INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 4

DUE: MONDAY, OCTOBER 7

(1) (from [M&S, §8.2]) Let S be the strip $\{(x,y) \in \mathbb{R} \times [1,2]\}$ with the symplectic form $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Suppose that

$$\psi(x,y) = \big(h(x,y), g(x,y)\big) : S \to S$$

is a diffeomorphism satisfying the following conditions.

- ψ comes from a diffeomorphism on the cylinder $C = \{(x,y) \in \mathbb{R}/\mathbb{Z} \times [1,2]\}$; namely,

$$h(x+1,y) = h(x,y) + 1 \ , \qquad\qquad g(x+1,y) = g(x,y) \ . \tag{0.1}$$

- ψ preserves the symplectic form: $\psi^*\omega = \omega$.
- ψ preserves the two boundary components:

$$g(x,1) = 1 \ , \qquad\qquad g(x,2) = 2 \tag{0.2}$$

  for any $x \in \mathbb{R}$.
- ψ *twists the boundaries in opposite directions*:

$$h(x,1) < x \ , \qquad\qquad h(x,2) > x \tag{0.3}$$

  for any $x \in \mathbb{R}$.

Then, the Poincaré–Birkhoff theorem asserts that ψ has at least two geometrically distinct fixed points.

Here are some questions about the Poincaré–Birkhoff theorem.

(a) Does the map $\psi_1(x,y) = (x + \frac{1}{2}, y)$ admit any fixed point? Can we apply the Poincaré–Birkhoff theorem on ψ_1?

(b) Does the map $\psi_2(x,y) = (x + y - \frac{3}{2}, y^2 - 2y + 2)$ admit any fixed point? Can we apply the Poincaré–Birkhoff theorem on ψ_2?

(c) Prove that $(C, \omega = \mathrm{d}x \wedge \mathrm{d}y)$ is symplectomorphic to the annulus

$$A = \{(u,v) \in \mathbb{R}^2 \mid 1 \leq u^2 + v^2 \leq 2\} \ \text{ with } \omega = \frac{1}{\pi}\mathrm{d}u \wedge \mathrm{d}v \ .$$

(d) The standard map $\psi_0(x,y) = (x + y - \frac{3}{2}, y)$ fixes $\{y = \frac{3}{2}\}$. When a map ψ is *close* to the standard map, it is easier to show that ψ has at least two geometrically distinct point.

Suppose that ψ satisfies all the conditions of the Poincaré–Birkhoff theorem. Moreover, suppose that ψ satisfies the *monotone twist* condition:

$$y < y' \qquad \Longrightarrow \qquad h(x,y) < h(x,y') \tag{0.4}$$

for any $x \in \mathbb{R}$.

(i) It follows from (0.4) that for every $x \in \mathbb{R}$, there exists a unique $y = w(x) \in (1, 2)$ such that $h(x, w(x)) = x$. Prove that $w(x + 1) = w(x)$.

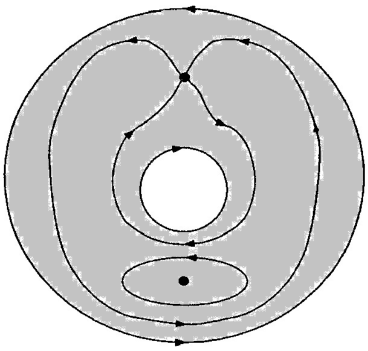
(ii) It is not hard to show that $w(x)$ is continuous. With the help of Item (i),

$$\Gamma = \{(x, w(x)) \mid x \in \mathbb{R}\}$$

is a *closed curve* in the cylinder C. We can also regard Γ as a closed curve in the annulus A. Prove that Γ must intersect $\psi(\Gamma)$ at least two points. (*Hint.* The map ψ is area preserving.)

(iii) Prove that each intersection point $(x, w(x)) \in \Gamma \cap \psi(\Gamma)$ is a fixed point of ψ.

The following picture is a vector field on the annulus A. There is a symplectomorphism associated with this vector field. Note that the vector field has exactly two zeros, and they are the fixed point of the symplectomorphism. This example shows that it is *impossible* to improve the lower bound to be three points. And the Poincaré–Birkhoff theorem is sharp. (Later on, you will be asked to construct this symplectomorphism.)



(2) For the following vector fields on $\mathbb{R}^2$, find out the corresponding one-parameter group of diffeomorphisms.
  (a) $v = (x, y)$.
  (b) $v = (y, -x)$.
  (c) $v = ((y + 2)x, 1)$.

(3) Consider the following two symplectic forms

$$\omega_0 = (25 + \sin x \cos y)\mathrm{d}x \wedge \mathrm{d}y \qquad \text{and} \qquad \omega_1 = 25\mathrm{d}x \wedge \mathrm{d}y$$

on the torus $T^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z})$. Apply the Moser's trick to construct an isotopy ρ_t such that $\rho_1^*\omega_1 = \omega_0$.
  (a) Check that ω_0 and ω_1 meet the requirement of [CdS1, Theorem 7.2].
  (b) Find a 1-form μ such that $\omega_1 - \omega_0 = \mathrm{d}\mu$. Note that μ is not unique.
  (c) Write down the t-dependent vector field $v_t = -\omega_t^{-1}(\mu)$.
  (d) Solve the isotopy generated by v_t. This is harder than Question (2). The vector field is no longer independent of t. (*Hint.* Try to construct some quantity whose derivative in t is zero. Thus, it is a conserved quantity along the flow.)