INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 14

DUE: MONDAY, DECEMBER 23

(1) (from [CdS1, Homework 10]) Let (M, J) be an almost complex manifold. Its *Nijenhuis tensor* $\mathcal{N}$ is defined by

$$\mathcal{N}(v,w) = [Jv, Jw] - J[v, Jw] - J[Jv, w] - [v, w] \ ,$$

where v and w are vector fields on M.

(a) Show that $\mathcal{N}$ is actually a tensor.

(b) Show that if M is a complex manifold and J is the corresponding complex structure, then $\mathcal{N} \equiv 0$.

(c) Compute $\mathcal{N}(v, Jv)$. Deduce that, if M is a surface, then $\mathcal{N} \equiv 0$.

(2) We calculated the Lie algebra cohomology of $\mathrm{SL}(2;\mathbb{R})$ in Homework 9. Its Lie algebra consists of traceless matrices:

$$\mathfrak{sl}(2;\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \mathrm{trace}(A) = 0\} \ .$$

The following matrices form a basis for $\mathfrak{sl}(2;\mathbb{R})$:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ , \qquad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \ .$$

They satisfy the following relations:

$$[H, X] = 2X \ , \qquad [H, Y] = -2Y \ , \qquad [X, Y] = H \ . \tag{0.1}$$

Now, consider a symplectic manifold (M^{2n}, ω). Let $\Omega^*(M) = \oplus_k \Omega^k(M)$ be the space of all differential forms on M. Remember that ω gives an isomorphism between T^*M and TM, and then induces an isomorphism between $\Lambda^2 T^*M$ and $\Lambda^2 TM$. Let Λ be the image of ω under this isomorphism, and it is called the *Poisson bivector*. With this understood, introduce three operators $\mathcal{L}$, Λ and $\mathcal{H}$ on $\Omega^*(M)$:

$$\begin{aligned}
\mathcal{L}(A) &= \omega \wedge A \ , \\
\Lambda(A) &= \text{contract } A \text{ with the Poisson bivector } \Lambda \ , \\
\mathcal{H}(A) &= \sum_{k=0}^{2n} (n-k)(\Pi^k A)
\end{aligned}$$

where Π^k is the projection $\Omega^*(M) \to \Omega^k(M)$. Note that $\mathcal{L}$ increases the degree of A by 2, Λ decreases the degree by 2, and $\mathcal{H}$ preserves the degree.

(a) Write down the Poisson bivector Λ in terms of a Darboux coordinate.

(b) Show that $\mathcal{L}$, Λ and $\mathcal{H}$ form an $\mathfrak{sl}_2$-triple. That is to say, their brackets satisfy the relations (0.1).

If (M^{2n}, ω, J, g) is a *compact Kähler manifold*, one can show that $\mathcal{L}$, Λ and $\mathcal{H}$ commute with the Laplace operator Δ_d. Then one can apply the representation theory of $\mathfrak{sl}(2;\mathbb{R})$ to study the de Rham cohomology of M. As a result, one can prove the *Hard Lefschetz theorem.* It says that

$$\mathcal{L}^k : \mathrm{H}^{n-k}_{\mathrm{dR}}(M) \longrightarrow \mathrm{H}^{n+k}_{\mathrm{dR}}(M)$$

is an isomorphism for any $k \in \{0, 1, \dots, n\}$.