INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 13

DUE: MONDAY, DECEMBER 16

This Homework set is taken from [CdS1, Homework 8].

(1) Let V be a $2n$-dimensional vector space. Let $\mathcal{S}(V)$ and $\mathcal{J}(V)$ be the spaces of (linear) symplectic forms and (linear) complex structures on V, respectively.

  (a) Take an $\omega \in \mathcal{S}(V)$. Let Sp be the group of symplectomorphisms of (V, ω). Show that $\mathcal{S}(V) = \mathrm{GL}(V)/\mathrm{Sp}(V, \omega)$. Here, $\mathrm{GL}(V)$ is the group of $\mathbb{R}$-linear isomorphisms of V.

  (b) Take a $J \in \mathcal{J}(V)$. Let $\mathrm{GL}(V, J)$ be the group of complex isomorphisms of (V, J). Show that $\mathcal{J}(V) = \mathrm{GL}(V)/\mathrm{GL}(V, J)$.

(2) Consider $\mathbb{R}^{2n} \cong \mathbb{C}^n$ endowed with the standard metric, complex structure, and the symplectic form. The groups leave these structures invariant are $\mathrm{O}(2n)$, $\mathrm{GL}(n;\mathbb{C})$ and $\mathrm{Sp}(n)$, respectively. You can find the definition of $\mathrm{Sp}(n)$ in Homework 2. We can identify a complex $n \times n$ matrix with a $(2n) \times (2n)$ matrix by

$$X + iY \;\mapsto \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix} .$$

With this understood, consider the following subgroups of $\mathrm{GL}(2n;\mathbb{R})$:

$$\mathrm{O}(2n),\ \mathrm{GL}(n;\mathbb{C}),\ \mathrm{Sp}(n)\ \text{ and }\ \mathrm{U}(n)\ .$$

Show that the intersection of any two of them is $\mathrm{U}(n)$.

(3) Let (V, ω) be a symplectic vector space of dimension $2n$, and let $J : V \to V$ be a complex structure on V.

  (a) Prove that, if J is ω-compatible and L is a Lagrangian subspace of (V, ω), then JL is also Lagrangian and $JL = L^{\perp}$, where $\perp$ denotes orthogonality with respect to the positive inner product $g(u, v) = \omega(u, Jv)$.

  (b) Deduce that J is ω-compatible if and only if there exists a symplectic basis for V of the form

$$e_1,\, e_2,\, \dots,\, e_n,\ f_1 = Je_1,\, f_2 = Je_2,\, \dots,\, f_n = Je_n$$

  where $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$ and $\omega(e_i, f_j) = \delta_{ij}$.