MEAN CURVATURE FLOWS OF TWO-CONVEX LAGRANGIANS

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Abstract. We prove regularity, global existence, and convergence of Lagrangian mean curvature flows in the two-convex case (1.6). Such results were previously only known in the convex case, of which the current work represents a significant improvement. The proof relies on a newly discovered monotone quantity (2.6) that controls two-convexity. Through a unitary transformation, same result for the mean curvature flow of area-decreasing Lagrangian submanifolds (1.10) were established.

1. Introduction

Let M be a $2n$ dimensional Kähler manifold. Throughout this paper, the Riemannian metric on M is assumed to be flat. The symplectic form ω on M is given by $\omega(\,\cdot\,,\,\cdot\,) = \langle J(\,\cdot\,),\,\cdot\,\rangle$ where J is the (almost) complex structure and $\langle\,\cdot\,,\,\cdot\,\rangle$ is the Riemannian metric. We also assume that there exist parallel bundle maps $\pi_1 : TM \to TM$ and $\pi_2 : TM \to TM$ such that the following conditions are satisfied.

(i) Both π_1 and π_2 are orthogonal projections on each fiber.
(ii) $\pi_1 + \pi_2$ is the identity map on TM.
(iii) The kernels of π_1 and π_2 on each fiber are Lagrangian subspaces.

It follows that $\ker \pi_1$ and $\ker \pi_2$ are everywhere orthogonal, and J maps one to the other. Moreover, $J\pi_1 = \pi_2 J$ and $J\pi_2 = \pi_1 J$. A typical example is $M = \mathbb{C}^n$ (or any quotient of $\mathbb{C}^n$ such as a complex torus) on which π_1 is projection from $\mathbb{C}^n$ onto $\mathbb{R}^n$ and π_2 is the projection from $\mathbb{C}^n$ onto $J(\mathbb{R}^n)$ where J is the standard complex structure on $\mathbb{C}^n$.

Given such a splitting structure on TM, one can define the following parallel 2-tensor S (see [19]):

$$S(X,Y) = \langle J\pi_1(X), \pi_2(Y)\rangle \tag{1.1}$$

for any $X, Y \in T_pM$ at any $p \in M$.

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Suppose L_p is a Lagrangian subspace of T_pM, it is not hard to check that the restriction of S to L_p is symmetric, i.e. $S(X,Y) = S(Y,X)$ if $X, Y \in L_p$. Moreover, if $\pi_1 : L_p \to T_pM$ is injective, one can apply the singular value decomposition theorem to find an orthonormal basis $\{a_i\}$ for $\pi_1(T_pM)$ and real numbers $\{\lambda_i\}$ such that

$$\left\{ e_i = \frac{1}{\sqrt{1+\lambda_i^2}}(a_i + \lambda_i J(a_i)) \right\}_{i=1}^{n} \tag{1.2}$$

forms an orthonormal basis for L_p. Note that $\{J(a_i)\}$ constitutes an orthonormal basis for $\pi_2(T_pM)$. In terms of this basis,

$$S_{ij} = S(e_i, e_j) = \frac{\lambda_i}{1+\lambda_i^2}\delta_{ij} \ . \tag{1.3}$$

1.1. **Two-Convex Lagrangians.** For a Lagrangian submanifold $F : L \hookrightarrow M$, we consider several geometric conditions that are characterized by the projection map π_1 and the tensor S defined in (1.1).

Definition 1.1. A Lagrangian submanifold $L \subset M$ is said to be *graphical* if $\pi_1 : T_pL \to T_pM$ is injective for any $p \in L$;

A typical example is when $M = \mathbb{C}^n$ and L is the graph of ∇u for a function u defined on $\mathbb{R}^n$, $\lambda_i' s$ in (1.2) are exactly the eigenvalues of D^2u, the Hessian of u, see [22, Section 2].

The graphical condition can be characterized by the positivity of a geometric quantity introduced in [26]. Fix an orientation for $\pi_1(TM)$. Under π_1^*, the volume form of $\pi_1(TM)$ gives a parallel n-form on M, and denote it by Ω. It is clear that a Lagrangian submanifold L is graphical if and only if $*\Omega$ is nowhere zero, where $*\Omega$ denotes the Hodge star of the restriction of the n-form Ω to L. If so, orient the Lagrangian so that $*\Omega > 0$. With respect to (1.2),

$$*\Omega = \frac{1}{\sqrt{\prod_{i=1}^n (1+\lambda_i^2)}} \ . \tag{1.4}$$

We also consider the restriction of S (1.1) to L, F^*S, as a symmetric 2-tensor on L. By (1.3), if the sum of any two eigenvalues of F^*S is positive, then

$$S_{ii} + S_{jj} = \frac{(\lambda_i + \lambda_j)(1+\lambda_i\lambda_j)}{(1+\lambda_i^2)(1+\lambda_j^2)} > 0 \tag{1.5}$$

for any $i \neq j$, or equivalently, $(\lambda_i + \lambda_j)(1+\lambda_i\lambda_j) > 0$ for any $i \neq j$. The region is not a connected region, and we always focus on the connected component where

$$\lambda_i + \lambda_j > 0 \ \text{ and } \ 1 + \lambda_i\lambda_j > 0 \ \text{ for any } i \neq j \ . \tag{1.6}$$

Definition 1.2. A graphical Lagrangian submanifold $L \subset M$ is said to be

(i) *convex* if $\lambda_i > 0$ for each i on L.

(ii) *two-convex* if $\lambda_i + \lambda_j > 0$ and $1 + \lambda_i\lambda_j > 0$ for any $i \neq j$, or (1.6) holds, on L

It is known that the Lagrangian condition is preserved by the mean curvature flow [17]. The main theorem of this paper is that (1.6) implies the long-time existence and convergence of the Lagrangian mean curvature flow.

Theorem 1.3. *Let $L \subset M$ be compact Lagrangian submanifold. If L is graphical and two-convex, then the mean curvature flow of L exists for all time, and remains graphical and two-convex. Moreover, it converges smoothly to a flat Lagrangian submanifold as $t \to \infty$.*

This theorem generalizes [19, Theorem A], which assumes L is convex, or $\lambda_i > 0$ for all i. In fact, all results of Lagrangian mean curvature flows [19, 4, 5] known to us are in the following cases: (1) the convex case, (2) cases that are equivalent to the convex case through unitary transformations (see the next subsection), or (3) cases that are perturbations of (1) and (2).

Remark 1.4. In the proof of Theorem 1.3, we implicitly assume that the ambient space M is also compact. It follows that M is topologically a torus, or its finite quotient. The theorem holds true for some non-compact M as well. For instance, if M is the cotangent bundle of a flat n-torus, one can prove by using the distance (squared) to the zero section that the mean curvature flow of L remains in a compact subset. See for instance [21, Theorem A].

We briefly describe the steps involved the proof as follows:

(i) We start with a compact two-convex Lagrangian that satisfies (1.6). We derive the evolution equation of the following quantity (see (2.6))

$$\log \prod_{i<j} \frac{(\lambda_i + \lambda_j)(1 + \lambda_i\lambda_j)}{(1 + \lambda_i^2)(1 + \lambda_j^2)},$$

show that it is monotone non-decreasing along the mean curvature flow, and therefore (1.6) is preserved.

(ii) We derive the evolution equation of

$$\log(*\Omega) = -\frac{1}{2} \log(\prod_{i=1}^{n} (1 + \lambda_i^2))$$

and show that it is monotone non-decreasing along the mean curvature flow as long as $1 + \lambda_i\lambda_j > 0$, which was established in the last step. This in particular shows that each λ_i remains uniformly bounded.

(iii) We prove that the second fundamental forms are bounded by contradiction. Suppose the second fundamental forms are unbounded, through a blow-up argument we obtain a non-flat ancient solution of the graphical Lagrangian mean curvature flow. A Liouville theorem ([15], see Section 3) which applies the Krylov-Safonov estimate to the equation of the Lagrangian angle θ,

$$\theta = \sum_i^n \arctan \lambda_i,$$

allows us to conclude that the ancient solution must be stationary. Finally, we apply the Bernstein theorem of [22] that asserts any stationary solution satisfying the condition $1 + \lambda_i \lambda_j > 0$ must be affine and arrive at the contradiction.

We remark that the underlying parabolic equation is the following equation for the potential function u:

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{-1}} \log \frac{\det(\mathbf{I} + \sqrt{-1} D^2 u)}{\sqrt{\det(\mathbf{I} + (D^2 u)^2)}} \ . \tag{1.7}$$

The estimates of $\lambda_i' s$ correspond to the C^2 estimates of the solution u and the estimates of the second fundamental forms correspond to the C^3 estimates.

The convex assumption implies that the right hand side of (1.7) , i.e. the Lagrangian angle θ, as a function of $D^2 u$ is concave in the space of symmetric matrices and thus PDE theories of fully nonlinear elliptic and parabolic equations [3, 12, 1] are applicable. The two-convex assumption (and the area-decreasing assumption in the next subsection) arises naturally in the study of the Lagrangian Grassmannian [22] and the Gauss map of the mean curvature flow [28]. It is interesting to see if some similar approach would work for related problems such as the deformed Hermitian–Yang–Mills equation considered in [11, 6] or the curvature type equations considered in [7]. On the other hand, it is a natural question to ask if two-convexity can replace the convex assumption in the work of Caffarelli-Nirenberg-Spruck [3].

1.2. **Area-Decreasing Lagrangians.** It is known that when $M = \mathbb{C}^n$ the convex case $\lambda_i > 0$ for each i is essentially equivalent to the case $|\lambda_i| < 1$ for each i through a unitary transformation $U(n)$ of $\mathbb{C}^n$ ([22], or the Lewy transformation in [30]). The two-convex case is essentially equivalent to the following area-decreasing case through the same unitary transformation.

One can consider another parallel 2-tensor P:

$$P(X, Y) = \langle \pi_1(X), \pi_1(Y) \rangle - \langle \pi_2(X), \pi_2(Y) \rangle \ . \tag{1.8}$$

With respect to the frame (1.2),

$$P_{ij} = P(e_i, e_j) = \frac{1 - \lambda_i^2}{1 + \lambda_i^2} \delta_{ij} \ .$$

For a Lagrangian submanifold $F : L \hookrightarrow M$, F^*P being 2-positive means that

$$P_{ii} + P_{jj} = \frac{1 - \lambda_i^2 \lambda_j^2}{(1 + \lambda_i^2)(1 + \lambda_j^2)} > 0 \tag{1.9}$$

for any $i \neq j$.

Definition 1.5. A graphical Lagrangian submanifold $L \subset M$ is said to be *area-decreasing* if

$$|\lambda_i \lambda_j| < 1 \text{ for any } i \neq j \tag{1.10}$$

holds true at every $p \in L$.

When $M = \mathbb{C}^n$ and L is the graph of ∇f, the condition corresponds to ∇f as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$ is area-decreasing.

The same results in Theorem 1.3 hold true for the mean curvature flow of area-decreasing Lagrangians.

Theorem 1.6. *Let $L \subset M$ be compact Lagrangian submanifold. If it is graphical and area-decreasing, then the mean curvature flow of L exists for all time, and remains graphical and area-decreasing. Moreover, it converges to a flat Lagrangian submanifold as $t \to \infty$.*

This theorem generalizes [18, Theorem 2], which assumes $\dim L = 2$.

The paper is organized as follows. In section 2, we derive the evolution equations and provide quantitative bounds of relevant quantities. In section 3, a Liouville Theorem for ancient solutions of Lagrangian mean curvature flows is discussed. Section 4 is devoted to prove Theorem 1.3 and Theorem 1.6.

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2. Evolution Equations

For a Lagrangian submanifold, J induces an isometry between its tangent bundle and its normal bundle. As a consequence, its second fundamental form is totally symmetric. That is to say,

$$h_{ijk} = \langle \bar{\nabla}_{e_i} e_j, J(e_k) \rangle \tag{2.1}$$

is totally symmetric in i, j, k, where $\{e_i\}$ is an orthonormal basis for its tangent space. Here $\bar{\nabla}$ is the covariant derivative of M.

Suppose that $F : L \times [0, T) \to M$ is a Lagrangian mean curvature flow. From [19, section 3.2], F^*S satisfies

$$(\frac{\partial}{\partial t} - \Delta) S_{ij} = h_{mki} h_{mk\ell} S_{\ell j} + h_{mkj} h_{mk\ell} S_{i\ell} + 2 h_{ki\ell} h_{kjm} S_{\ell m} \ , \tag{2.2}$$

where the equation is in terms of an evolving orthonormal frame and repeated indexes are summed.

Since S is a parallel tensor on M, $\bar{\nabla} S = 0$, the (spatial) gradient of F^*S is (see for example [23, p.1121])

$$\begin{aligned} S_{ij;k} &= e_k(S(e_i, e_j)) - S(\nabla_{e_k} e_i, e_j) - S(e_i, \nabla_{e_k} e_j) \\ &= S(\bar{\nabla}_{e_k} e_i - \nabla_{e_k} e_i, e_j) + S(e_i, \bar{\nabla}_{e_k} e_j - \nabla_{e_k} e_j), \end{aligned} \tag{2.3}$$

where ∇ is the covariant derivative on L. The definition of second fundamental forms (2) implies $\bar{\nabla}_{e_k} e_i - \nabla_{e_k} e_i = h_{ki\ell} J(e_\ell)$ and

$$S_{ij;k} = h_{ki\ell}\, S(J(e_\ell), e_j) + h_{kj\ell}\, S(e_i, J(e_\ell)).$$

At a space-time point p, $S_{ij;k}$ can thus be expressed in terms the frame (1.2) as

$$S_{ij;k} = h_{kij} \left(\frac{\lambda_j^2}{1+\lambda_j^2} - \frac{1}{1+\lambda_i^2} \right) = -\frac{1}{2} h_{kij} \left(\frac{1-\lambda_i^2}{1+\lambda_i^2} + \frac{1-\lambda_j^2}{1+\lambda_j^2} \right). \tag{2.4}$$

2.1. **The Logarithmic Determinant of $S^{[2]}$.** In [3], another tensor $S^{[2]}$ is introduced to study the two-positivity of F^*S; see also [23, section 5]. Similar to [20], we consider the equation of the logarithmic determinant of $S^{[2]}$.

With respect to an orthonormal frame, $S^{[2]}$ is defined by

$$S^{[2]}_{(ij)(k\ell)} = S_{ik}\delta_{j\ell} + S_{j\ell}\delta_{ik} - S_{i\ell}\delta_{jk} - S_{jk}\delta_{i\ell} \tag{2.5}$$

for any $i < j$ and $k < \ell$. It can be regarded as a symmetric endomorphism on $\Lambda^2 TL$.

At a space-time point p, suppose that S is diagonal in terms of the frame (1.2). It follows from (2.5) that $S^{[2]}_{(ij)(k\ell)}|_p = (S_{ii} + S_{jj})\delta_{ik}\delta_{j\ell}$. Thus, the 2-positivity of F^*S is equivalent to the positivity of $S^{[2]}$.

It can be proved that the positivity of $S^{[2]}$ is preserved along the flow in the same way as in [23]. However, in this article we consider another quantity that also controls the two-convexity condition: the logarithmic determinant of $S^{[2]}$:

$$\log\det(S^{[2]}) = \log \prod_{i<j} \frac{(\lambda_i + \lambda_j)(1 + \lambda_i\lambda_j)}{(1+\lambda_i^2)(1+\lambda_j^2)}. \tag{2.6}$$

It is a straightforward computation to show that the function $\log\det(S^{[2]})$ satisfies

$$\begin{aligned} &(\frac{\partial}{\partial t} - \Delta) \ln(\det S^{[2]}) \\ \overset{\text{at } p}{=} &\sum_{1\le i<j\le n} \left[(S_{ii}+S_{jj})^{-1} (\frac{\partial}{\partial t} - \Delta)(S_{ii}+S_{jj}) + (S_{ii}+S_{jj})^{-2} |\nabla(S_{ii}+S_{jj})|^2 \right] \\ &+ 2 \sum_{1\le i\le n} \sum_{\substack{1\le j<k\le n \\ j\ne i,\, k\ne i}} (S_{ii}+S_{jj})^{-1}(S_{ii}+S_{kk})^{-1} |\nabla S_{jk}|^2 \;. \end{aligned} \tag{2.7}$$

The main task is to calculate the first term on the right hand side of (2.7). According to (2.2),

$$
\begin{aligned}
(\frac{\partial}{\partial t}-\Delta)(S_{ii}+S_{jj}) &= 2\sum_{k,\ell}\left[h_{k\ell i}^2(S_{ii}+S_{\ell\ell})+h_{k\ell j}^2(S_{jj}+S_{\ell\ell})\right] \\
&= 4\sum_k\left[h_{kii}^2\,S_{ii}+h_{kjj}^2\,S_{jj}\right]+4(S_{ii}+S_{jj})\sum_k h_{kij}^2 \\
&\quad +2\sum_k\sum_{\ell\neq\{i,j\}}\left[h_{k\ell i}^2(S_{ii}+S_{\ell\ell})+h_{k\ell j}^2(S_{jj}+S_{\ell\ell})\right]\ .
\end{aligned}
$$

By (2.4),

$$
|\nabla(S_{ii}+S_{jj})|^2=\sum_k\left(h_{kii}\frac{1-\lambda_i^2}{1+\lambda_i^2}+h_{kjj}\frac{1-\lambda_j^2}{1+\lambda_j^2}\right)^2\ . \tag{2.8}
$$

It follows that

$$
\begin{aligned}
&(S_{ii}+S_{jj})\Delta(S_{ii}+S_{jj})+|\nabla(S_{ii}+S_{jj})|^2 \\
&\quad -4(S_{ii}+S_{jj})^2\sum_k h_{kij}^2-2(S_{ii}+S_{jj})\sum_k\sum_{\ell\neq\{i,j\}}\left[h_{k\ell i}^2(S_{ii}+S_{\ell\ell})+h_{k\ell j}^2(S_{jj}+S_{\ell\ell})\right] \\
&=4(S_{ii}+S_{jj})\sum_k\left[h_{kii}^2\,S_{ii}+h_{kjj}^2\,S_{jj}\right]+\sum_k\left(h_{kii}\frac{1-\lambda_i^2}{1+\lambda_i^2}+h_{kjj}\frac{1-\lambda_j^2}{1+\lambda_j^2}\right)^2 \\
&=\sum_k\left[\frac{(\lambda_i+\lambda_j)^2+(1+\lambda_i\lambda_j)^2}{(1+\lambda_i^2)(1+\lambda_j^2)}(h_{kii}^2+h_{kjj}^2)+\frac{(1-\lambda_i^2)(1-\lambda_j^2)}{(1+\lambda_i^2)(1+\lambda_j^2)}2h_{kii}h_{kjj}\right] \\
&=\sum_k\left[\frac{(1+\lambda_i\lambda_j)^2}{(1+\lambda_i^2)(1+\lambda_j^2)}(h_{kii}+h_{kjj})^2+\frac{(\lambda_i+\lambda_j)^2}{(1+\lambda_i^2)(1+\lambda_j^2)}(h_{kii}-h_{kjj})^2\right]\ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(S_{ii}+S_{jj})^{-1}(\frac{\partial}{\partial t}-\Delta)(S_{ii}+S_{jj})+(S_{ii}+S_{jj})^{-2}|\nabla(S_{ii}+S_{jj})|^2 \\
&=4\sum_k h_{kij}^2+2(S_{ii}+S_{jj})^{-1}\sum_k\sum_{\ell\neq\{i,j\}}\left[h_{k\ell i}^2(S_{ii}+S_{\ell\ell})+h_{k\ell j}^2(S_{jj}+S_{\ell\ell})\right] \\
&\quad +\sum_k\left[\frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(\lambda_i+\lambda_j)^2}(h_{kii}+h_{kjj})^2+\frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(1+\lambda_i\lambda_j)^2}(h_{kii}-h_{kjj})^2\right]\ .
\end{aligned}
$$

With (2.7), it leads to the following Proposition.

Proposition 2.1. *Suppose that a graphical Lagrangian mean curvature flow is two-convex, then the function* $\log(\det S^{[2]})$ *satisfies*

$$
\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta)\log(\det S^{[2]}) \\
&\geq \sum_{i<j}\sum_{k}\left[4h_{kij}^2 + \frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(\lambda_i+\lambda_j)^2}(h_{kii}+h_{kjj})^2 + \frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(1+\lambda_i\lambda_j)^2}(h_{kii}-h_{kjj})^2\right] \quad (2.9)\\
&\geq 2|A|^2 \geq 0\ . \quad (2.10)
\end{aligned}
$$

In particular, $\min\log(\det S^{[2]})$ *is monotone non-decreasing along the flow and two-convexity is preserved.*

Proof. It remains to show that (2.10) is no less than $2|A|^2$ under the condition (1.6). It is straightforward to verify that under the condition (1.6),

$$\frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(\lambda_i+\lambda_j)^2} \geq 1 \quad \text{and} \quad \frac{(1+\lambda_i^2)(1+\lambda_j^2)}{(1+\lambda_i\lambda_j)^2} \geq 1\ . \tag{2.11}$$

Note that both equalities are attained when $\lambda_i = 1 = \lambda_j$. Hence, the right hand side of (2.9) is no less than

$$\sum_{i<j}\sum_{k}\left[4h_{kij}^2 + 2h_{kii}^2 + 2h_{kjj}^2\right] = 2|A|^2 + 2(n-2)\sum_{i,k}h_{kii}^2\ . \tag{2.12}$$

It finishes the proof of this proposition. □

2.2. **The Logarithmic Determinant of the Jacobian of** π_1**.** In the minimal Lagrangian case, the equation for $\log(*\Omega)$ is derived in [22, (2.4)]. The computation in the parabolic case is essentially the same, and

$$(\frac{\partial}{\partial t} - \Delta)\log(*\Omega) = \sum_{i,j,k}h_{ijk}^2 + \sum_{i,j}\lambda_i^2\, h_{iij}^2 + 2\sum_{\substack{i,j,k\\ i<j}}\lambda_i\lambda_j\, h_{ijk}^2\ .$$

By re-grouping the summations, one finds the following proposition.

Proposition 2.2. *Along a graphical Lagrangian mean curvature flow, the function* $\log(*\Omega)$ *satisfies*

$$
\begin{aligned}
(\frac{\partial}{\partial t} - \Delta)\log(*\Omega) &= \sum_{i}(1+\lambda_i^2)h_{iii}^2 + \sum_{i\neq j}(3+\lambda_i^2+2\lambda_i\lambda_j)h_{iij}^2 \\
&\quad + \sum_{i<j<k}(6+2\lambda_i\lambda_j+2\lambda_j\lambda_k+2\lambda_k\lambda_i)h_{ijk}^2\ .
\end{aligned} \tag{2.13}
$$

If the flow is in addition two-convex, then $(\frac{\partial}{\partial t} - \Delta)\log(*\Omega) \geq 0$ *and* $\min\log(*\Omega)$ *in monotone non-decreasing along the flow.*

Proof. It is clear that the right hand side is non-negative if $1 + \lambda_i \lambda_j > 0$ which is part of the two-convexity assumption.

$\square$

2.3. **Some Quantitative Bounds.** Since $\frac{\lambda}{1+\lambda^2}$ takes value within $[-\frac{1}{2}, \frac{1}{2}]$, the expression (1.5), $S_{ii} + S_{jj}$, is always no greater than 1. It follows that for a two-convex Lagrangian, $\det S^{[2]}$ takes value within $(0, 1]$. Hence, $\log(\det S^{[2]}) \in (-\infty, 0]$. Since $*\Omega = 1/\sqrt{\prod_i (1+\lambda_i^2)}$, $\log(*\Omega) \in (-\infty, 0]$.

For a two-convex Lagrangian submanifold, suppose that $\log(*\Omega) \geq -\delta_1$ and $\log(\det S^{[2]}) \geq -\delta_2$ for some $\delta_1, \delta_2 > 0$. It follows that

$$\sum_i \lambda_i^2 \leq e^{2\delta_1} - 1 \tag{2.14}$$

for all i. From $\log(\det S^{[2]}) \geq -\delta_2$,

$$\frac{(\lambda_i + \lambda_j)(1 + \lambda_i \lambda_j)}{(1+\lambda_i^2)(1+\lambda_j^2)} \geq \prod_{k<\ell} (S_{kk} + S_{\ell\ell}) \geq e^{-\delta_2}$$

for any $i \neq j$. Therefore,

$$(\lambda_i + \lambda_j)(1 + \lambda_i \lambda_j) \geq e^{-\delta_2} \ . \tag{2.15}$$

Under the condition (1.6), (2.14) and (2.15) lead to that

$$1 + \lambda_i \lambda_j \geq \frac{e^{-\delta_2}}{\sqrt{2(e^{2\delta_1} - 1)}} \quad \text{and} \quad \lambda_i + \lambda_j \geq \frac{2e^{-\delta_2}}{e^{2\delta_1} + 1} \tag{2.16}$$

for any $i \neq j$.

3. A Liouville Theorem

In this section, we state a Liouville theorem for ancient solutions of the Lagrangian mean curvature flow in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ under the graphical condition. For discussions of ancient solutions of the Lagrangian mean curvature flow under other assumptions, see [13]. The theorem is due to Nguyen and Yuan [15, Proposition 2.1] and is a direct consequence of the Krylov–Safonov estimate [12]. We include the proof here for completeness.

Theorem 3.1. *Let u be a smooth solution to*

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{-1}} \log \frac{\det(\mathbf{I} + \sqrt{-1} D^2 u)}{\sqrt{\det(\mathbf{I} + (D^2 u)^2)}} \tag{3.1}$$

in $Q = \mathbb{R}^n \times (-\infty, t_0]$ for some $t_0 > 0$. Denote by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the Hessian of u, $D^2 u$. Suppose that every $|\lambda_i|$ is bounded over Q. Then, u is stationary, i.e. u satisfies the special Lagrangian equation.

Proof. Denote the right hand side of (3.1) by θ. It is the argument of the complex number $\det(\mathbf{I}+\sqrt{-1}D^2u)$. Note that θ is a smooth function over Q, and takes value within $(-n\pi/2, n\pi/2)$. According to [8, §III.2.D], the differential of θ is equivalent to the mean curvature of the graph of Du. As a consequence, (3.1) means that the Lagrangian $\{(x, Du)\}$ evolves under the mean curvature flow.

The induced metric on the graph of Du has the first fundamental form given by

$$g = \mathbf{I} + (D^2u)^2 = (\mathbf{I}+\sqrt{-1}D^2u)(\mathbf{I}-\sqrt{-1}D^2u) \ .$$

In particular, $(\mathbf{I}+\sqrt{-1}D^2u)^{-1}$ is $(\mathbf{I}-\sqrt{-1}D^2u)g^{-1} = g^{-1}(\mathbf{I}-\sqrt{-1}D^2u)$. With this understood, the derivative of (3.1) in t gives

$$\frac{\partial\theta}{\partial t} = g^{ij}\,\partial_t u_{ij} = g^{ij}\,\partial_i\partial_j(\partial_t u) = g^{ij}\,\partial_i\partial_j\theta \ , \tag{3.2}$$

where $[g^{ij}]$ is the inverse of $g = \mathbf{I}+(D^2u)^2$. Since $|\lambda_i|$'s are uniformly bounded, (3.2) is a uniformly parabolic equation. As the right hand side of (3.2) has no first and zeroth order terms, the Krylov–Safonov estimate [12, Lemma 2 on p.133] implies that there exist positive α and C depending on n, $\sup_Q |u|$ and $\sup_Q |D^2u|$ such that

$$\sup\left\{\frac{|\theta(\tilde{x},\tilde{t})-\theta(x,t)|}{\max\{|\tilde{x}-x|^\alpha, |\tilde{t}-t|^{\frac{\alpha}{2}}\}} : (\tilde{x},\tilde{t}),(x,t)\in B_r\times[t_0-r^2,t_0] \ , \ (\tilde{x},\tilde{t})\neq(x,t)\right\} \leq C\,\frac{1}{r^\alpha}$$

for any $r>0$. By letting $r\to\infty$, one finds that $\theta(x,t)=\theta(\tilde{x},\tilde{t})$ for any $(x,t)\neq(\tilde{x},\tilde{t})$.

It follows that the graph of Du is a time-independent minimal/special Lagrangian submanifold. It finishes the proof of this theorem. □

4. Proof of the Main Theorems

This section is devoted to the proof of Theorem 1.3. The proof of Theorem 1.6 is almost the same, and we only address the key ingredient at the end of this section.

4.1. **Preserving the Graphical and Two-Convexity Condition.** Suppose that L is a compact, oriented n-dimensional manifold, and $F_0 : L \to M$ is a two-convex Lagrangian submanifold. Consider the Lagrangian mean curvature flow $F : L\times[0,T)\to M$ with $F(\cdot,0)=F_0(\cdot)$, where T is the maximal existence time. Let

$$\bar{\tau} = \sup\{\tau\in(0,T) : \text{the flow remains two-convex in } [0,\tau)\} \ .$$

Denote by L_t the image of $L\times\{t\}$ under F. Since L is compact, $\log(\det S^{[2]}) \geq -\delta_2$ and $\log(*\Omega)\geq-\delta_1$ for some $\delta_1,\delta_2>0$ on L_0. Due to Proposition 2.1, Proposition 2.2 and the maximum principle, both $\min_{L_t}\log(\det S^{[2]})$ and $\min_{L_t}\log(*\Omega)$ are non-decreasing in $t\in[0,\bar{\tau})$. If $\bar{\tau}<T$, it follows from (2.14) and (2.15) that $L\times\{\bar{\tau}\}$ is two-convex. Because of the openness

of the two-convexity condition, this is a contradiction, and $\bar{\tau}$ must be the maximal existence time.

4.2. **Long-time Existence.** With (2.10), one may use the same argument as that in the proof of [26, Theorem A] to prove the mean curvature flow exists for all time. It is based on Huisken's monotonicity formula [10] and White's regularity theorem [29].

Below, we present another argument based on the Liouville theorem (Theorem 3.1). Recall that Huisken proved that if the maximal existence time $T < \infty$, then it is characterized by

$$\limsup_{t\to T} \max_{L_t} |A|^2 = \infty \ ; \tag{4.1}$$

see [9, Theorem 8.1] for the hypersurface case.

Assume (4.1) for some $\bar{t} < \infty$. There exist sequences $\{t_k\} \in (0, T)$ and $\{\mathbf{x}_k\} \in L$ such that

- $t_k \to T$ monotonically as $k \to \infty$;
- $|A|^2(\mathbf{x}_k, t_k) = \max\{|A|^2(x, t) : (\mathbf{x}, t) \in L \times [0, t_k]\}$;
- $|A|^2(\mathbf{x}_k, t_k) \to \infty$ monotonically as $k \to \infty$.

Denote $|A|^2(\mathbf{x}_k, t_k)$ by ρ_k. Due to [18, Lemma 3.1] (see also [14, Theorem 1]), the second fundamental form obeys

$$(\frac{\partial}{\partial t} - \Delta)|A|^2 \leq -2|\nabla A|^2 + 3|A|^4 \ . \tag{4.2}$$

By the maximum principle, there exists $c > 0$ such that $T - t_k \geq c\,\rho_k^{-2}$ for all $k \in \mathbb{N}$. Identify a neighborhood of $F(\mathbf{x}_k, t_k)$ with a subset of $\mathbb{R}^{2n}$, and consider

$$\tilde{F}_k(\cdot, s) = \rho_k \left[F(\cdot, t_k + \frac{s}{\rho_k^2}) - F(\mathbf{x}_k, t_k) \right] \ . \tag{4.3}$$

The image of $\tilde{F}_k$ is given by the graph of $D\tilde{u}_k$ for some $\tilde{u}_k : \mathcal{U}_k \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ with $\tilde{u}_k(0, 0) = 0$ and $D\tilde{u}_k(0, 0) = 0$. It follows from $\rho_k \to \infty$ that any compact subset of $\mathbb{R}^n \times (-\infty, c)$ is contained in $\mathcal{U}_k$ for any $k >> 1$. By the standard blow-up argument[1], $\tilde{u}_k$ converges to $u : \mathbb{R}^n \times (-\infty, c) \to \mathbb{R}$ satisfying (3.1), and the convergence is smooth on any compact subset of $\mathbb{R}^n \times (-\infty, c)$.

Note that the slope is invariant under the rescaling (4.3), and λ_i's remain unchanged. In particular, the eigenvalues of $D^2 u$ satisfy (2.14) and (2.16) everywhere. Hence, Theorem 3.1 implies that the graph of Du is a special Lagrangian submanifold that satisfies the condition that for any i, j, $3 + 2\lambda_i\lambda_j \geq \delta$ over Q; we conclude that the graph of Du in $\mathbb{R}^{2n}$, $\{(x, Du) : x \in \mathbb{R}^n\}$, is an affine n-plane by [22, Corollary C]. However, the second fundamental form of the graph of $D\tilde{u}_k$ has norm 1 at $(0, 0)$. This is a contradiction.

[1]Since $|\lambda_i|'s$ are uniformly bounded, so is $D^2\tilde{u}_k$. The third order derivative $D^3\tilde{u}_k$ is equivalent to the second fundamental form, which is bounded. The higher order derivatives are also bounded; see [2, Proposition 4.8].

4.3. **Convergence.** The key to conclude the convergence as $t \to \infty$ is to show that $\max_{L_t} |A|^2 \to 0$ as $t \to \infty$.

4.3.1. *Uniform Boundedness of* $|A|^2$. The first task is to show that $|A|^2$ is uniformly bounded. Suppose not, then

$$\limsup_{t\to\infty} \max_{L_t} |A|^2 = \infty \ . \tag{4.4}$$

With the same argument as that in section 4.2, one can extract a blow-up limit, which is a non-trivial ancient solution to (3.1). By the same token, it contradicts to Theorem 3.1 (i).

4.3.2. L^2*-Convergence of* $|A|^2$. With the uniformly boundedness of $|A|^2$, [2, Proposition 4.8] asserts that $|\nabla^\ell A|^2$ is uniformly bounded for all $\ell \in \mathbb{N}$.

Denote $\log(\det S^{[2]})$ by v, whose value belongs to $[-\delta_2, 0)$. By Proposition 2.1, it obeys $(\frac{\partial}{\partial t} - \Delta)v \geq 2|A|^2$. Denote by $\mathrm{d}\mu_t$ the volume form of L_t. Since $\frac{\partial}{\partial t}\mathrm{d}\mu_t = -|H|^2\mathrm{d}\mu_t$,

$$2\int_{L_t} |A|^2 \,\mathrm{d}\mu_t \leq \int_{L_t} \left[(\frac{\partial}{\partial t} - \Delta)v - v|H|^2\right] \mathrm{d}\mu_t = \frac{\mathrm{d}}{\mathrm{d}t}\int_{L_t} v\,\mathrm{d}\mu_t \ .$$

This together with $|\int_{L_t} v\,\mu_t| \leq \delta_2 \,\mathrm{vol}(L_t) \leq \delta_2\,\mathrm{vol}(L_0)$ implies that

$$\int_0^\infty (\int_{L_t} |A|^2 \mathrm{d}\mu_t)\,\mathrm{d}t < \infty \ . \tag{4.5}$$

By (4.2) and the uniform boundedness of $|A|^2$ and $|\nabla A|^2$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{L_t} |A|^2 \mathrm{d}\mu_t &= \int_{L_t} \left[(\frac{\partial}{\partial t} - \Delta)|A|^2 - |A|^2|H|^2\right] \mathrm{d}\mu_t \\ &\leq \int_{L_t} \left[3|A|^4 - 2|\nabla A|^2 - |A|^2|H|^2\right] \mathrm{d}\mu_t \leq c_1 \ . \end{aligned} \tag{4.6}$$

According to [21, Lemma 6.3], (4.5) and (4.6) imply that $\int_{L_t} |A|^2 \mathrm{d}\mu_t \to 0$ as $t \to \infty$.

4.3.3. *Convergence of the Flow.* Fix $t \geq 0$; suppose that $\max_{L_t} |A|^2$ is achieved at $\mathbf{x}_0$. On a fixed size neighborhood of $\mathbf{x}_0$, L_t is the graph over $\pi_1(T_{\mathbf{x}_0}M)$, whose higher derivatives are all bounded. It follows that there exists a $c_2 > 0$ such that $\int_{L_t} |A|^2 \mathrm{d}\mu_t \geq c_2 \max_{L_t} |A|^2$. Therefore,

$$\lim_{t\to 0} \max_{L_t} |A|^2 = 0 \ . \tag{4.7}$$

Since the mean curvature flow is a gradient flow and the metrics are analytic, it follows from the theorem of Simon [16] that the flow converges to a unique limit as $t \to \infty$. This finishes the proof of Theorem 1.3.

4.4. **About Theorem 1.6.** Analogous to (2.5), we introduce $P^{[2]}$ to study the 2-positivity of F^*P. According to [20, Theorem 3.2], the logarithmic determinant[2] of $P^{[2]}$ obeys

$$(\frac{\partial}{\partial t} - \Delta)\log(\det P^{[2]}) \geq 2|A|^2 \tag{4.8}$$

along the mean curvature flow. In terms of λ_i, we have

$$\log\det(P^{[2]}) = \log\prod_{i<j}\frac{1-\lambda_i^2\lambda_j^2}{(1+\lambda_i^2)(1+\lambda_j^2)}\ . \tag{4.9}$$

The proof of Theorem 1.6 is very similar to that of Theorem 1.3, and is sketched below. As in section 4.1, denote by T the maximal existence time. Let

$$\bar{\tau} = \sup\{\tau\in(0,T) : \text{the flow remains graphical and area-decreasing in } [0,\tau)\}\ .$$

By the maximum principle on (4.8), $\log(\det P^{[2]})$ is uniformly bounded from below. It follows from [20, Lemma 3.3] that L_t remains graphical and area-decreasing as long as the flow exists.

For the long-time existence, suppose that $T<\infty$, and perform the same blow-up argument as that in 4.2 to get a non-trivial ancient solution of (3.1). Here, we rely on [27, Theorem 1.1] to conclude that any entire, graphical minimal submanifold that satisfies the condition $|\lambda_i\lambda_j|\leq 1-\delta$ must be an affine n-plane. It is a contradiction, and thus $T=\infty$.

By a similar blow-up argument, the second fundamental form cannot tend to infinity as $t\to\infty$. As in section 4.3.2, one deduces that $\int_{L_t}|A|^2\mathrm{d}\mu_t\to 0$ as $t\to\infty$ by considering the integration of (4.8) over L_t. The same argument as that in section 4.3.3 implies that $\sup_{L_t}|A|^2\to 0$ as $t\to\infty$. Finally, one invokes the theorem of [16] to finish the proof.

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