INTRODUCTION TO SYMPLECTIC GEOMETRY
HOMEWORK 1

DUE: MONDAY, SEPTEMBER 16

(1) A *line complex* is a three-dimensional family of lines in the three-dimensional projective space. In this exercise, we will construct a *linear line complex*, and connect it with symplectic geometry.

This exercise focuses on $V = \mathbb{R}^4$. Denote by V^* the dual space of $\mathbb{R}^4$, and denote by $\Lambda^2 V^*$ the exterior square of V^*. As a vector space, $V^* \cong \mathbb{R}^4$ and $\Lambda^2 V^* \cong \mathbb{R}^6$.

The manifold $\mathbb{P}(V) = \mathbb{RP}^3$ is the space of lines in V. The *lines*[1] in $\mathbb{P}(V)$ are the projectification of two-dimensional linear subspaces of V.

(a) For any nonzero $\psi \in \Lambda^2 V^*$, prove that the subspace $\ker(\psi) = \{v \in V \mid \psi(v, \cdot) = 0\}$ is either $\{0\}$ or a two dimensional subspace. Moreover, suppose that $\psi_1, \psi_2 \in \Lambda^2 V^*$ are two nonzero elements whose kernel define the same two-dimensional subspace. Prove that ψ_1 and ψ_2 are the constant multiple of each other. (*Hint.* Regard elements in V as column vectors. Then any $\tau \in V^* \otimes V^*$ can be represent by a 4×4 matrix A_τ. Namely, $\tau(v, w) = v^T A_\tau w$ where T means transpose. What is special about the matrix A_ψ when $\psi \in \Lambda^2 V^* \subset V^* \otimes V^*$?)

(b) For any two-dimensional subspace W of V, show that

$$W = \ker(\psi)$$

for some $\psi \in \Lambda^2 V^*$ with $\psi \wedge \psi = 0$. (*Hint.* W can be thought as the intersection of two three-dimensional subspaces of V.)

(c) For any nonzero $\psi \in \Lambda^2 V^*$ with $\psi \wedge \psi = 0$, show that $\dim \ker(\psi) = 2$ and ψ is decomposable[2].

The above discussion gives an identification

$$\big\{\text{lines in } \mathbb{P}(V)\big\} \cong \big\{[\psi] \in \mathbb{P}(\Lambda^2 V^*) \bigm| \psi \wedge \psi = 0\big\} \tag{0.1}$$

where $[\psi]$ means the equivalent class of ψ under the identification $\psi \sim \lambda\psi$ for $\lambda \in \mathbb{R}\backslash\{0\}$. Since the latter expression is the zero locus of an equation in $\mathbb{RP}^5$, the space of lines in $\mathbb{P}(V)$ is four-dimensional.

To construct a *line complex* (see the very beginning), we can intersect

$$\mathcal{C} = \big\{[\psi] \in \mathbb{P}(\Lambda^2 V^*) \bigm| \psi \wedge \psi = 0\big\} \tag{0.2}$$

[1] Equivalently, a one-dimensional submanifold of $\mathbb{P}(V)$ is called a *line* if it is an affine line in each standard coordinate chart of $\mathbb{P}(V)$.

[2] Namely, $\psi = \alpha \wedge \beta$ for some $\alpha, \beta \in V^*$.

with a hypersurface in $\mathbb{P}(\Lambda^2 V^*)$, and the dimension will be cut down by one. If the hypersurface is a hyperplane[3] in $\mathbb{P}(\Lambda^2 V^*)$, the intersection is called a *linear line complex.*

(d) To be more precise, let x_1, y_1, x_2, y_2 be the coordinate on $V = \mathbb{R}^4$, and consider

$$\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \mathrm{d}x_2 \wedge \mathrm{d}y_2 \in \Lambda^2 V^* \ .$$

It defines a hyperplane in $\mathbb{P}(\Lambda^2 V^*)$ by

$$\mathcal{D} = \big\{ [\psi] \in \mathbb{P}(\Lambda^2 V^*) \ \big| \ \psi \wedge \omega = 0 \big\} \ . \tag{0.3}$$

By intersecting $\mathcal{C}$ with $\mathcal{D}$, we obtain a linear line complex. Prove the following geometric characterization of $\mathcal{L} = \mathcal{C} \cap \mathcal{D}$:

$$[\psi] \in \mathcal{L} \quad \Longleftrightarrow \quad \ker(\psi) \text{ is two-dimensional, and } \omega|_{\ker(\psi)} = 0 \ . \tag{0.4}$$

(*Remark.* Since $\dim \ker(\psi) = 2$, the last statement is equivalent to that $\omega(v_1, v_2) = 0$ for a basis v_1, v_2 of $\ker(\psi)$. It does not mean that $\ker(\psi) \subset \ker(\omega)$. In fact, $\ker(\omega) = \{0\}$.)

We now make a summary. The lines in $\mathbb{P}(V)$ are two planes in V. In other words, $\{\text{lines in } \mathbb{P}(V)\}$ is exactly the Grassmannian $G(2,4)$. The identification (0.1) is the so-called *Plücker embedding*, which is a canonical embedding of a Grassmannian into a projective space. The linear line complex $\mathcal{L} \subset G(2,4)$ consists of all those two planes on which ω vanishes. We will talk more about this ω vanishing condition later.

So far we have explained all the geometric ingredients. The last item asks you to check that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{L}$ are smooth manifold.

Let $u_1, u_2, v_1, v_2, w_1, w_2$ be the coordinate for $\Lambda^2 V^*$ with respect to $\mathrm{d}x_1 \wedge \mathrm{d}y_1$, $\mathrm{d}x_2 \wedge \mathrm{d}y_2$, $\mathrm{d}x_1 \wedge \mathrm{d}x_2$, $-\mathrm{d}y_1 \wedge \mathrm{d}y_2$, $\mathrm{d}x_1 \wedge \mathrm{d}y_2$, $\mathrm{d}y_1 \wedge \mathrm{d}x_2$. Namely, write any $\psi \in \Lambda^2 V^*$ as

$$\begin{aligned} \psi = {} & u_1 \, \mathrm{d}x_1 \wedge \mathrm{d}y_1 + u_2 \, \mathrm{d}x_2 \wedge \mathrm{d}y_2 + v_1 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 - v_2 \, \mathrm{d}y_1 \wedge \mathrm{d}y_2 \\ & + w_1 \, \mathrm{d}x_1 \wedge \mathrm{d}y_2 + w_2 \, \mathrm{d}y_1 \wedge \mathrm{d}x_2 \ . \end{aligned}$$

(e) Write down the defining equation for $\mathcal{C}$ and $\mathcal{D}$ in terms of this homogeneous coordinate. Prove that $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{L} = \mathcal{C} \cap \mathcal{D}$ are smooth manifolds.

(2) The purpose of this exercise is to give the geometric interpretation of the Legendre transform in the simplest case. Suppose that $f(v)$ is a strongly convex[4], smooth function on $\mathbb{R}$. Let

$$y = \frac{\mathrm{d}f}{\mathrm{d}v} \ . \tag{0.5}$$

The strongly convexity guarantees that $v \to y(v)$ is a change of variable for $\mathbb{R}$, and we can also regard v as a function of y.

(a) Let $\Gamma_f = \{(v, f(v)) \mid v \in \mathbb{R}\}$ be the graph of f. Consider the tangent line of Γ_f at $(v_0, f(v_0))$. Explain its relation to $y(v_0)$.

[3]Namely, it is the projectification of a hyperplane in $\Lambda^2 V^* \cong \mathbb{R}^6$.

[4]Strongly convexity means that there exists some $\epsilon > 0$ such that $f''(v) \geq \epsilon$ for all $v \in \mathbb{R}$.

(b) Define $h(y)$ to be

$$h(y) = vy - f(v) \ . \tag{0.6}$$

Is $h(y(v_0))$ related to the tangent line of Γ_f at $(v_0, f(v_0))$? (*Hint.* A tangent line/plane of a graph always intersects the second/last axis.)

(c) Find out the inverse Legendre transform, i.e. express v in terms of y and $h(y)$. (*Hint.* Regard (0.6) as a function of v. What is its derivative?)

(d) Prove that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}v^2} \frac{\mathrm{d}^2 h}{\mathrm{d}y^2} = 1 \ .$$

(*Hint.* Consider the Jacobians of $y(v)$ and $v(y)$.)

This exercise works in higher dimensions as well. We will discuss more about it later.