INTRODUCTION TO SYMPLECTIC GEOMETRY HOMEWORK 3

DUE: MONDAY, SEPTEMBER 30

(1) Consider the standard sphere S^n in $\mathbb{R}^{n+1}$:

$$S^n = \Big\{(p_0, p_1, \ldots, p_n) \in \mathbb{R}^{n+1} \;\Big|\; \sum_{j=0}^{n} p_j^2 = 1\Big\} \ .$$

Show that the following *immersion* of S^n into $(\mathbb{R}^{2n}, \omega_0)$ is Lagrangian:

$$\iota :\ (p_o,\, p_1,\, p_2,\, \ldots,\, p_n) \mapsto (p_1,\, p_0p_1,\, p_2,\, p_0p_2,\, \ldots,\, p_n,\, p_0p_n) \tag{0.1}$$

where the coordinate for $\mathbb{R}^{2n}$ is arranged as $(x_1,\, y_1,\, x_2,\, y_2,\, \ldots,\, x_n,\, y_n)$. Note that $\iota(\pm 1, 0, \ldots, 0) = \mathbf{0}$.

(2) (from [CdS1, Homework 2]) The purpose of this problem set is to give a criterion for which the self-symplectomorphisms of $(T^*X, \omega_{\text{can}})$ arise as liftings of self-diffeomorphisms of X.

(a) Let (M, ω) be a symplectic manifold with $\omega = -\mathrm{d}\alpha$ for some 1-form α. According to [CdS1, §1.2], there exists a unique vector field v such that

$$\omega(v, \,\cdot\,) = -\alpha(\,\cdot\,) \ . \tag{0.2}$$

Remember that any vector field v generates a one-parameter group of diffeomorphisms, $\exp tv$ for $t \in \mathbb{R}$. The diffeomorphisms are characterized by the following property: for any $p \in M$, $(\exp tv)(p)$ is the unique curve solving

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\exp tv)(p) = v|_{(\exp tv)(p)} \ , \\ (\exp tv)(p)|_{t=0} = p \ . \end{cases}$$

Suppose that τ is a self-diffeomorphism of M which preserves α (hence τ is a symplectomorphism). Prove that

$$(\exp tv) \circ \tau = \tau \circ (\exp tv) \ . \tag{0.3}$$

(b) Let $M = T^*X$ and α be the tautological 1-form.. Find the explicit formula for $\exp tv$ where v is defined by (0.2).

(c) Let $M = T^*X$ and α be the tautological 1-form. Suppose that τ is a symplectomorphism of (M, ω_{can}) which perserves α. Show that

$$\tau(x, \xi) = (y, \eta) \quad \Longrightarrow \quad \tau(x, \lambda\xi) = (y, \lambda\eta) \tag{0.4}$$

for all $(x, \xi) \in M$ and $\lambda \in \mathbb{R}$.

(d) Let $M = T^*X$ and α be the tautological 1-form. Suppose that τ is a symplectomorphism of (M, ω_{can}) which preserves α. Show that there exists a diffeomorphism $f : X \to X$ such that

$$\pi \circ \tau = f \circ \pi \tag{0.5}$$

where $\pi : M \to X$ is the projection. Then, prove that τ is the lifting of f, i.e. $\tau = f_\sharp$.

Together with [CdS1, Theorem 2.1], we find that a symplectomorphism τ of T^*X is the lifting of a diffeomorphism f of X if and only if τ preserves the tautological 1-form.

There are many symplectomorphisms of T^*X which do not come from diffeomorphisms of X, see [CdS1, #4 of Homework 3].

(3) [CdS1, §4] describes the method of generating functions to construct symplectomorphisms between cotangent bundles.

(a) Suppose that $X_1 = X_2 = \mathbb{R}$. Work out the symplectomorphisms of $T^*\mathbb{R}$ induced by the following generating functions. Also double-check that the map you find actually defines a symplectomorphism of $T^*\mathbb{R}$.

(i) $f(x, y) = xy$;

(ii) $f(x, y) = (x^3 + 10x)y$.

(b) Is it possible that the method of generating functions produces $f_\sharp$ for some diffeomorphism $f : X \to X$? Justify your answer.