

AN ISOPERIMETRIC-TYPE INEQUALITY FOR SPACELIKE SUBMANIFOLD IN THE MINKOWSKI SPACE

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ABSTRACT. We prove an isoperimetric-type inequality for maximal, spacelike submanifold in the Minkowski space. The argument is based on the recent work of Brendle.

1. INTRODUCTION

The isoperimetric inequality asserts that there exists a constant c_n such that

$$\text{Vol}(\Sigma)^{n-1} \leq c_n \text{Vol}(\partial\Sigma)^n \quad (1.1)$$

for any compact, n -dimensional, minimal submanifold Σ in the Euclidean space, \mathbb{R}^{n+m} . The sharp constant of c_n is believed to be

$$\frac{\text{Vol}(B^n)^{n-1}}{\text{Vol}(\partial B^n)^n} = \frac{1}{n^n \text{Vol}(B^n)} \quad (1.2)$$

where B^n is the unit ball in \mathbb{R}^n . The sharp inequality is known to be true in the full dimensional case ($m = 0$), and the surface case ($n = 2$) with some topological condition. We refer to [10, Problem 109] and [5] for the developments. Recently, Brendle [3] generalizes brilliantly the argument of [4], and proves the sharp inequality for codimension at most 2 ($m \leq 2$).

The purpose of this note is to investigate the isoperimetric inequality for spacelike submanifolds in the Minkowski space based on the approach of Brendle. What follows are the main results.

Theorem 1. *For any $k \geq 2$ and $\tau \in [1, \infty)$, suppose that Σ is a compact, connected, n -dimensional, spacelike submanifold in $\mathbb{R}^{n,k}$ whose space-time slope is no greater than τ . Then, it satisfies*

$$\text{Vol}(\Sigma)^{n-1} \leq c(n, k, \tau) \left(\text{Vol}(\partial\Sigma) + \int_{\Sigma} \sqrt{-|H|^2} d\text{vol} \right)^n,$$

where H is the (timelike) mean curvature vector of Σ , and

$$c(n, k, \tau) = \frac{n+k-2}{n} (\tau + \sqrt{\tau^2 - 1})^{n+k-2} \frac{1}{n^n \text{Vol}(B^n)}.$$

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Here are some remarks.

- (1) The quantity $\tau \geq 1$ measures the deviation of Σ from belonging to an Euclidean slice. If there is some tangent direction close to being null, τ is large. See Definition 3.
- (2) For $k = 1$, one can embed $\mathbb{R}^{n,1}$ into $\mathbb{R}^{n,2}$, and Theorem 1 applies.
- (3) Suppose that Σ is an n -dimensional domain in \mathbb{R}^n . In this case, τ can be taken to be 1. By embedding \mathbb{R}^n into $\mathbb{R}^{n,2}$, Theorem 1 recovers the sharp form of (1.1) (the full dimension case).

Unlike the Euclidean case, our constant has an additional dependence on τ . This issue will be addressed in Section 5. The observation there illustrates the difficulty to have the dependence only on the dimensions.

Theorem 2. *For any $k \geq 2$, $m \geq 1$ and $\tau \in [1, \infty)$, suppose that Σ is a compact, connected, n -dimensional, spacelike submanifold in $\mathbb{R}^{n+m,k}$ whose space-time slope is no greater than τ . Then, it obeys*

$$\text{Vol}(\Sigma)^{n-1} \leq c(n, m, k, \tau) \left(\text{Vol}(\partial\Sigma) + \int_{\Sigma} \left(\frac{1 + \sqrt{\tau^4 - 1}}{\tau} |\pi_s(H)| + \sqrt{\tau^2 + 1} |\pi_t(H)| \right) \text{dvol} \right)^n .$$

Here, π_s is the orthogonal projection onto $\mathbb{R}^{n+m,0}$, π_t is the orthogonal projection onto $\mathbb{R}^{0,k}$, and

$$c(n, m, k, \tau) = \frac{n + m + k - 2}{n + m} \frac{(\tau + \sqrt{\tau^2 - 1})^{n+m+k-2} (\tau^2 + 1)^{\frac{k-2}{2}}}{\tau^{m+k-2}} \frac{\text{Vol}(B^m)}{n^n \text{Vol}(B^{n+m})} .$$

The convention here is that $|\pi_t(H)| \geq 0$.

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2. SOME LINEAR ALGEBRA

For the Minkowski space $\mathbb{R}^{n+m,k}$, denote by π_s the projection onto the $\mathbb{R}^{n+m,0}$ -summand, and by π_t the projection onto the $\mathbb{R}^{0,k}$ -summand. For vectors after π_t , the convention here is to take the *positive definite* inner product. That is to say, $|v|^2 = |\pi_s(v)|^2 - |\pi_t(v)|^2$.

Definition 3. For a compact, spacelike submanifold Σ of $\mathbb{R}^{n+m,k}$, let

$$\tau(x) = \max\{ |\pi_s(v)| : v \in T_x \Sigma, |v| = 1 \} \geq 1 ,$$

and let τ be $\max\{ \tau(x) : x \in \Sigma \}$.

Since $|\pi_t(v)|^2 = |\pi_s(v)|^2 - |v|^2$, $\max\{ |\pi_t(v)| : v \in T_x \Sigma, |v| = 1 \}$ is equal to $\sqrt{(\tau(x))^2 - 1}$.

Remark 4. This notion depends on the choice of the Euclidean slice, $\mathbb{R}^{n+m,0}$. In other words, τ is *not* invariant under the isometries of $\mathbb{R}^{n+m,k}$. The infimum of τ over $\{A(\Sigma) : A \in \text{O}(n+m, k)\}$ measures the deviation of Σ from belonging to an Euclidean slice.

Lemma 5. Let $L \subset \mathbb{R}^{n+m,k}$ be an n -dimensional spacelike subspace. Denote $\max\{|\pi_s(v)| : v \in L, |v| = 1\}$ by $\tau(L)$. Then,

- (1) $|\pi_L(v)| \leq \tau(L)|\pi_s(v)| + \sqrt{\tau(L)^2 - 1}|\pi_t(v)|$ for any $v \in \mathbb{R}^{n+m,k}$;
- (2) there exists an orthogonal decomposition of L^\perp , $N^+ \oplus N^-$ such that
 - N^+ is spacelike, and N^- is timelike;
 - $|\pi_{N^+}(v)| \leq |\pi_s(v)|$ for any $v \in \mathbb{R}^{n+m,k}$;
 - $|\pi_{N^-}(v)| \leq \sqrt{\tau(L)^2 - 1}|\pi_s(v)| + \tau(L)|\pi_t(v)|$ for any $v \in \mathbb{R}^{n+m,k}$.

As mentioned above, the norm on N^- is taken to be the positive definite one.

Proof. Since L is spacelike, $\pi_s|_L$ is injective. It follows that $H = \pi_s(L)$ is an n -dimensional subspace in $\mathbb{R}^{n+m,0} \subset \mathbb{R}^{n+m,k}$. Let $\{\mathbf{e}_j^+\}_{j=n+1}^{n+m}$ be an orthonormal basis for the orthogonal complement of H in $\mathbb{R}^{n+m,0}$

The subspace L is a graph of a transform, B , from H to $\mathbb{R}^{0,k} \subset \mathbb{R}^{n+m,k}$. By using the positive definite metric on $\mathbb{R}^{0,k}$, and applying the singular value decomposition, there exist orthonormal basis $\{\mathbf{e}_j^+\}_{j=1}^n$ for H , orthonormal basis $\{\mathbf{e}_j^-\}_{j=1}^k$ for $\mathbb{R}^{0,k}$, singular values $\{\lambda_j\}_{j=1}^n$ such that

$$\left\{ \frac{1}{\sqrt{1-\lambda_j^2}}(\mathbf{e}_j^+ + \lambda_j \mathbf{e}_j^-) \right\}_{j=1}^n \text{ forms an orthogonal basis for } L.$$

Note that $\lambda_j^2 < 1$ follows from the spacelike condition of L . When $n > k$, $\lambda_j = 0$ for $j > k$. By re-labeling, assume $|\lambda_1| \geq |\lambda_j|$ for any j . It is easy to see that $\tau(L) = 1/\sqrt{1-\lambda_1^2}$.

For any $v \in \mathbb{R}^{n+m,k}$, write $v = v_\mu^+ \mathbf{e}_\mu^+ + v_\nu^- \mathbf{e}_\nu^-$. Its orthogonal projection onto L is

$$\pi_L(v) = \frac{1}{1-\lambda_j^2}(v_j^+ - \lambda_j v_j^-)(\mathbf{e}_j^+ + \lambda_j \mathbf{e}_j^-),$$

and thus

$$\begin{aligned} |\pi_L(v)|^2 &= \sum_j \frac{1}{1-\lambda_j^2} (v_j^+ - \lambda_j v_j^-)^2 \\ &= \sum_j \frac{1}{1-\lambda_j^2} \left((v_j^+)^2 + \lambda_j^2 (v_j^-)^2 - 2\lambda_j v_j^+ v_j^- \right) \\ &\leq \frac{1}{1-\lambda_1^2} \left(\sum_j (v_j^+)^2 \right) + \frac{\lambda_1^2}{1-\lambda_1^2} \left(\sum_j (v_j^-)^2 \right) + 2 \frac{\lambda_1}{1-\lambda_1^2} \sqrt{\sum_j (v_j^+)^2} \sqrt{\sum_j (v_j^-)^2} \\ &\leq \tau(L)^2 |\pi_s(v)|^2 + \left((\tau(L))^2 - 1 \right) (|\pi_t(v)|^2) + 2\tau(L) \sqrt{\tau(L)^2 - 1} |\pi_s(v)| |\pi_t(v)|. \end{aligned} \tag{2.1}$$

It proves assertion (1) of this Lemma.

For the second part, let

$$N^+ = \text{span}\{\mathbf{e}_j^+\}_{j=n+1}^{n+m} \quad \text{and} \quad N^- = \text{span}\left\{\frac{1}{\sqrt{1-\lambda_j^2}}(\lambda_j\mathbf{e}_j^+ + \mathbf{e}_j^-)\right\}_{j=1}^k.$$

Assertion (2) can be obtained by a similar argument as (2.1). \square

3. THE COMPARISON MAP

The main ingredient of the argument in [3] is a map from the normal bundle of Σ to the unit ball in \mathbb{R}^{n+m} . This section is devoted to study analogous properties of the map.

Throughout the rest of this paper, Σ always denotes a compact, connected, n -dimensional, spacelike submanifold in $\mathbb{R}^{n+m,k}$, with boundary $\partial\Sigma$.

For a positive constant c_0 and a function f on Σ , let u be the solution to the following Neumann problem:

$$\begin{cases} \Delta^\Sigma u = c_0(c_f - f) & \text{on } \Sigma, \\ \langle \nabla^\Sigma u, \eta \rangle = c_0 & \text{on } \partial\Sigma \end{cases} \quad (3.1)$$

where η is the unit outer normal of $\partial\Sigma$ with respect to Σ , and

$$c_f = \frac{\text{Vol}(\partial\Sigma) + \int_\Sigma f \, \text{dvol}}{\text{Vol}(\Sigma)}. \quad (3.2)$$

The value of c_f is determined by the Green's identity. We will use Lipschitz continuous f , and thus u will be of class $C^{2,\alpha}$ for any $\alpha \in (0, 1)$ (see [7, section 6.7]). With this function u , introduce the map $\Phi = \Phi_{\Sigma, c_0, f}$ as follows:

$$\begin{aligned} \Phi : N\Sigma &\rightarrow \mathbb{R}^{n+m,k} \\ (x, y) &\mapsto \nabla^\Sigma u(x) + y \end{aligned} \quad (3.3)$$

where $x \in \Sigma$ and $y \in N_x\Sigma = (T_x\Sigma)^\perp$.

With τ given by Definition 3, consider the following regions:

$$\begin{aligned} D &= \{ \xi \in \mathbb{R}^{n+m,k} : c_0 - \tau|\pi_s(\xi)| - \sqrt{\tau^2 - 1}|\pi_t(\xi)| > 0 \}, \\ U &= \{ x \in \Sigma \setminus \partial\Sigma : |\nabla^\Sigma u(x)| < c_0 \}, \\ \Omega &= \{ (x, y) : x \in U, y \in N_x\Sigma, \nabla^\Sigma u(x) + y \in D \}, \\ A &= \{ (x, y) \in \Omega : \nabla_\Sigma^2 u(x) - \langle \text{II}|_x, y \rangle \geq 0 \} \end{aligned} \quad (3.4)$$

Lemma 6. *The image of A under the map Φ (3.3) is exactly D .*

Proof. It follows from definition that $\Phi(A) \subset D$. For surjectivity, define $w_\xi(x) = u(x) - \langle x, \xi \rangle$ for any $\xi \in D$. When $x \in \partial\Sigma$, it follows from Definition 3 and Cauchy–Schwarz inequality that

$$\begin{aligned} \langle \nabla^\Sigma w_\xi(x), \eta(x) \rangle &= \langle \nabla^\Sigma u(x), \eta(x) \rangle - \langle \xi, \eta(x) \rangle \\ &= c_0 - \langle \pi_s(\xi), \pi_s(\eta(x)) \rangle + \langle \pi_t(\xi), \pi_t(\eta(x)) \rangle \\ &\geq c_0 - \tau |\pi_s(\xi)| - \sqrt{\tau^2 - 1} |\pi_t(\xi)| > 0 . \end{aligned}$$

Therefore, there is an $x_\xi \in \Sigma \setminus \partial\Sigma$ attaining the minimum of w_ξ . Since $\nabla^\Sigma w_\xi(x_\xi) = 0$, $\nabla^\Sigma u(x_\xi) = \xi^T$ where ξ^T is the orthogonal projection of ξ onto $T_x\Sigma$. Equivalently, $\xi = \nabla^\Sigma u(x_\xi) + y_\xi$ for some $y_\xi \in N_x\Sigma$. According to (1) of Lemma 5,

$$|\nabla^\Sigma u(x_\xi)| \leq \tau(x_\xi) |\pi_s(\xi)| + \sqrt{\tau(x_\xi)^2 - 1} |\pi_t(\xi)| \leq \tau |\pi_s(\xi)| + \sqrt{\tau^2 - 1} |\pi_t(\xi)| < c_0 ,$$

and hence $x_\xi \in U$. This shows that $\Phi(\Omega) = D$.

Since x_ξ achieves the minimum of w_ξ ,

$$0 \leq \nabla_\Sigma^2 w_\xi(x_\xi) = \nabla_\Sigma^2 u(x_\xi) - \langle \Pi|_{x_\xi}, \xi \rangle .$$

Note that $\langle \Pi|_{x_\xi}, \xi \rangle = \langle \Pi|_{x_\xi}, \nabla^\Sigma u(x_\xi) + y_\xi \rangle = \langle \Pi|_{x_\xi}, y_\xi \rangle$, and thus $(x_\xi, y_\xi) \in A$. \square

3.1. The Jacobian Determinant. To extract an isoperimetric-type inequality from Lemma 6, some estimate on the Jacobian determinant of Φ is needed. The properties in this subsection are essentially the same as [3, Lemma 5 and 6]. The arguments are included for completeness.

Lemma 7. *At any $(x, y) \in \Omega$, $(\det d\Phi)(x, y) = \det(\nabla_\Sigma^2 u(x) - \langle \Pi|_x, y \rangle)$.*

Proof. Fix a point $(\bar{x}, \bar{y}) \in \Omega$. Let (x_1, \dots, x_n) be a normal coordinate system for Σ near \bar{x} , and let $\{\mathbf{e}_i\}_{i=1}^n$ be a local orthonormal frame for $T\Sigma$ near \bar{x} such that $\mathbf{e}_i = \frac{\partial}{\partial x_i}$ at \bar{x} for $i \in \{1, \dots, n\}$. Also choose a local orthonormal frame $\{\mathbf{f}_i^+\}_{i=1}^m \cup \{\mathbf{f}_j^-\}_{j=1}^k$ of $N\Sigma$ near \bar{x} , where \mathbf{f}_i^+ 's are spacelike, and \mathbf{f}_j^- 's are timelike.

By writing a normal vector to Σ as $y = \sum_{i=1}^m y_i \mathbf{f}_i^+ + \sum_{j=1}^k y_{m+j} \mathbf{f}_j^-$,

$$(x_1, \dots, x_n, y_1, \dots, y_m, y_{m+1}, \dots, y_{m+k})$$

constitutes a local coordinate system on the total space of $N\Sigma$. We compute the differential of Φ at (\bar{x}, \bar{y}) :

$$\begin{aligned} \left\langle \frac{\partial \Phi}{\partial x_i}(\bar{x}, \bar{y}), \mathbf{e}_j \right\rangle &= \langle \nabla_\Sigma^2 u(\mathbf{e}_i, \mathbf{e}_j) - \langle \Pi|_x(\mathbf{e}_i, \mathbf{e}_j), \bar{y} \rangle , & \left\langle \frac{\partial \Phi}{\partial y_i}(\bar{x}, \bar{y}), \mathbf{f}_j^+ \right\rangle &= \langle \mathbf{f}_i^+, \mathbf{f}_j^+ \rangle = \delta_{ij} , \\ \left\langle \frac{\partial \Phi}{\partial y_i}(\bar{x}, \bar{y}), \mathbf{e}_j \right\rangle &= \langle \mathbf{f}_i^+, \mathbf{e}_j \rangle = 0 , & \left\langle \frac{\partial \Phi}{\partial y_{m+i}}(\bar{x}, \bar{y}), \mathbf{f}_j^+ \right\rangle &= \langle \mathbf{f}_i^-, \mathbf{f}_j^+ \rangle = 0 , \\ \left\langle \frac{\partial \Phi}{\partial y_{m+i}}(\bar{x}, \bar{y}), \mathbf{e}_j \right\rangle &= \langle \mathbf{f}_i^-, \mathbf{e}_j \rangle = 0 , & \left\langle \frac{\partial \Phi}{\partial y_{m+i}}(\bar{x}, \bar{y}), \mathbf{f}_j^- \right\rangle &= \langle \mathbf{f}_i^-, \mathbf{f}_j^- \rangle = -\delta_{ij} . \end{aligned}$$

It follows that the Jacobian of Φ at (\bar{x}, \bar{y}) in terms of the frame $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1^+, \dots, \mathbf{f}_m^+, \mathbf{f}_1^-, \dots, \mathbf{f}_k^-\}$ takes the following form

$$\begin{bmatrix} \nabla_{\Sigma}^2 u(\bar{x}) - \langle \Pi |_{\bar{x}, \bar{y}} \rangle & 0 & 0 \\ * & \mathbf{I}_m & 0 \\ * & * & \mathbf{I}_k \end{bmatrix}.$$

Taking determinant finishes the proof of this Lemma. \square

Lemma 8. *At any $(x, y) \in A$, $0 \leq (\det d\Phi)(x, y) \leq \left(\frac{1}{n}(c_0(c_f - f(x)) - \langle H(x), y \rangle)\right)^n$. In particular, $c_0(c_f - f(x)) - \langle H(x), y \rangle \geq 0$ at any $(x, y) \in A$.*

Proof. For $(x, y) \in A$, $\nabla_{\Sigma}^2 u(x) - \langle \Pi |_{x, y} \rangle$ is positive semi-definite. By the arithmetic-geometric inequality,

$$0 \leq \det(\nabla_{\Sigma}^2 u(x) - \langle \Pi |_{x, y} \rangle) \leq \left(\frac{\text{tr}(\nabla_{\Sigma}^2 u(x) - \langle \Pi |_{x, y} \rangle)}{n}\right)^n = \left(\frac{\Delta^{\Sigma} u(x) - \langle H(x), y \rangle}{n}\right)^n.$$

Combining this with Lemma 7 and (3.1) finishes the proof of this Lemma. \square

4. PROOF OF MAIN THEOREMS

This section is devoted to the proof of Theorem 1 and 2. It will be done by analyzing the total (usual) Lebesgue measure of D via Φ . In this section, k is always assumed to be greater than or equal to 2.

4.1. Measure Estimate on the Target. It follows from Lemma 6 that

$$\int_{\xi \in D} \rho(\xi) d\xi \leq \int_{x \in U} \int_{\substack{y \in N_x \Sigma \\ y + \nabla^{\Sigma} u(x) \in D}} (\det d\Phi)(x, y) \cdot 1_A(x, y) \cdot \rho(y + \nabla^{\Sigma} u(x)) dy d\text{vol}_x, \quad (4.1)$$

for any non-negative $\rho \in L^1(D)$, where 1_A is the characteristic function of A . The strategy is to choose ρ to be the characteristic function of the set

$$S_{\varepsilon} = \{\xi \in \mathbb{R}^{n+m, k} : -\varepsilon^2 < |\xi|^2 = |\pi_s(\xi)|^2 - |\pi_t(\xi)|^2 < 0\}, \quad (4.2)$$

and examine the limiting inequality of (4.1) over ε^2 as $\varepsilon \rightarrow 0$. For $\varepsilon \ll 1$, it follows from

$$S_{\varepsilon} \cap D \supset \{\xi : |\pi_s(\xi)|^2 < |\pi_t(\xi)|^2 < |\pi_s(\xi)|^2 + \varepsilon^2, |\pi_s(\xi)| < c_0\tau - \sqrt{(\tau^2 - 1)(c_0^2 + \varepsilon^2)}\}$$

that

$$\int_{\xi \in D} 1_{S_{\varepsilon}}(\xi) d\xi \geq \text{Vol}(B^k) \text{Vol}(\partial B^{n+m}) \int_0^{c_0\tau - \sqrt{(\tau^2 - 1)(c_0^2 + \varepsilon^2)}} \left((s^2 + \varepsilon^2)^{\frac{k}{2}} - s^k\right) s^{n+m-1} ds$$

where $s = |\pi_s(\xi)|$. By the L'Hôpital's rule,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_0^{c_0 \tau - \sqrt{(\tau^2 - 1)(c_0^2 + \varepsilon^2)}} \left((s^2 + \varepsilon^2)^{\frac{k}{2}} - s^k \right) s^{n+m-1} ds \\ &= \int_0^{c_0(\tau - \sqrt{\tau^2 - 1})} \frac{k}{2} s^{k-2} s^{n+m-1} ds = \frac{k}{2(n+m+k-2)} \left(c_0(\tau - \sqrt{\tau^2 - 1}) \right)^{n+m+k-2}. \end{aligned}$$

Putting these together gives

$$\begin{aligned} & \frac{k \operatorname{Vol}(B^k) \operatorname{Vol}(\partial B^{n+m})}{2(n+m+k-2)} \left(c_0(\tau - \sqrt{\tau^2 - 1}) \right)^{n+m+k-2} \\ & \leq \limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \int_{x \in U} \int_{\substack{y \in N_x \Sigma, y + \nabla^\Sigma u(x) \in D, \\ -\varepsilon^2 - |\nabla^\Sigma u(x)|^2 < |y|^2 < -|\nabla^\Sigma u(x)|^2}} (\det d\Phi)(x, y) \cdot 1_A(x, y) dy \operatorname{dvol}_x. \end{aligned} \quad (4.3)$$

4.2. Proof of Theorem 1. Now, suppose that $m = 0$. Let f be the length of the mean curvature vector of Σ ,

$$f(x) = \sqrt{-|H|^2}. \quad (4.4)$$

Since $m = 0$, $N\Sigma$ consists of only timelike directions, and hence $|\langle H(x), y \rangle| \leq \sqrt{-|H(x)|^2} \sqrt{-|y|^2}$ for any $y \in N_x \Sigma$. According to Lemma 8,

$$\begin{aligned} (\det d\Phi)(x, y) \cdot 1_A(x, y) & \leq \left(\frac{c_0(c_f - \sqrt{-|H|^2}) - \langle H(x), y \rangle}{n} \right)^n \cdot 1_A(x, y) \\ & \leq n^{-n} \left(\left(c_0(c_f - \sqrt{-|H|^2}) + \sqrt{-|H|^2} \sqrt{-|y|^2} \right)_+ \right)^n \end{aligned}$$

where $(\cdot)_+$ means $\max\{(\cdot), 0\}$. It follows that the inner integral of (4.3) is no greater than

$$\begin{aligned} & n^{-n} \int_{-\varepsilon^2 - |\nabla^\Sigma u(x)|^2 < |y|^2 < -|\nabla^\Sigma u(x)|^2} \left(\left(c_0(c_f - \sqrt{-|H|^2}) + \sqrt{-|H|^2} \sqrt{-|y|^2} \right)_+ \right)^n dy \\ &= \frac{1}{2} n^{-n} \operatorname{Vol}(\partial B^k) \int_{|\nabla^\Sigma u(x)|^2}^{|\nabla^\Sigma u(x)|^2 + \varepsilon^2} \left(\left(c_0(c_f - \sqrt{-|H|^2}) + \sqrt{-|H|^2} t^{\frac{1}{2}} \right)_+ \right)^n t^{\frac{k-2}{2}} dt \end{aligned}$$

where the parameter t is $-|y|^2$. Since $k \geq 2$, the integrand is continuous for $t \geq 0$. It is clear that the limit of the above expression over ε^2 as $\varepsilon \rightarrow 0$ is

$$\frac{1}{2} n^{-n} \operatorname{Vol}(\partial B^k) \left(\left(c_0(c_f - \sqrt{-|H|^2}) + \sqrt{-|H|^2} |\nabla^\Sigma u(x)| \right)_+ \right)^n |\nabla^\Sigma u(x)|^{k-2}.$$

By combining these with (4.3) and (3.4),

$$\begin{aligned}
& \frac{k \operatorname{Vol}(B^k) \operatorname{Vol}(\partial B^n)}{2(n+k-2)} \left(c_0(\tau - \sqrt{\tau^2 - 1}) \right)^{n+k-2} \\
& \leq \frac{1}{2} n^{-n} \operatorname{Vol}(\partial B^k) \int_{x \in U} \left(\left(c_0(c_f - \sqrt{-|H|^2}) + \sqrt{-|H|^2} |\nabla^\Sigma u(x)| \right)_+ \right)^n |\nabla^\Sigma u(x)|^{k-2} \operatorname{dvol}_x . \\
& \leq \frac{1}{2} n^{-n} \operatorname{Vol}(\partial B^k) (c_0)^{n+k-2} (c_f)^n \operatorname{Vol}(U)
\end{aligned}$$

This together with (3.2) and the fact that $\operatorname{Vol}(U) \leq \operatorname{Vol}(\Sigma)$ finishes the proof of Theorem 1.

4.3. Proof of Theorem 2. When $m \geq 1$, let

$$f(x) = \left(\frac{1}{\tau} + \sqrt{\tau^2 - \frac{1}{\tau^2}} \right) |\pi_s(H)| + \sqrt{\tau^2 + 1} |\pi_t(H)| . \quad (4.5)$$

By Lemma 8 and Lemma 5,

$$\begin{aligned}
& (\det d\Phi)(x, y) \cdot 1_A(x, y) \\
& \leq \left(\frac{c_0(c_f - f) - \langle H(x), y \rangle}{n} \right)^n \cdot 1_A(x, y) \\
& \leq n^{-n} \left(\left(c_0(c_f - f) + |\pi_{N_x^+}(H)| |\pi_{N_x^+}(y)| + |\pi_{N_x^-}(H)| |\pi_{N_x^-}(y)| \right)_+ \right)^n \\
& \leq n^{-n} \left(\left(c_0(c_f - f) + |\pi_s(H)| |\pi_{N_x^+}(y)| + (\sqrt{\tau^2 - 1} |\pi_s(H)| + \tau |\pi_t(H)|) |\pi_{N_x^-}(y)| \right)_+ \right)^n .
\end{aligned}$$

For brevity, denote $|\pi_s(H)|$ by $h_1(x)$ and $\sqrt{\tau^2 - 1} |\pi_s(H)| + \tau |\pi_t(H)|$ by $h_2(x)$. In this case, $|y|^2 = |\pi_{N_x^+}(y)|^2 - |\pi_{N_x^-}(y)|^2$. The region of the inner integral of (4.3) is contained in

$$\left\{ y \in N_x \Sigma : |\pi_{N_x^+}(y)| < \frac{c_0}{\tau}, |\pi_{N_x^+}(y)|^2 + |\nabla^\Sigma u(x)|^2 < |\pi_{N_x^-}(y)|^2 < |\pi_{N_x^+}(y)|^2 + |\nabla^\Sigma u(x)|^2 + \varepsilon^2 \right\} .$$

It follows that the inner integral of (4.3) is no greater than

$$\frac{\operatorname{Vol}(\partial B^m) \operatorname{Vol}(\partial B^k)}{2n^n} \int_0^{\frac{c_0}{\tau}} \int_{s^2 + |\nabla^\Sigma u(x)|^2}^{s^2 + |\nabla^\Sigma u(x)|^2 + \varepsilon^2} \left(\left(c_0(c_f - f) + h_1 s + h_2 t^{\frac{1}{2}} \right)_+ \right)^n s^{m-1} t^{\frac{k-2}{2}} \operatorname{d}t \operatorname{d}s$$

where $s = |\pi_{N_x^+}(y)|$ and $t = |\pi_{N_x^-}(y)|^2$.

Since $k \geq 2$, the limit of the above integral over ε^2 as $\varepsilon \rightarrow 0$ is

$$\begin{aligned}
& \int_0^{\frac{c_0}{\tau}} \left(\left(c_0(c_f - f) + h_1 s + h_2 \sqrt{s^2 + |\nabla^\Sigma u(x)|^2} \right)_+ \right)^n (s^2 + |\nabla^\Sigma u(x)|^2)^{\frac{k-2}{2}} s^{m-1} \operatorname{d}s \\
& \leq \left(\left(c_0(c_f - f) + h_1 \frac{c_0}{\tau} + h_2 \sqrt{\left(\frac{c_0}{\tau} \right)^2 + |\nabla^\Sigma u(x)|^2} \right)_+ \right)^n \left(\left(\frac{c_0}{\tau} \right)^2 + |\nabla^\Sigma u(x)|^2 \right)^{\frac{k-2}{2}} \cdot \frac{1}{m} \left(\frac{c_0}{\tau} \right)^m .
\end{aligned}$$

At any $x \in U$, it follows from $|\nabla^\Sigma u(x)| < c_0$ that

$$\begin{aligned} \left(\frac{c_0}{\tau}\right)^2 + |\nabla^\Sigma u(x)|^2 &\leq \left(\frac{c_0}{\tau}\right)^2(\tau^2 + 1) \quad \text{and} \\ c_0(c_f - f) + h_1 \frac{c_0}{\tau} + h_2 \sqrt{\left(\frac{c_0}{\tau}\right)^2 + |\nabla^\Sigma u(x)|^2} &\leq c_0(c_f - f + f) = c_0 c_f . \end{aligned}$$

With these estimates, (4.3) leads to

$$\begin{aligned} &\frac{k \operatorname{Vol}(B^k) \operatorname{Vol}(\partial B^{n+m})}{2(n+m+k-2)} \left(c_0(\tau - \sqrt{\tau^2 - 1})\right)^{n+m+k-2} \\ &\leq \frac{\operatorname{Vol}(\partial B^m) \operatorname{Vol}(\partial B^k)}{2m n^n} (c_0 c_f)^n \left(\frac{c_0}{\tau}\right)^{k-2} (\tau^2 + 1)^{\frac{k-2}{2}} \left(\frac{c_0}{\tau}\right)^m \operatorname{Vol}(\Sigma) . \end{aligned}$$

After some simple manipulations, this is exactly the inequality asserted in Theorem 2.

5. A REMARK ON THE CONSTANT

In the Euclidean isoperimetric inequality (1.1), the constant depends only on the dimension of the minimal submanifold, and there is a conjectural sharp constant (1.2). One may be wondering

- (1) whether it is possible to remove the dependence of τ in Theorem 1;
- (2) intuitively the volume is “smaller” in the Minkowski space, and maybe the constant (1.2) could do the job.

The observation below implies that (2) cannot be true, and also suggests that it might be impossible to have the constant solely depends on the dimensions.

For a disk type domain of a minimal surface in \mathbb{R}^n , the sharp isoperimetric inequality, $A \leq \frac{1}{4\pi} L^2$, is known. Here, A stands for the area of the domain, and L stands for the length of its boundary.

Now, consider a domain of a spacelike, maximal surface in $\mathbb{R}^{2,1}$. Note that there are a plethora of such surfaces; see for instance [9] and [1]. By $\mathbb{R}^{2,1} \hookrightarrow \mathbb{R}^{2,2}$, Theorem 1 says that

$$A \leq \frac{(\tau + \sqrt{\tau^2 - 1})^2}{4\pi} L^2 .$$

Since the normal vector is timelike, the Gauss equation implies that the Gauss curvature of a spacelike, maximal surface in $\mathbb{R}^{2,1}$ is always non-negative. However, it is known that the constant for the isoperimetric inequality becomes larger for surfaces with non-negative Gauss curvature. One can consider a geodesic ball at some point p with radius $\rho \ll 1$. By using the geodesic polar coordinate, one finds that

$$\frac{4\pi A}{L^2} = 1 + \frac{K_p}{4} \rho^2 + o(\rho^2) .$$

There are also some studies on the isoperimetric inequality for non-negatively curved surfaces. According to [6, 8],

$$A \leq \frac{1}{4\pi - 2 \iint K d\sigma} L^2 ,$$

where the total Gaussian curvature on the domain is a defect. In [2], this inequality is proved to be sharp.

To sum up, $\frac{1}{4\pi}$ does not work for spacelike, maximal surface in $\mathbb{R}^{2,1}$. It also suggests that to remove the dependence on τ , one may have to introduce certain dependence on the intrinsic curvature of the minimal submanifold.

REFERENCES

- [1] L. J. Alías, R. M. B. Chaves, and P. Mira, *Björling problem for maximal surfaces in Lorentz-Minkowski space*, Math. Proc. Cambridge Philos. Soc. **134** (2003), no. 2, 289–316.
- [2] J. L. Barbosa and M. do Carmo, *A proof of a general isoperimetric inequality for surfaces*, Math. Z. **162** (1978), no. 3, 245–261.
- [3] S. Brendle, *The isoperimetric inequality for a minimal submanifold in Euclidean space*, preprint, available at [arXiv:1907.09446](https://arxiv.org/abs/1907.09446).
- [4] X. Cabré, *Elliptic PDE's in probability and geometry: symmetry and regularity of solutions*, Discrete Contin. Dyn. Syst. **20** (2008), no. 3, 425–457.
- [5] J. Choe, *Isoperimetric inequalities of minimal submanifolds*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 325–369.
- [6] F. Fiala, *Le problème des isopérimètres sur les surfaces ouvertes à courbure positive*, Comment. Math. Helv. **13** (1941), 293–346 (French).
- [7] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [8] A. Huber, *On the isoperimetric inequality on surfaces of variable Gaussian curvature*, Ann. of Math. (2) **60** (1954), 237–247.
- [9] O. Kobayashi, *Maximal surfaces in the 3-dimensional Minkowski space L^3* , Tokyo J. Math. **6** (1983), no. 2, 297–309.
- [10] S. T. Yau, *Problem section*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 669–706.

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