INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 8

DUE: MONDAY, NOVEMBER 4

(1) (From [CdS1. Homework 16]) Let $\mathcal{H}$ be the vector space of $n \times n$ Hermitian matrices. The unitary group $\mathrm{U}(n)$ acts on $\mathcal{H}$ by conjugation:

$$A \cdot \xi = A\xi A^{-1} \ , \tag{0.1}$$

for $A \in \mathrm{U}(n)$, $\xi \in \mathcal{H}$. For each (unordered) n-tuple of real numbers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, let $\mathcal{H}_\lambda$ be the set of all $n \times n$ Hermitian matrices whose spectrum is λ.

(a) Show that the orbits of the $\mathrm{U}(n)$-action are $\mathcal{H}_\lambda$. For a fixed λ, what is the stabilizer of a point in $\mathcal{H}_\lambda$? (Since the stabilizers of different points are conjugate to each other, it suffices to find the stabilizer of some particular point.)

(b) Show that the symmetric bilinear form on $\mathcal{H}$,

$$(V, W) \ \mapsto \ \mathrm{trace}(VW) \ ,$$

is non-degenerate.

(c) Note that the Lie algebra of $\mathrm{U}(n)$ is the set of skew-Hermitian matrices. Namely, $\mathfrak{u}(n) = i\mathcal{H}$. For $\xi \in \mathcal{H}$, define a skew symmetric bilinear form ω_ξ on $\mathfrak{u}(n)$ by

$$\omega_\xi(X, Y) = i\,\mathrm{trace}([X, Y]\xi)$$

for any $X, Y \in i\mathcal{H}$. Check that

$$\omega_\xi(X, Y) = i\,\mathrm{trace}(X[Y, \xi]) \quad \text{and} \quad [Y, \xi] \in \mathcal{H} \ .$$

(d) Show that the kernel of ω_ξ is

$$K_\xi = \{Y \in \mathfrak{u}(n) \mid [Y, \xi] = 0\} \ ,$$

and show that K_ξ is the Lie algebra of the stabilizer of the action (0.1) at ξ.

It follows that ω_ξ's induce a *non-degenerate* 2-form on the orbits $\mathcal{H}_\lambda$. More precisely, the tangent space of $\mathcal{H}_\lambda$ at ξ is identified with $\mathfrak{u}(n)/K_\xi$, and ω_ξ is a non-degenerate, skew-symmetric bilinear map on this vector space. Denote this two form by ω.

(e) Prove that $\mathrm{d}\omega = 0$. Here are two facts that can help you:

- $(\mathrm{d}\omega)(X, Y, Z) = X(\omega(Y, Z)) + Z(\omega(X, Y)) + Y(\omega(Z, X)) - \omega([Y, Z], X) - \omega([X, Y], Z) - \omega([Z, X], Y)$ for any *vector fields* X, Y, Z. Note that for X, Y, Z, the left hand side involves only the evaluation.

- Given $X, Y, Z \in \mathfrak{u}(n)$, they can be regarded as *vector field* on $\mathcal{H}_\lambda$ by $T_\xi \mathcal{H}_\lambda = \mathfrak{u}(n)/K_\xi$. With this understood, the derivative of the function $\xi \mapsto \operatorname{trace}([X,Y]\xi)$ with respect to Z is

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \operatorname{trace}([X,Y](e^{tZ}\xi e^{-tZ})) = \operatorname{trace}([X,Y][Z,\xi]) \ .$$

(f) With the above discussions, each orbit $\mathcal{H}_\lambda$ is a *compact symplectic manifold*. Identify the manifold $\mathcal{H}_\lambda$ for

$$\lambda_1 \neq \lambda_2 = \cdots = \cdots = \lambda_n \ ?$$

Namely, there are only two distinct eigenvalues: one has multiplicity 1, and another has multiplicity $n-1$. In general, the manifold $\mathcal{H}_\lambda$ is the so-called *partial flag variety*.

This construction is a special case of the *coadjoint orbits*: the orbits of the action $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ are symplectic manifolds. The coadjoint orbits appear in [CdS1, Homework 17].

(g) Show that, for any skew-Hermitian matrix $X \in \mathfrak{u}(n)$, the vector field on $\mathcal{H}$ generated by X for the action (0.1) is $X^\sharp_\xi = [X, \xi]$ at any $\xi \in \mathcal{H}$. (*Hint.* This is basically the second fact in Item (e).)

(2) Compute the moment map for the following group actions on $\mathbb{C}^3$ with the standard symplectic form $\omega_0 = \frac{i}{2}\sum_{j=1}^3 \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$.

(a) $G = \mathbf{T}^2$: the action is

$$(e^{i\theta_1}, e^{i\theta_2}) : \ (z_1, z_2, z_3) \ \mapsto \ (e^{i\theta_1} z_1, e^{i\theta_2} z_2, e^{-i(\theta_1+\theta_2)} z_3) \ .$$

(b) $G = \mathrm{U}(1) \times \mathbb{R}$: the action is

$$(e^{i\theta}, x) : \ (z_1, z_2, z_3) \ \mapsto \ (e^{i\theta} z_1, e^{-i\theta} z_2, z_3 + x) \ .$$

(3) A *complex manifold* is a manifold with an atlas of charts to the open subsets in $\mathbb{C}^n$, such that the transition maps are holomorphic. On a complex manifold, the exterior derivative d can be decomposed as the following. Let $\{z_1, z_2, \ldots, z_n\}$ be the local complex coordinate. Introduce the vector fields and differential forms:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big) \ , \qquad \mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j \ ,$$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big) \ , \qquad \mathrm{d}\bar{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j$$

where $x_j = \mathrm{Re}(z_j)$ and $y_j = \mathrm{Im}(z_j)$. Define the operators ∂ and $\bar\partial$ by

$$\partial = \sum_{j=1}^n \frac{\partial}{\partial z_j}\mathrm{d}z_j \qquad \text{and} \qquad \bar\partial = \sum_{j=1}^n \frac{\partial}{\partial \bar{z}_j}\mathrm{d}\bar{z}_j \ .$$

Since the transition maps are holomorphic, ∂ and $\bar\partial$ are invariant under coordiante change.

By writing everything in terms of x_j and y_j, we find that $\mathrm{d} = \partial + \bar{\partial}$. The condition $\mathrm{d}^2 = 0$ implies that

$$\partial^2 = 0 \ , \qquad \bar{\partial}^2 = 0 \ , \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0 \ . \tag{0.2}$$

Note that the standard symplectic form on $\mathbb{C}^n$ can be written as $\frac{i}{2}\partial\bar{\partial}(\sum_{j=1}^n |z_j|^2)$.

The complex projective space is defined to be

$$\mathbb{CP}^n = \frac{\big\{(u_0, u_1, \ldots, u_n) \in \mathbb{C}^{n+1}\backslash\{0\}\big\}}{(u_0, u_1, \ldots, u_n) \sim (\lambda u_0, \lambda u_1, \ldots, \lambda u_n) \ \text{ for } \lambda \in \mathbb{C}\backslash\{0\}}$$

(a) Consider the coordinate chart $U_0 : (z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto [(1, z_1, z_2, \ldots, z_n)] \in \mathbb{CP}^n$ and $U_1 : (w_1, \ldots, w_n) \in \mathbb{C}^n \mapsto [(w_1, 1, w_2, \ldots, w_n)] \in \mathbb{CP}^n$. Check that the transition function between U_0 and U_1 is holomorphic[1]. We can then construct an atlas using similar charts, and $\mathbb{CP}^n$ is a complex manifold.

(b) On U_0, consider that 2-form defined by

$$\omega = \frac{i}{2}\partial\bar{\partial}\log(1 + \sum_{j=1}^{n} |z_j|^2) \ .$$

Consider the similar expression on U_1 (subsitute z_j by w_j). Check that the expression defines the same 2-form. (*Hint.* Suppose that h is a holomorphic function. Then $\partial\bar{\partial}\log|h|^2 = \partial\big(|h|^{-2}(h\,\bar{\partial}\bar{h})\big) = \partial\big(\bar{h}^{-1}\bar{\partial}\bar{h}\big) = 0$.)

It follows that ω is a 2-form on $\mathbb{CP}^n$.

(c) Use (0.2) to show that $\mathrm{d}\omega = 0$.

(d) Check that ω is non-degenerate at $(0, 0, \ldots, 0) \in U_0$.

It is not hard to argue from Item (d) that ω is non-degenerate everywhere. It follows that ω defines a symplectic form on $\mathbb{CP}^n$. It is called the *Fubini–Study form*.

[1] Namely, it only involves z_j but not $\bar{z}_j$ Or, it is annihilated by $\frac{\partial}{\partial \bar{z}_j}$ for all j.