INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 6

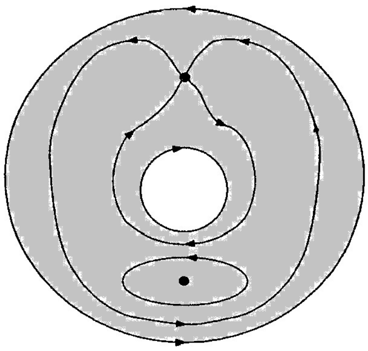
DUE: MONDAY, OCTOBER 21

(1) Consider $\mathbb{R}^2$ with the standard symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y$.

(a) Let $f(x, y) = x^2 + y^2$. Write down X_f, and draw the picture of X_f.

(b) Let $g(x, y) = x^2 - y^2$. Write down X_g, and draw the picture of X_g.

(c) Explain a strategy to construct a symplectomorphism of the the annulus corresponding to the following picture.



(d) With the standard metric $\mathrm{d}x^2 + \mathrm{d}y^2$, we can consider the gradient vector field of a function. For the functions in (a) and (b), compare the gradient vector field of f and g with X_f and X_g.

(2) Let $\mathbf{D}^2$ be the unit disk in $\mathbb{R}^2$. Let (M^4, ω) be a symplectic manifold. Suppose that $f : M \to \mathbf{D}^2$ is a completely integrable system whose fibers are Lagrangian $\mathbf{T}^2$. The purpose of this exercise is give you some idea about the construction of the *action-angle coordinate* for M.

To start, note that M must be diffeomorphic to $\mathbf{D}^2 \times \mathbf{T}^2$. Let u_1, u_2 be the coordinate for $\mathbf{D}^2$, and let ψ_1, ψ_2 be the coordinate for $\mathbf{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Consider the following symplectic form on M:

$$\begin{aligned}\omega = &\big(2 - 2\sin(2\pi\psi_1)\big)\mathrm{d}u_1 \wedge \mathrm{d}u_2 + \big(2 + 4\pi u_1 \cos(2\pi\psi_1)\big)\mathrm{d}u_1 \wedge \mathrm{d}\psi_1 + \mathrm{d}u_2 \wedge \mathrm{d}\psi_2 \\ &+ \mathrm{d}u_1 \wedge \mathrm{d}\psi_2 + 4\pi u_1 \cos(2\pi\psi_1)\mathrm{d}u_2 \wedge \mathrm{d}\psi_1 \ .\end{aligned}$$

The absence of $\mathrm{d}\psi_1 \wedge \mathrm{d}\psi_2$ is equivalent to that the fibers are Lagrangian.

(a) Let γ_1 and γ_2 be the image of the two axes of $\mathbb{R}^2$ in $\mathbf{T}^2$. They form a basis for $\mathrm{H}_1(\mathbf{T}^2; \mathbb{Z})$. For each j, define a 1-form η_j on $\mathbf{D}^2$ by

$$\eta_j(v) = \int_{\gamma_j} \iota_v \omega \tag{0.1}$$

where v is a tangent vector on $\mathbf{D}^2$. (More precisely, the vector on the right hand side of (0.1) should be a lifting $\tilde{v}$ of v. Namely, $f_*(\tilde{v}) = v$, and such a lifting is not unique. The Lagrangian fibration condition implies that the integration over γ_j is independent of the choice of the lifting.) Find out η_j, and check that η_j is d-closed.

(b) Since η_1 and η_2 are d-closed 1-forms on $\mathbf{D}^2$. They are the exterior derivative of some functions. Find out these two functions[1]. We will denote them by x_1 and x_2, and they serve as the *action coordinate*.

Then, compute X_{x_1} and X_{x_2}, and check that the Poisson bracket between x_1 and x_2 vanishes, i.e. $\{x_1, x_2\} = 0$.

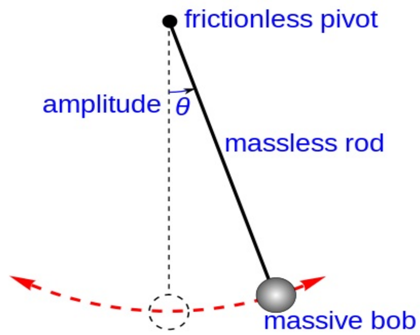
(c) The *angle coordinate* can be constructed inductively. At first, find an angle function θ_1 such that $\{x_1, \theta_1\} = 1$ and $\{x_2, \theta_1\} = 0$. These conditions are equivalent to that $X_{x_1}(\theta_1) = -1$ and $X_{x_2}(\theta_1) = 0$.

(d) Secondly, find an angle function θ_2 such that $\{x_1, \theta_2\} = 0$, $\{x_2, \theta_2\} = 1$ and $\{\theta_1, \theta_2\} = 0$.

(e) Express the symplectic form ω in terms of $x_1, x_2, \theta_1, \theta_2$.

(f) The central fiber $f^{-1}(\mathbf{0})$ is a Lagrangian torus in M. Can we obtain the action-angle coordinate by invoking the Weinstein tubular neighborhood theorem on $f^{-1}(\mathbf{0})$?

(3) (from [CdS1, Howework 13]) The *simple pendulum* is a mechanical system consisting of a massless rigid rod of length ℓ, fixed at one end, whereas the other end has a plumb bob of mass m, which may oscillate in the vertical plane. Assume that the force of gravity is constant pointing vertically downwards, and that this is the only external force acting on this system.



Let θ be the oriented angle between the rod (regarded as a line segment) and the vertical direction. Let ξ be the coordinate along the fibers of T^*S^1 induced by the standard angle coordinate on S^1. The simple pendulum turns out to be describe by the following Hamiltonian function:

$$H(\theta, \xi) = \underbrace{\frac{\xi^2}{2m\ell^2}}_{K} + \underbrace{m\ell(1 - \cos\theta)}_{V} . \tag{0.2}$$

[1]They are not unique. Simply choose one representative.

Here, V is the potential energy, and K is the kinetic energy. Note that $K = \frac{1}{2}m\ell^2 v^2$ in terms of the coordinate $v = \frac{\partial H}{\partial \xi}$.

(a) For simplicity assume that $m = \ell = 1$.

  (i) Plot the level curves of H in the (θ, ξ) plane.

  (ii) Show that there exists a number c such that for $0 < h < c$ the level curve $H = h$ is a disjoint union of closed curves. Show that the projection of each of these curves onto the θ-axis is an interval of length less than π.

  (iii) Show that neither of the assertions in (ii) is true if $h > c$.

  (iv) What types of motion are described by these two types of curves?

  (v) What about the case $H = c$?

(b) Compute the critical points of the function H. Show that, modulo 2π in θ, there are exactly two critical points: a critical point s where H vanishes, and a critical point u where H equals c. These points are called the *stable* and *unstable* points of H, respectively. Justify this terminology, i.e. explain that a trajectory of the Hamiltonian vector field of H whose initial point is close to s stays close to s forever, and show that this is not the case for u. What is happening physically?

In the first lecture we explained that the Hamiltonian mechanics is a reformulation of the Lagrangian mechanics through the Legendre transform. In a way this exercise tells you that it is easier to work with the Hamiltonian formulation.