}(x, y, z) &= (a_2 z - a_3 y, a_3 x - a_1 z, a_1 y - a_2 x) \\ &= (a_1, a_2, a_3) \times (x, y, z)\ .\end{aligned}$$

Hence,

$$\begin{aligned}\iota_{A^\sharp}\tilde{\omega} = (a_1 y^2 - a_2 xy - a_3 xz + a_1 z^2)\mathrm{d}x \\ + (-a_1 xy + a_2 x^2 + a_2 z^2 - a_3 yz)\mathrm{d}y \\ + (a_3 x^2 - a_1 xz - a_2 yz + a_3 y^2)\mathrm{d}z\ .\end{aligned}$$

(e) Restrict $\iota_{A^\sharp}\tilde{\omega}$ on $\mathbf{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$, and use it to find out the moment map.

We first use $x^2 + y^2 + z^2 = 1$ to rewrite $\iota_{A^\sharp}\omega$ as

$$\iota_{A^\sharp}\omega = (a_1\mathrm{d}x + a_2\mathrm{d}y + a_3\mathrm{d}z) - (a_1 x + a_2 y + a_3 z)(x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z)\ .$$

Since $\mathrm{d}(x^2 + y^2 + z^2) = 0$, the second term vanishes on $\mathbf{S}^2$, and

$$\iota_{A^\sharp}\omega = (a_1\mathrm{d}x + a_2\mathrm{d}y + a_3\mathrm{d}z) = \mathrm{d}\langle (x, y, z), (a_1, a_2, a_3)\rangle\ .$$

It follows that $\mu(x, y, z) = (x, y, z) + (c_1, c_2, c_3)$ via the identification of $\mathfrak{so}^*(3)$ with $\mathbb{R}^3$. The SO(3)-equivariant condition implies that (c_1, c_2, c_3) must be the zero vector. Therefore, $\mu : \mathbf{S}^2 \to \mathbb{R}^3$ is the original embedding.

$(1 + 1 + 1 + 1 + 3 = 7\text{points})$