INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 10

DUE: MONDAY, NOVEMBER 25

(1) Let $A \in \mathrm{GL}(2n;\mathbb{R})$ be skew-symmetric, $A = -A^T$. Consider its polar factorization[1] $A = PJ$. Show that A and P commute: $AP = PA$.

(2) For a Lie group G, there exists, up to a constant multiple, a unique left-invariant volume form $\mathrm{d}V_L$. There is also a right-invariant volume form $\mathrm{d}V_R$, unique up to a constant multiple.

With any nonzero $\mu \in \Lambda^{\dim W} W^*$, we can define a group homomorphism

$$\begin{array}{rccc} \det : & \mathrm{GL}(W) & \to & (\mathbb{R}\backslash\{0\}, \times) \\ & A & \mapsto & \dfrac{\mu(Aw_1, Aw_2, \ldots)}{\mu(w_1, w_2, \ldots)} \end{array}$$

where $\{w_j\}_{j=1}^{\dim W}$ is a basis for W. Since μ is unique up to a constant multiple, det does not depend on the choice of μ.

(a) Show that

$$(R^*_{g^{-1}}\mathrm{d}V_L)|_e = (\det \mathrm{Ad}_g)\,(\mathrm{d}V_L)|_e$$

for any $g \in G$.

(b) When G is compact and connected, prove that $\mathrm{d}V_L$ is also right-invariant.

(c) Consider the following Lie group

$$G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x, y \in \mathbb{R},\ y > 0 \right\} .$$

Check that $\mathrm{d}V_L = y^{-2}\mathrm{d}x \wedge \mathrm{d}y$ and $\mathrm{d}V_R = y^{-1}\mathrm{d}x \wedge \mathrm{d}y$.

(3) Consider $\mathbb{CP}^2$ with the Fubini–Study form ω_{FS}. Consider the following $\mathbf{T}^3$-action:

$$(e^{i\theta_0}, e^{i\theta_1}, e^{i\theta_2}) \bullet [z_0 : z_1 : z_2] = [e^{i\theta_0} z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2] .$$

(a) Calculate the moment map μ of this $\mathbf{T}^3$-action, and draw the moment polytope.

(b) If you did Part (a) correctly, the moment polytope should be a triangle. Denote it by $\triangle$. What are the pre-images under μ of the vertices and edges of $\triangle$?

(c) Find out the stabilizer of $p \in \mathbb{CP}^2$ for p in:

(i) μ^{-1}(vertex of $\triangle$);

(ii) μ^{-1}(edge of $\triangle$);

(iii) μ^{-1}(interior of $\triangle$).

[1]See (2.d) of Homework 9, or the lecture note.

(4) Let $\mathcal{H}$ be the vector space of $n \times n$ Hermitian matrices. The unitary group $\mathrm{U}(n)$ acts on $\mathcal{H}$ by conjugation:

$$A \bullet \xi = A \xi A^{-1} \tag{0.1}$$

for $A \in \mathrm{U}(n)$, $\xi \in \mathcal{H}$.

For each (unordered) n-tuple of real numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ Hermitian matrices whose spectrum is λ. In (1.e) of Homework 8, we have shown that $\mathcal{H}_\lambda$ admits a symplectic form defined by

$$\omega_\xi(X^\sharp, Y^\sharp) = i \,\mathrm{trace}([X, Y]\xi)$$

for any $\xi \in \mathcal{H}_\lambda$ and $X, Y \in \mathfrak{u}(n)$. The vector field $X^\sharp$ on $\mathcal{H}_\lambda$ is induced by the action (0.1).

Now, prove *Schur's theorem*:

(a) for any $\xi \in \mathcal{H}_\lambda$,

$$\mathrm{diag}(\xi) \in \text{the convex hull of } \{(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(n)}) \,|\, \sigma \in S_n\} \tag{0.2}$$

where $\mathrm{diag} : \mathcal{H} \to \mathbb{R}^n$ is the diagonal map, and S_n is the symmetric group;

(b) conversely, every point in the convex hull (0.2) is $\mathrm{diag}(\xi)$ for some $\xi \in \mathcal{H}_\lambda$.

(*Hint.* Consider $\mathbf{T}^n \subset \mathrm{U}(n)$. It acts on $\mathcal{H}_\lambda$ by (0.1). What is its moment map? Where are the fixed points of this $\mathbf{T}^n$-action? What does the convexity theorem say in this case? You may also see note1104.)

(c) When $n = 3$, draw the convex hull (0.2) for $\lambda_1 = -\frac{1}{2}$ and $\lambda_2 = \lambda_3 = 0$. (*Remark.* In (1.f) of Homework 8, we recognized that the corresponding $\mathcal{H}_\lambda$ is $\mathbb{CP}^2$. You can compare this part with (3.a) above.)

(d) When $n = 3$, draw the convex hull (0.2) for $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 2$.

(e) When $n = 2$, prove Schur's theorem directly. Namely, prove Part (a) and (b) for $n = 2$ using only linear algebra.