INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 12

DUE: MONDAY, DECEMBER 9

(1) Let $P \subset \mathbb{R}^2$ be a 2-dimensional Delzant polytope with vertices $p_1, \dots, p_d$ ordered in a counterclockwise fashion in the plane and edges given by the primitive[1] vectors $\alpha_1, \dots, \alpha_d \in \mathbb{Z}^2$ where α_i points from p_i to p_{i+1}, $p_{d+1} := p_1$. Show that there are integers e_i for $i = 1, \dots, d$ such that $e_i \alpha_i = \alpha_{i-1} + \alpha_{i+1}$, for $i = 1, \dots, d$ with $\alpha_{d+1} := \alpha_1$ and $\alpha_0 := \alpha_d$. (*Hint.* α_i and α_{i+1} constitute an integral basis for any i.)

(2) Let $P \subset (\mathbb{R}^2)^*$ be the convex hull of $(0,0)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$ and $(\frac{1}{2}, \frac{1}{2})$. Work out the Delzant construction for P. What follows is basically what you need to do.
- Find out u_j and λ_j for $j = 1, 2, 3, 4$.
- Work out $0 \to H \xrightarrow{i} \mathbf{T}^4 \xrightarrow{\pi} \mathbf{T}^2 \to 0$.
- Denote by μ be the moment map of the standard action of $\mathbf{T}^4$ on $\mathbb{C}^4$, and normalize it by $\mu(0) = (-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4)$.
- Check *directly* that H acts on $(\iota^* \circ \mu)^{-1}(0) = \mu^{-1}(\pi^* P)$ freely.
- Choose a splitting $\mathbf{T}^4 \xleftarrow{j} \mathbf{T}^2$, and then $\mathbf{T}^2$ acts on $(\iota^* \circ \mu)^{-1}(0)/H$ via j. Justify that the moment polytope is exactly P.

[1] A primitive vector $\mathbf{v}$ is a vector in $\mathbb{Z}^n$, and cannot be written as $s\mathbf{w}$ for $\mathbf{w} \in \mathbb{Z}^n$, $s \in \mathbb{Z}$ and $|s| > 1$.