# INTRODUCTION TO SYMPLECTIC GEOMETRY
# HOMEWORK 2

DUE: MONDAY, SEPTEMBER 23

(1) (Symplectic linear algebra, from [CdS1, Homework 1]) Let $(V^{2n}, \omega)$ be a symplectic vector space, and $U \subset V$ be a subspace. The *symplectic orthogonal* of $U$ is defined to be

$$U^\omega = \left\{ v \in V \;\middle|\; \omega(v, u) = 0 \quad \text{for any } u \in U \right\} .$$

Prove the following statements.

(a) $\dim U + \dim U^\omega = \dim V$.

(b) $(U^\omega)^\omega = U$.

(c) For any two subspaces $U_1$ and $U_2$, $U_1 \subseteq U_2$ if and only if $U_1^\omega \supseteq U_2^\omega$.

(d) $U$ is *symplectic*[1] if and only if $V = U \oplus U^\omega$.

A subspace $U$ is called *isotropic* if $U \subseteq U^\omega$. That is to say, $\omega$ vanishes on $U \times U$. A subspace $U$ is called *coisotropic* if $U^\omega \subseteq U$.

(e) $U$ is *Lagrangian*[2] if and only if $U$ is isotropic and $\dim U = \frac{1}{2} \dim V$.

(2) (Symplectic group, from [M&S, Lemma 2.20]) Denote by $I_n$ the $n \times n$ identity matrix. Let $J_n$ be the following $2n \times 2n$ matrix

$$J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} .$$

The symplectic group is defined to be

$$\mathrm{Sp}(n) = \left\{ B \in \mathrm{GL}(2n; \mathbb{R}) \;\middle|\; B^T J_n B = J_n \right\} .$$

Prove the following statements. (The *multiplicity* here means the *algebraic* multiplicity, not the *geometric* multiplicity.)

(a) For any $B \in \mathrm{Sp}(n)$, $\lambda$ is an eigenvalue of $B$ if and only if $\lambda^{-1}$ is an eigenvalue of $B$. Moreover, they have the same multiplicities. (*Hint.* Is $B^T$ related to $B^{-1}$ in some way?)

(b) If $\pm 1$ is an eigenvalue of $B \in \mathrm{Sp}(n)$, then it must occur with even multiplicities. (*Hint.* Start with $-1$.)

(c) Find an element of $\mathrm{SL}(4; \mathbb{R})$ that does not belong to $\mathrm{Sp}(2)$.

(3) True or False. No justications are needed.

(a) [T] [F] There exists a one dimensional subspace of $(\mathbb{R}^2, \omega_0)$ that is not Lagrangian.

[1] The definition introduced in class is that $\omega|_U$ is non-degenerate.

[2] The definition introduced in class is that $U = U^\omega$.

(b) T F Any hyperplane (codimension one subspace) of $(\mathbb{R}^{2n}, \omega_0)$ is coisotropic.

(c) T F Any element of $\mathrm{Sp}(n)$ is diagonalizable.

(4) Suppose that $W$ is a finite dimensional vector space. Let $W^*$ be the dual space. For any subspace $U$ of $W$, set

$$\mathcal{A}(U) = \{f \in W^* \mid f(u) = 0 \quad \text{for any } u \in U\}$$

to be the annihilator of $U$. Show that $U \times \mathcal{A}(U)$ is a Lagrangian subspace of $(W \times W^*, \omega_{\mathrm{can}})$. You can find the definition of $(W \times W^*, \omega_{\mathrm{can}})$ in [CdS1, #9 of Homework 1].

(5) (**Bonus**) Prove that $\mathrm{Sp}(n) \subseteq \mathrm{SL}(2n; \mathbb{R})$ using block matrix. You can find some properties of the determinant of a block matrix on
`http://en.wikipedia.org/wiki/Determinant#Block_matrices`.