# INTRODUCTION TO SYMPLECTIC GEOMETRY
# HOMEWORK 7

DUE: MONDAY, OCTOBER 28

(1) Let $(M, \omega)$ be a symplectic manifold. Let $(\mathcal{C}^\infty(M), \{\,,\})$ be the Poisson algebra of $M$. We have proved that

$$X_{\{f,g\}} = -[X_f, X_g] \ . \tag{0.1}$$

We compute

$$\begin{aligned}
\{f,\{g,h\}\} + \{g,\{h,f\}\} &= \omega(X_f, X_{\{g,h\}}) + \omega(X_g, X_{\{h,f\}}) \\
&= -X_f(\{g,h\}) - X_g(\{h,f\}) \\
&= X_f(X_g(h)) - X_g(X_f(h)) \\
&= [X_f, X_g](h) \\
&= -X_{\{f,g\}}(h) \qquad \text{by (0.1)} \\
&= -\{h,\{f,g\}\} \ ,
\end{aligned}$$

and this proves the Jacobi identity. In this exercise, you are asked to prove the Jacobi identity using another argument.

(a) Let $\varphi_t = \exp(tX_h)$ be the flow of $X_h$. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} X_{\varphi_t^* f} = X_{X_h(f)} \ . \tag{0.2}$$

(b) Note that $\varphi_t^*\{f,g\} = \{\varphi_t^* f, \varphi_t^* g\}$. Differentiate the equation with respect to $t$ and evaluate at $t = 0$. Use it to conclude the Jacobi identity.

(2) (from [CdS1, #1 of Howework 15]) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. $f$ is called *strictly convex* if $f''(x) > 0$ for all $x \in \mathbb{R}$.

(a) Assuming that $f$ is strictly convex, prove that the following four conditions are equivalent:
- $f' = 0$ at some point $x_0$,
- $f$ has a local minimum at some point $x_0$,
- $f$ has a unique (global) minimum at some point $x_0$,
- $f(x) \to \infty$ as $x \to \pm\infty$. (*Hint.* Strictly convexity implies that the graph of $f$ must lie above its tangent line (of any point).)

The function $f$ is *stable* if it satisfies one (and hence all) of these conditions. The same story works in higher dimensions as well, see [CdS1, #2 of Howework 15].

(b) For what values of $a$ is the function $e^x + ax$ stable? For those values of $a$ for which it is not stable, what does the graph look like?

(3) (from [CdS1, #3 of Howework 15]) let $V$ be an $n$-dimensional vector space and $F : V \to \mathbb{R}$ be a smooth function. Since $V$ is a vector space, there is a canonical identification $T_p^*V \cong V^*$, for every $p \in V$. Therefore, we can define a map

$$L_F : V \to V^* \qquad (\textit{Legendre transform})$$

by setting

$$L_F(p) = (dF)|_p \in T_p^*V \cong V^* \ .$$

Show that, if $F$ is strictly convex, then, for every point $p \in V$, $L_F$ maps a neighborhood of $p$ diffeomorphically onto a neighborhood of $L_F(p)$.

(4) (from [CdS1, #4, #6 of Howework 15]) Given any strictly convex function $F : V \to \mathbb{R}$, we will denote by $S_F$ the set of $\ell \in V^*$ for which the function

$$\begin{array}{rccc} F_\ell : & V & \to & \mathbb{R} \\ & p & \mapsto & F(p) - \ell(p) \end{array}$$

is stable. Prove that:

(a) The set $S_F$ is open and convex.
(b) $L_F$ maps $V$ diffeomorphically onto $S_F$.
(c) If $\ell \in S_F$ and $p_0 = L_F^{-1}(\ell)$, then $p_0$ is the unique minimum point of the function $F_\ell$.

Let $F^* : S_F \to \mathbb{R}$ be the function defined by

$$F^*(\ell) = -\min_{p \in V} F_\ell(p) \ . \tag{0.3}$$

It is not hard to prove that the function $F^*$ is also a smooth function, and is called the *dual* of the function $F$.

(d) Prove the Young's inequality: $F(p) + F^*(\ell) \geq \ell(p)$ for any $p \in V$ and $\ell \in S_F$.
(e) In Exercise (3), the Legendre transform is defined using derivative. Give a definition of $L_F(p)$ that does not involve taking derivative. (*Hint.* The construction of $F^*$ does not use derivatives.)

(5) (from [CdS1, #5 of Howework 15]) For the following strictly convex functions on $\mathbb{R}^1$, find out the set $S_f$ defined in Exercise (4).

(a) $f(x) = x^4$.
(b) $f(x) = e^x - x$.

In general, $S_F$ may not be the whole space $V^*$. A sufficient condition for $S_F = V^*$ is that $F$ has *quadratic growth at infinity.* That is to say, there exists a positive-definite quadratic form $Q$ on $V$ and a constant $K$ such that $F(p) \geq Q(p) - K$, for all $p \in V$.