INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 9

DUE: MONDAY, NOVEMBER 18

(1) Consider $\mathbb{CP}^3$ with the Fubini–Study form ω_{FS}. Consider the following $\mathbf{S}^1$-action:

$$e^{i\phi} \bullet [z_0 : z_1 : z_2 : z_3] = [z_0 : e^{i\phi} z_1 : e^{i\phi} z_2 : z^{-i\phi} z_3] \ .$$

With #3 of Homework 8, it is not to see that this $\mathbf{S}^1$-action is symplectic.

(a) Show that its moment map is

$$\mu([z_0 : z_1 : z_2 : z_3]) = -\frac{1}{2}\frac{|z_1|^2 + |z_2|^2 - |z_3|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2} \ . \tag{0.1}$$

We identify the Lie algebra of $\mathbf{S}^1$ with $\mathbb{R}$, and also its dual. (*Hint*. Remember that $(\mathbb{CP}^3, \omega_{\mathrm{FS}})$ can be constructed as the symplectic reduction of $(\mathbb{C}^4, \omega_0)$.)

The denominator of the expression makes it homogeneous of degree 0. And we do not have to impose the constraint condition $|z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 = 1$.

(b) Find out those points in $\mathbb{CP}^3$ where the stabilizer is non-trivial.

From (0.1), the image of μ is $[-\frac{1}{2}, \frac{1}{2}]$. Namely, the moment polytope is $[-\frac{1}{2}, \frac{1}{2}]$. If you did Part (b) correctly, $\mathbf{S}^1$ acts freely on $\mu^{-1}(c)$ for any $c \in (-\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$. Therefore, we can perform symplectic reduction on these values of c. Let us try to visualize $\mu^{-1}(\frac{1}{4})/\mathbf{S}^1$. Consider the map

$$\begin{array}{rccc} \pi : & \mu^{-1}(\frac{1}{4}) \subset \mathbb{CP}^3 & \to & \mathbb{CP}^2 \\ & [z_0 : z_1 : z_2 : z_3] & \mapsto & [\dfrac{z_0^2}{z_3} : z_1 : z_2] \ . \end{array} \tag{0.2}$$

(c) Check that π is well-defined on $\mu^{-1}(\frac{1}{4})$.

(d) Prove that for any $p, q \in \mu^{-1}(\frac{1}{4})$,

$$\pi(p) = \pi(q) \quad \text{if and only if} \quad p = e^{i\phi} \bullet q$$

for some $e^{i\phi} \in \mathbf{S}^1$.

If you keep working on this, you can use π to show that $\mu^{-1}(c)/\mathbf{S}^1$ is diffeomorphic to $\mathbb{CP}^2$ for any $c \in (0, \frac{1}{2})$. However, the topology of $\mu^{-1}(c)/\mathbf{S}^1$ is different when $c \in (-\frac{1}{2}, 0)$. It is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. (The notation # means connected sum in manifold topology, and $\overline{\mathbb{CP}^2}$ is $\mathbb{CP}^2$ with opposite orientation. In algebraic geometry, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is $\mathbb{CP}^2$ blow-up at one point.)

(2) Consider the degree 2 special linear group:

$$\mathrm{SL}(2;\mathbb{R}) = \{G \in \mathrm{GL}(2;\mathbb{R}) \mid \det(G) = 1\} \ .$$

(a) Show that its Lie algebra consists of traceless matrices. Namely,

$$\mathfrak{sl}(2;\mathbb{R}) = \{A \in M_{2\times 2}(\mathbb{R}) \mid \mathrm{trace}(A) = 0\} \ .$$

(b) With Part (a), we may choose the following basis for $\mathfrak{sl}(2;\mathbb{R})$:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \qquad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} .$$

Calculate their Lie brackets.

(c) Compute the Lie algebra cohomology of $\mathfrak{sl}(2;\mathbb{R})$.

(d) Show that $\mathrm{SL}(2;\mathbb{R})$ is homeomorphic to $\mathbb{H} \times \mathbf{S}^1$ where $\mathbb{H}$ is the upper-half plane. (*Hint.* Any $G \in \mathrm{GL}(2;\mathbb{R})$ has a unique polar factorization: $G = PO$ where P is a positive-definite symmetric matrix, and O is orthogonal. The matrix P is the square root of GG^T, and $O = P^{-1}G$.)

If you solve this question correctly, it provides an example that the de Rham cohomology of the Lie group is different from the Lie algebra cohomology. Surely $(\mathbb{R}^n, +)$ is the simplest example, but this one is more interesting.

(3) Draw the moment polytope for the following $\mathbf{T}^2$-action on $(\mathbb{CP}^1 \times \mathbb{CP}^1, \pi_1^*\omega_{\mathrm{FS}} + \pi_2^*\omega_{\mathrm{FS}})$, where π_1 and π_2 are the projection maps onto the two $\mathbb{CP}^1$-components.

$$(e^{i\alpha}, e^{i\beta}) \bullet ([z_0, z_1], [w_0, w_1]) = ([z_0, e^{i\alpha}z_1], [w_0, e^{3i\beta}w_1]) \ .$$