INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 5

DUE: MONDAY, OCTOBER 14

(1) (from [CdS1, §6.3]) Let $U = \mathbb{R}^2$, and $X = \mathbb{R} \times \{0\} \subset U$ be the x-axis. For any $t \in [0,1]$, define the map

$$\begin{array}{cccc} \rho_t : & U & \to & U \\ & (x,y) & \mapsto & (x,ty) \end{array} .$$

(a) Write down the vector field v_t. (*Hint.* If you take the derivative of $\rho_t(x,y)$ in t, you will obtain the vector field v_t at $\rho_t(x,y)$ but not at (x,y).)

(b) Let $\eta = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$ be a smooth 1-form on U. Write down $\rho_t^*(\iota_{v_t}\eta)$. (*Hint.* $\iota_{v_t}\eta$ should be evaluated at $\rho_t(x,y)$. To avoid confusion about the domain and the target, you may use (x,z) as the coordinate for the target, and $z = ty$.)

(c) Let $\eta = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$ be a smooth 1-form on U. Write down $\rho_t^*(\iota_{v_t}\mathrm{d}\eta)$.

(d) Let $\eta = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$ be a smooth 1-form on U. Show that

$$\eta - f(x,0)\mathrm{d}x = \mathrm{d}\big(\int_0^1 \rho_t^*(\iota_{v_t}\eta)\mathrm{d}t\big) + \int_0^1 \rho_t^*(\iota_{v_t}\mathrm{d}\eta)\mathrm{d}t \ .$$

(2) (from [CdS1, #1 of Homework 6]) Think $\mathbf{S}^2$ as the unit sphere in $\mathbb{R}^3$. For any $p \in \mathbf{S}^2$, $T_p\mathbf{S}^2$ consists of all vectors orthogonal to p. Define a symplectic form on $\mathbf{S}^2$ by

$$\omega_p(u,v) = \langle p, u \times v\rangle$$

where $\langle\ ,\ \rangle$ is the standard inner product, and $\times$ is the exterior product. Parametrize $\mathbf{S}^2$ by the cylindrical coordinate

$$(\theta, z) \ \mapsto \ \big((1-z^2)^{\frac{1}{2}}\cos\theta, (1-z^2)^{\frac{1}{2}}\sin\theta, z\big)$$

where $\theta \in [0, 2\pi]$ and $z \in (-1,1)$. Write down ω in this coordinate.

(3) (from [CdS1, #2 of Homework 6]) Prove Darboux theorem in dimension two. Locally, a symplectic form (area form in this case) is $A(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$ for some positive function $A(x,y)$. Note that it is the exterior derivative of $-(\int_0^y A(x,s)\mathrm{d}s)\mathrm{d}x$. Use this 1-form to construct the Darboux coordinate.

(4) (from [CdS1, #3 of Homework 6]) In dimension two, suppose that ω_0 and ω_1 are symplectic forms that induce the same orientation. Then, their convex combination[1] still defines a symplectic form. This is no longer true in higher dimensions. Consider the following questions on $\mathbb{R}^4$.

[1]It means $(1-t)\omega_0 + t\omega_1$ for some $t \in [0,1]$.

(a) Let $\omega_0 = \mathrm{d}x^1 \wedge \mathrm{d}y^1 + \mathrm{d}x^2 \wedge \mathrm{d}y^2$, and $\omega_1 = -\omega_0$. Check that they induce the same orientation on $\mathbb{R}^4$, but some convex combination degenerates.

(b) Show that ω_0 and ω_1 are *deformation equivalent*[2]. (*Hint.* This 2-form $\mathrm{d}x^1 \wedge \mathrm{d}y^2 + \mathrm{d}y^1 \wedge \mathrm{d}x^2$ might help you.)

(5) **Proposition 8.2** of [CdS1]. Suppose that (V^{2n}, ω) is a symplectic vector space, and $U \subset V$ is a Lagrangian vector subspace. Let W be a vector subspace of V such that $W \oplus U = V$. Then from W, we can *canonically* build a Lagrangian complement to V.

Proof. (a) Prove that $\omega : U \times W \to \mathbb{R}$ is non-degenerate. (Namely, $\forall u \in U\backslash\{0\}$, $\exists v \in W$ such that $\omega(u, v) \neq 0$, and $\forall v \in W\backslash\{0\}$, $\exists u \in U$ such that $\omega(u, v) \neq 0$.)

Hence, it induces an isomorphism $\omega' : U \to W^*$. In order to get a complement to V, consider

$$W' = \big\{ v + A(v) \mid v \in W \big\}$$

where $A : W \to U$ is a linear map.

(b) Show that W' is Lagrangian if and only if

$$\omega(v_1, v_2) = \big(\omega'(A(v_2))\big)(v_1) - \big(\omega'(A(v_1))\big)(v_2) \tag{0.1}$$

for any $v_1, v_2 \in W$.

Note that we can write $\omega(v_1, v_2)$ as

$$\omega(v_1, v_2) = -s\,\omega(v_2, v_1) + (1-s)\,\omega(v_1, v_2)\ . \tag{0.2}$$

It follows that $\omega'(A(v_2)) = -s\,\omega(v_2, \cdot\,)$ and $\omega'(A(v_1)) = (s-1)\,\omega(v_1, \cdot\,)$. Therefore, the canonical choice of s is $\frac{1}{2}$. The coefficient $\frac{1}{2}$ is the canonical choice. With (0.1) and (0.2), we take $A(v)$ to be $(\omega')^{-1}\big(-\frac{1}{2}\omega(v, \cdot\,)\big)$. This finishes the proof of the proposition. □

(6) **Proposition 8.3** of [CdS1]. Suppose that ω_0 and ω_1 are two linear symplectic structures on V^{2n}. Suppose that $U \subset V$ is a Lagrangian vector subspace with respect to both ω_0 and ω_1. Let W be a vector subspace of V such that $W \oplus U = V$. Then from W, we can *canonically* construct a linear isomorphism $L : V \to V$ such that $L|_U = \mathbf{Id}_U$ and $L^*\omega_1 = \omega_0$.

Proof. Let W_0 and W_1 are the canonical complement to U given by Proposition 8.2, with respect to ω_0 and ω_1. It follows from #5(a) that we can define a linear isomorphism $B : W_0 \to W_1$ by

$$B :\ W_0 \xrightarrow{\omega_0'} U^* \xrightarrow{(\omega_1')^{-1}} W_1\ .$$

We can extend it to a linear isomorphism on V by

$$L = \mathbf{Id}_U \oplus B :\ U \oplus W_0 \longrightarrow U \oplus W_1\ .$$

(a) Check that $L^*\omega_1 = \omega_0$.

It is clear that $L|_U = \mathbf{Id}_U$. This completes the proof of the proposition. □

[2] See [CdS1, Definition 7.1]