INTRODUCTION TO SYMPLECTIC GEOMETRY HOMEWORK 11

DUE: MONDAY, DECEMBER 2

(1) Let V be a finite dimensional topological vector space. Suppose that $S \subset V$ be a *closed* set with *non-empty interior*. The (*converse*) *supporting hyperplane theorem* asserts that if for any $p \in \partial S$, there exists an $\ell_p \in V^*$ such that

$$\ell_p(q) \leq \ell_p(p) \quad \text{for any } q \in S \ .$$

Then, S is convex.

Now, prove this theorem by the following steps.

(a) Let q_0 be an interior point of S, and q_1 be a point in S. Show that $q_t = (1-t)q_0 + tq_1$ belongs to S for any $t \in (0, 1)$. (*Hint.* Suppose that it is not true. Consider $t_0 = \inf\{t \in [0, 1] \,|\, q_t \notin S\}$. Since q_0 is an interior point and S is closed, $t_0 \in (0, 1)$. The point q_{t_0} must be a boundary point of S. What about its supporting hyperplane?)

(b) Let p_0 and p_1 be two points of S. Show that $(1 - t)p_0 + tp_1$ belongs to S for any $t \in (0, 1)$. (*Hint.* Again prove by contradiction. Choose an interior point $q \in S$. Let $p_{t,s} = (1 - t)((1 - s)q + sp_0) + tp_1$. Consider $s_0 = \inf\{s \in [0, 1] \,|\, \{p_{t,s}\}_{t\in[0,1]} \not\subset S\}$. Due to Part (a) and the closedness of S, $s_0 \in (0, 1)$. There must exist some point of $\{p_{t,s_0}\}_{t\in[0,1]}$ which is the boundary point of S. What about its supporting hyperplane?)

(2) For any non-negative integer m, define a 4-dimensional manifold as follows. Take two copies of $\mathbb{C} \times \mathbb{CP}^1$; let $(z, [U_0 : U_1])$ and $(w, [V_0 : V_1])$ be their coordinate. Glue them together by

$$(w, [V_0 : V_1]) = (z^{-1}, [z^m U_0 : U_1]) \tag{0.1}$$

when $zw \neq 0$. Denote the resulting manifold by $\mathbf{F}_m$. It admits a map to $\mathbb{CP}^1$ induced by $[z : 1] = [1 : w]$, and each fiber is $\mathbb{CP}^1$. That is to say, $\mathbf{F}_m$ is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. The second component of the above charts is the fiber.

Let ω_F be the following 2-form

$$\begin{aligned}\omega_F &= \frac{i}{2}\partial\bar{\partial} \log\Big(|U_1|^2 + (1 + |z|^2)^m |U_0|^2\Big) \\ &= \frac{i}{2}\partial\bar{\partial} \log\Big(1 + (1 + |z|^2)^m |u|^2\Big) \end{aligned} \tag{0.2}$$

where the expression is on the first chart using the coordinate $(z, [u : 1])$. A straightforward computation shows that ω_F is equal to

$$\omega_F = \frac{i}{2}\partial\bar{\partial} \log\Big(1 + (1 + |w|^2)^m |v|^2\Big)$$

on the second chart using the coordinate $(w, [v:1])$.

(a) Write down (0.2) explicitly. More precisely, find out the coefficient functions in front of $\mathrm{d}u \wedge \mathrm{d}\bar{u}$, $\mathrm{d}z \wedge \mathrm{d}\bar{u}$, $\mathrm{d}u \wedge \mathrm{d}\bar{z}$ and $\mathrm{d}z \wedge \mathrm{d}\bar{z}$.

Let ω_B be

$$\omega_B = \frac{i}{2}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2} = \frac{i}{2}\frac{\mathrm{d}w \wedge \mathrm{d}\bar{w}}{(1+|w|^2)^2} \ .$$

It is the pull-back of the Fubini–Study form through the fibration map $\mathbf{F}_m \to \mathbb{CP}^1$. Define

$$\omega = \omega_F + \omega_B \ . \tag{0.3}$$

(b) Use Part (a) to show that ω is symplectic. By construction, it is always d-closed. You only need to check that it is non-degenerate on the coordinate chart $(z, [u:1])$.

Theses manifold $\mathbf{F}_m$ are called *Hirzebruch surfaces*[1]. Note that $\mathbf{F}_0$ is nothing more than $\mathbb{CP}^1 \times \mathbb{CP}^1$. The symplectic form ω defined by (0.3) is the sum of the pull-back of the Fubini–Study forms from each factor.

[1]The term 'surface' is due to the fact that they are complex manifolds of complex dimension two.