INFINITE-TIME SINGULARITIES OF THE LAGRANGIAN MEAN CURVATURE FLOW

WEI-BO SU, CHUNG-JUN TSAI, AND ALBERT WOOD

ABSTRACT. In this paper, we exhibit examples of Lagrangian mean curvature flow which exist and are embedded for all time, but form an infinite-time singularity and converge to an immersed special Lagrangian as $t \to \infty$. This result shows that infinite-time singularities can form in the Thomas–Yau [33] 'semi-stable' situation.

Our work is a parabolic analogue of the results of Joyce [15] and Lee [19] regarding desingularisation of special Lagrangians with conical singularities. The gluing construction that we employ is inspired by the work of Brendle and Kapouleas [3] regarding ancient solutions of the Ricci flow.

CONTENTS

1. INTRODUCTION

1.1. **Singularities of Lagrangian Mean Curvature Flow.** The celebrated theorem of Yau [35] states that if the canonical bundle of a Kähler manifold is holomorphically trivial, then it admits a Ricci flat Kähler metric, which is referred to as the Calabi–Yau metric. Over the past four decades, understanding the Lagrangian submanifolds minimal with respect to

Date: January 4, 2024.

W.-B. Su is supported in part by the Taiwan MOST postdoctoral research abroad program 111-2917-I-564-002. C.-J. Tsai is supported in part by the Taiwan NSTC grants 111-2636-M-002-022, 112-2636-M-002-003 and 112-2628-M-002-004-MY4. A. Wood is supported in part by the Taiwan NSTC grant 111-2636-M-002-022, and by a Simons Collaboration Grant on "Special holonomy in Geometry, Analysis and Physics".

such a metric (known as special Lagrangians [9]) has been a major direction in differential geometry. Calabi–Yau manifolds and their special Lagrangians also appear in various proposals of theoretical physics. In particular, Strominger, Yau and Zaslow [30] have proposed to use special Lagrangian fibrations to understand the mirror symmetry of Calabi–Yau manifolds.

The basic question about the existence of a special Lagrangian representative for a given homology class in a Calabi–Yau manifold is still open. In contrast, the Lagrangian condition is symplectic-topological, and so it is easier to find Lagrangian submanifolds. Moreover, the Lagrangian condition is preserved along the mean curvature flow if the ambient metric is Calabi–Yau [27, 29]. One can therefore naturally deform a Lagrangian submanifold by its mean curvature vector to decrease its volume, and hope that the flow will exist forever and converge to a special Lagrangian submanifold. Motivated by mirror symmetry, Thomas and Yau [32, 33] proposed a conjectural picture for the Lagrangian mean curvature flow, relating the behavior of the flow to a "stability" property of the Lagrangian cycle; these conjectures have since been refined and reformulated by Joyce [16].

In practice, the Lagrangian mean curvature flow often forms singularities in finite-time. In fact, Neves [25, 26] constructed examples of Lagrangian mean curvature flow forming a *finite-time* singularity within *any* Hamiltonian isotopy class of Lagrangians, in the case of 2-dimensional Lagrangians. A resolution of the Thomas–Yau conjecture will therefore require a detailed understanding of singularity formation.

In this paper, we will study the complementary phenomenon of infinite-time singularities, in order to improve our understanding of the Thomas–Yau picture. Explicitly, we will show that there exist Lagrangian mean curvature flows that exist for time $t \in [\Lambda, \infty)$, for which the flow converges to a *singular* special Lagrangian as $t \to \infty$. Since the limit is not smooth, the second fundamental form cannot remain uniformly bounded, and therefore this is an example of an *infinite-time singularity* of Lagrangian mean curvature flow. To our knowledge, this is the first example of an infinite-time singularity of mean curvature flow in high codimension. Our construction is based on a parabolic gluing technique. Over the past five years, there have been several works based on this method, in particular the work of Brendle and Kapouleas [3] on the Ricci flow. See also [1, 7, 8, 34] for other geometric flows.

We remark that Chen and He [6] proved that the mean curvature flow cannot have an infinite-time singularity when the ambient manifold is non-compact satisfying some mild conditions. In contrast, Chen and Sun [5] recently constructed a non-compact example in $\mathbb{R}^3$ with an infinite-time singularity.

1.2. **Special Lagrangians with Isolated Conical Singularities.** Consider an immersed special Lagrangian, whose singular points are modelled on the transverse intersection of two half-dimensional planes. If the two half-dimensional planes at a singular point satisfy the *angle criterion* (also known as a type 1 intersection, see Definition 2.17), then there exists an asymptotically conical special Lagrangian (known as a *Lawlor Neck* [17]) with the same planes as asymptotes. One may therefore 'glue in' this Lawlor neck at scale ε at the singular point to produce an almost-minimal desingularisation N^ε (see Figure 1).

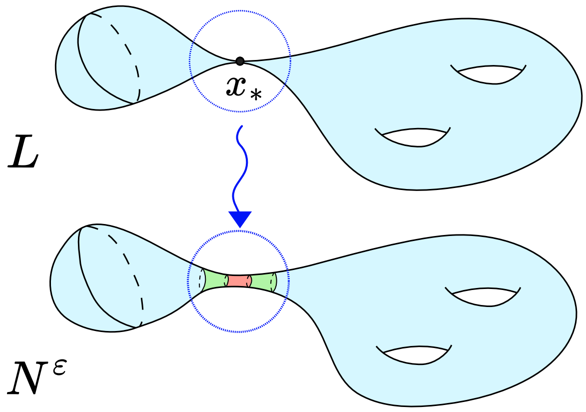

FIGURE 1. An diagram of a special Lagrangian with a single immersed point L such that $L\backslash\{x_\star\}$ is disconnected, along with its desingularisation N^ε. The desingularisation is obtained by 'gluing in' an asymptotically conical special Lagrangian (a Lawlor neck) at scale ε. The Laplacian on N^ε has a one-dimensional space of non-trivial eigenfunctions with small eigenvalues.

A natural question is whether the desingularisation N^ε can be perturbed to a smooth special Lagrangian submanifold. This question was studied thoroughly by Joyce in a series of papers [11, 12, 13, 14, 15], and also by Lee [19]. The upshot is that as long as the immersed special Lagrangian satisfies a "balancing" condition, the desingularisation can be perturbed into a special Lagrangian [15, Theorem 9.7]. These theorems can be applied to construct interesting examples of special Lagrangians; see for instance [10].

An overview of Joyce's construction is as follows. Firstly, nearby Lagrangians are represented as graphs of closed one forms in the Lagrangian neighbourhood of N^ε, and the special Lagrangian equation is expressed as a scalar equation on N^ε in the potential functions. The potential function of the mean curvature vector corresponds to the Lagrangian angle, and the linearised operator of the special Lagrangian equation is the Laplace operator on N^ε. In general, the linearised operator may have eigenfunctions with small eigenvalues (relative to the size of the neck), which means the inverse is not bounded independently of ε. However, the balancing condition guarantees that the orthogonal projection of the mean curvature potential to the small eigenspace is sufficiently small, and can be ignored. One can therefore construct an iteration map using the inverse of the linearised operator, and by applying this map iteratively converge to a solution.

1.3. **Desingularisation and Lagrangian Mean Curvature Flow.** Instead of stating the balancing condition of Joyce, let us illustrate the simplest case. Suppose that an immersed special Lagrangian L has only one singular point $x_\star$, such that the tangent cone at $x_\star$ satisfies the angle criterion. In this case, the balancing condition means that the complement of $x_\star$, $L\backslash\{x_\star\}$, is *connected*.

When $L\backslash\{x_\star\}$ is not connected (e.g. as in Figure 1), the linearised operator of the special Lagrangian equation on the desingularisation N^ε has a one-dimensional space of non-trivial eigenfunctions with small eigenvalues, which acts as an obstruction to finding a special Lagrangian nearby to N^ε. In [18], Lee proved that by allowing a perturbation of the ambient Calabi–Yau structure, N^ε can still be perturbed into a special Lagrangian. In this paper, we consider instead the Lagrangian mean curvature flow of N^ε with respect to a fixed Calabi–Yau structure on the ambient space.

Following an idea of Brendle and Kapouleas [3], we consider a family of desingularisations $N^{\varepsilon(t)}$, and imagine that $N^{\varepsilon(t)}$ is approximately a mean curvature flow. By projecting $\frac{\partial}{\partial t}N^{\varepsilon(t)}$ and the mean curvature potential onto the small eigenspace and heuristically dropping out smaller terms, one obtains an ODE for $\varepsilon(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon(t))^2 = -c\,(\varepsilon(t))^m,$$

where $c > 0$ is a constant, and m is the dimension of L. Throughout this paper, we assume m is no less than 3 (such a dimensional constraint[1] also appears in the work of Joyce [15], and of Lee [19]). The solution of the ODE is then $\varepsilon(t) = (\varepsilon_0^{2-m} + \frac{c}{2}(m-2)\,t)^{-\frac{1}{m-2}}$, which suggests that the family $\{N^{\varepsilon(t)} : \varepsilon(t) = (\varepsilon_0^{2-m} + \frac{c}{2}(m-2)\,t)^{-\frac{1}{m-2}}$ for $0 \leqslant t < \infty\}$ can be perturbed into a genuine Lagrangian mean curvature flow, where the neck pinches as $t \to \infty$ at the rate $O(t^{-\frac{1}{m-2}})$.

The main purpose of this paper is to make precise this parabolic gluing construction in a particular case. Our main result is as follows; the precise statement is given as Theorem 9.1.

Main Theorem. *Let $m \geqslant 3$, and endow $T^{2m} = \mathbb{C}^m/\Gamma$ with the Calabi–Yau structure induced from the standard one on $\mathbb{C}^m$. Suppose that L_1 and L_2 are two special Lagrangian sub-tori in T^{2m} intersecting transversely at a point $x_\star$, and suppose the tangent planes at $x_\star$ satisfy the angle criterion.*

Then for sufficiently small ε there exists a desingularisation N^ε of $L := L_1 \cup L_2$ constructed by gluing in a Lawlor neck, such that the Lagrangian mean curvature flow starting from N^ε exists for all time. As $t \to \infty$, the flow converges to L and forms an infinite-time singularity modelled on a shrinking Lawlor neck of size $O(t^{-\frac{1}{m-2}})$.

This paper is structured as follows. The main part of section 2 is devoted to the construction of the Weinstein neighbourhood of a Lagrangian cone of suitable rate. Instead of a single Lagrangian submanifold, the parabolic gluing construction is based on a one-parameter family of Lagrangians. Hence, we require a systematic construction of their Weinstein neighbourhoods in order to study the geometry of their Hamiltonian perturbations. In section 3, the desingularisation N^ε is introduced. It is formulated as an ε-dependent embedding ι^ε from a static manifold $\underline{N}$ to the Calabi–Yau manifold M. With these preparations, the non-parametric form of the Lagrangian mean curvature flow equation is derived in section 4. The condition under which the Lagrangian mean curvature flow equation can be integrated to the level of potentials is also discussed in section 4. In section 5, we compute the linearised operator, and introduce its approximate kernel. The spatial properties of the approximate kernel are established by Joyce; the materials in sections 2 and 3 allow us to study their parabolic properties. In section

[1] One reason for this constraint is that the Green's function in dimension 2 is different from that in higher dimensions.

6, we prove three Liouville theorems, and use them to establish the weighted Schauder estimate for the solution to the inhomogeneous heat equation. We note that the discussions in sections 2 to 6 are valid not only for the specific case of our main theorem, but for general immersed Lagrangians in a general ambient Calabi–Yau manifold, where the intersections satisfy the angle condition.

From section 7, we focus on the case of two intersecting Lagrangian tori in a complex torus, $L_1 \cup L_2 \subset \mathbb{C}^m/\Gamma$. The main purpose of section 7 is to establish an existence theorem for solutions to the heat equation on the L^2-orthogonal complement of the approximate kernel. In section 8, we derive the projection formula to the approximate kernel, and the estimates of the zeroth order and quadratic terms of the Lagrangian mean curvature flow equation. Finally, these materials are put together in section 9, and the main theorem is proven by a Schauder fixed point argument.

When the ambient space is a non-flat Calabi–Yau manifold, the error terms are larger than those of the flat case. The main theorem in the general case therefore requires more effort to establish, and is left for future investigation.

1.4. **Conventions.** Here are some conventions that will be used throughout this paper.

(1) The complex dimension of our Calabi–Yau is assumed to be 3 or greater, $m \geqslant 3$.
(2) In a Calabi–Yau manifold $(M^{2m}, g, J, \omega, \Omega)$, a half-dimensional submanifold L^m is called a *special Lagrangian* submanifold if it is calibrated by $\operatorname{Im}\Omega$. Namely, $\operatorname{Im}\Omega|_L$ coincides with the volume form of L. According to [9, p.89], this is equivalent to the vanishing of $\omega|_L$ (the Lagrangian condition) and the vanishing of $\operatorname{Re}\Omega|_L$ (the special condition).
(3) Unless otherwise specified, $\mathbb{C}^m \cong \mathbb{R}^{2m}$ is equipped with the standard Calabi–Yau structure $(g_0, J_0, \omega_0, \Omega_0)$, where $\omega_0 = \sum_{j=1}^m \mathrm{d}x_j \wedge \mathrm{d}y_j$ and $\Omega_0 = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_m$.
(4) Given a diffeomorphism $\varphi : L \to \tilde{L}$, it induces a diffeomorphism $(\varphi^*)^{-1} : T^*L \to T^*\tilde{L}$. Such a map will be denoted by $\varphi_\dagger$. Given a smooth function $u : \tilde{L} \to \mathbb{R}$, $\mathrm{d}u$ embeds $\tilde{L}$ into $T^*\tilde{L}$. It is straightforward to verify the following relation
$$(\mathrm{d}u) \circ \varphi = \varphi_\dagger \circ \mathrm{d}(u \circ \varphi) \ . \tag{1.1}$$
(5) The constant C in the estimates may change from line to line.

Acknowledgments. The authors would like to thank Dominic Joyce and Yng-Ing Lee for helpful discussions and interests in this work. The authors are grateful to Simon Brendle for answering our questions regarding [3].

2. Neighbourhood Theorems and Local Models

2.1. **Equivariant Neighbourhoods of Lagrangian Cones.** In this section, we consider a Lagrangian cone $C \subset \mathbb{C}^m \cong \mathbb{R}^{2m}$, where $\mathbb{C}^m \cong \mathbb{R}^{2m}$ is equipped with the standard Liouville form $\lambda_0 = \frac{1}{2}\sum_{j=1}^m (y_j \mathrm{d}x_j - x_j \mathrm{d}y_j)$, so that $\mathrm{d}\lambda_0 = -\omega_0$. Note that the link $\Sigma = C \cap S^{2m-1}$ is a Legendrian submanifold in the contact manifold $(S^{2m-1}, \lambda_0|_{S^{2m-1}})$. On the other hand, one can equip $T^*\Sigma \times \mathbb{R}$ with the contact form $\lambda_\Sigma - \mathrm{d}s$, where λ_Σ is the tautological 1-form on $T^*\Sigma$ and s is the coordinate on $\mathbb{R}$. It is clear that Σ, as the zero section of $T^*\Sigma$, is a Legendrian submanifold in $(T^*\Sigma \times \mathbb{R}, \lambda_\Sigma - \mathrm{d}s)$. By Moser's trick, one can show that the latter is the standard local model of Legendrian neighbourhoods.

Lemma 2.1 ([23])**.** *Denote by $\underline{0}$ the zero section in $T^*\Sigma \subset T^*\Sigma \times \mathbb{R}$. There exist an open neighbourhood W_Σ of $\underline{0}$ in $T^*\Sigma \times \mathbb{R}$ and an embedding $\Psi_\Sigma : W_\Sigma \to S^{2m-1}$ such that $\Psi_\Sigma|_{\underline{0}} = \iota_\Sigma$ and $\Psi_\Sigma^*(\lambda_0|_{S^{2m-1}}) = \lambda_\Sigma - \mathrm{d}s$, where $\iota_\Sigma : \Sigma \to S^{2m-1}$ is the inclusion map.*

Using Lemma 2.1, we construct an equivariant Lagrangian neighbourhood for C. Recall the notion of a Lagrangian neighbourhood.

Definition 2.2. Let $\iota : L^m \to (M^{2m}, \omega)$ be a Lagrangian embedding. A *Lagrangian neighbourhood* consists of an open neighbourhood $U \subset T^*L$ of the zero section $\underline{0}$, and an embedding $\Psi_L : U \to M$ such that $\Psi_L|_{\underline{0}} = \iota$ and $\Psi_L^*(\omega) = \omega_L$, where ω_L is the canonical symplectic form on T^*L.

We consider the natural $\mathbb{R}_+$-action on $\mathbb{C}^m\backslash\{0\}$ given by dilations, and the $\mathbb{R}_+$-action on $T^*C = T^*(\Sigma \times (0,\infty))$ defined as follows. Formally writing a point in $T^*(\Sigma \times (0,\infty))$ as $(\sigma, r, \varsigma, s)$ where $\sigma \in \Sigma$, $r \in (0,\infty)$, $\varsigma \in T^*_\sigma\Sigma$ and $s \in \mathbb{R} \cong T^*_r(0,\infty)$, and letting $\epsilon \in \mathbb{R}_+$, we define

$$\epsilon \cdot (\sigma, r, \varsigma, s) = (\sigma, \epsilon r, \epsilon^2\varsigma, \epsilon s) \ . \tag{2.1}$$

The following proposition gives not only the neighbourhood, but also the expression of the Liouville form on the neighbourhood. It is an extension of [11, Theorem 4.3].

Proposition 2.3. *Let Σ be a Legendrian link in $(S^{2m-1}, \lambda_0|_{S^{2m-1}})$, and let $C = \Sigma \times (0,\infty)$ be the corresponding Lagrangian cone in $(\mathbb{C}^m\backslash\{0\}, \omega_0)$. There exists a Lagrangian neighbourhood $\Phi_C : U_C \subset T^*C = T^*(\Sigma \times (0,\infty)) \to \mathbb{C}^m\backslash\{0\}$ such that*

$$\Phi_C^*\lambda_0 = \lambda_C - \mathrm{d}\left(\frac{rs}{2}\right) \ ,$$

where λ_C is the tautological 1-form on T^*C, $r \in (0,\infty)$, and $s \in \mathbb{R} \cong T^*_r(0,\infty)$. Moreover, U_C is invariant under the $\mathbb{R}_+$-action defined in (2.1), and Φ_C is equivariant with respect to it.

Proof. The *symplectization* of $(S^{2m-1}, \lambda_0|_{S^{2m-1}})$ is $(\mathbb{R}^{2m}\backslash\{0\}, \omega_0 = -\mathrm{d}\lambda_0)$. More precisely, identify $\mathbf{x} \in \mathbb{R}^{2m}\backslash\{0\}$ with $(\frac{\mathbf{x}}{|\mathbf{x}|}, |\mathbf{x}|) \in S^{2m-1} \times (0,\infty)$. Under this identification, λ_0 is the pull-back of $r^2\ \lambda_0|_{S^{2m-1}}$, where r is the coordinate on $(0,\infty)$. With this understood, it is equivalent to construct the embedding Φ_C to $S^{2m-1} \times (0,\infty)$ so that $\Phi_C^*(r^2\ \lambda_0|_{S^{2m-1}}) = \lambda_C - \mathrm{d}\left(\frac{rs}{2}\right)$. Note that on $S^{2m-1} \times (0,\infty)$, the dilation acts only on the $(0,\infty)$-summand.

Consider the diffeomorphism

$$\begin{array}{rccc} \psi : & T^*(\Sigma \times (0,\infty)) & \to & T^*\Sigma \times \mathbb{R} \times (0,\infty) \\ & (\sigma, r, \varsigma, s) & \mapsto & \big((\sigma, r^{-2}\varsigma), (2r)^{-1}s, r\big) \ . \end{array}$$

For the open set W_Σ given by Lemma 2.1, it is not hard to see that $U_C = \psi^{-1}(W_\Sigma \times (0,\infty))$ is an open neighbourhood of the zero section in $T^*(\Sigma \times (0,\infty))$. Let

$$\Phi_C = (\Psi_\Sigma \times \mathrm{id}_{(0,\infty)}) \circ \psi : U_C \subset T^*(\Sigma \times (0,\infty)) \to S^{2m-1} \times (0,\infty)$$

where Φ_C is given by Lemma 2.1. The pull-back of the Liouville form under Φ_C is

$$\begin{aligned}\Phi_C^*(r^2\,\lambda_0|_{S^{2m-1}}) &= \psi^*(r^2(\lambda_\Sigma - \mathrm{d}s)) \\ &= \lambda_\Sigma - r^2\mathrm{d}\left(\frac{s}{2r}\right) = (\lambda_\Sigma + s\mathrm{d}r) - \mathrm{d}\left(\frac{rs}{2}\right) .\end{aligned}$$

Note that $\lambda_\Sigma + s\mathrm{d}r$ is exactly the tautological 1-form on $T^*(\Sigma \times (0,\infty))$.

The invariance of U_C under (2.1) follows from the construction. It remains to check the $\mathbb{R}_+$-equivariance of Φ_C. For any $\epsilon > 0$,

$$\begin{aligned}\Phi_C(\epsilon\cdot(\sigma,r,\varsigma,s)) &= \Phi_C(\sigma,\epsilon r,\epsilon^2\varsigma,\epsilon s) \\ &= (\Psi_\Sigma \times \mathrm{id}_{(0,\infty)})\left((\sigma, r^{-2}\varsigma),(2r)^{-1}s,\epsilon r\right) = \epsilon\cdot\Phi_C(\sigma,r,\varsigma,s) .\end{aligned}$$

This finishes the proof of the proposition. □

2.2. **Asymptotically Conical Lagrangians.** Proposition 2.3 can be used to construct good neighbourhoods for asymptotically conical Lagrangians. We first recall their definition.

Definition 2.4. A Lagrangian $L \subset \mathbb{C}^m$ is called *asymptotically conical* with cone C and rate γ if the following holds. Let $\Sigma = C \cap S^{2m-1}$ be the link of C. There exist a compact subset $K \subset L$, a constant $R_1 > 0$, and a diffeomorphism $\varphi : \Sigma \times (R_1,\infty) \to L\backslash K$ such that for any non-negative integer k,

$$|\nabla^k(\varphi - \iota_C)|(\sigma,r) = O(r^{\gamma-1-k}) \quad \text{as } r \to \infty , \tag{2.2}$$

where ∇ and $|\cdot|$ are computed using the cone metric $g_C = \mathrm{d}r^2 + r^2 g_\Sigma$.

Remark 2.5.

- Later on, we will consider the "potential" of L over $\Sigma \times (R_1,\infty)$. The -1 in the power of r in (2.2) will imply that the potential is of order γ.
- In [13, Definition 4.1], (2.2) is only required for $k = 0,1$. Under suitable assumptions, it can be upgraded to all $k \geqslant 0$; see for instance Theorem 3.8 and Theorem 4.6 in [13]. Since we will only work with specific asymptotically conical Lagrangians, a more restrictive assumption is chosen here for convenience.

Suppose that the rate satisfies $\gamma < 0$. According to Proposition 2.3, $L\backslash K$ can be written as the graph of a smooth *closed* 1-form on $\Sigma \times (R_1,\infty)$, after taking R_1 larger if necessary. That is to say, $L\backslash K$ belongs to $\Phi_C(U_C)$. Thus, there exists a closed 1-form, $\mathfrak{e}$, on $\Sigma \times (R_1,\infty)$ such that

$$\varphi(\sigma,r) = \Phi_C(\sigma,r,\mathfrak{e}_1(\sigma,r),\mathfrak{e}_2(\sigma,r)) \tag{2.3}$$

for any $(\sigma,r) \in \Sigma \times (R_1,\infty)$, where $\mathfrak{e}_2 = \mathfrak{e}(\frac{\partial}{\partial r})$ and $\mathfrak{e}_1 = \mathfrak{e} - \mathfrak{e}_2\mathrm{d}r$. The condition (2.2) implies that

$$|\nabla^k\mathfrak{e}| = O(r^{\gamma-1-k}) \quad \text{as } r \to \infty \tag{2.4}$$

where ∇ and $|\cdot|$ are computed using the cone metric $g_C = \mathrm{d}r^2 + r^2 g_\Sigma$.

Recall that a Lagrangian submanifold $L \subset \mathbb{C}^m$ is said to be *exact* if the restriction of the Liouville form, $\iota_L^*\lambda_0$, is exact. Here is the neighbourhood theorem we will need.

Theorem 2.6. *Let $L \subset (\mathbb{C}^m, \omega_0)$ be an exact, connected, asymptotically conical Lagrangian submanifold with cone $C = \Sigma \times (0, \infty)$ and rate $\gamma < 0$. Then:*

- *There exist a Lagrangian neighbourhood $\Phi_L : U_L \subset T^*L \to \mathbb{C}^m$ and a function $\alpha_L : U_L \to \mathbb{R}$ such that*
$$\Phi_L^* \lambda_0 = \lambda_L - \mathrm{d}\alpha_L \ .$$
Moreover, Φ_L can be chosen so that
$$(\Phi_L \circ \varphi_\dagger)(\sigma, r, \varsigma, s) = \Phi_C(\sigma, r, \varsigma + \mathfrak{e}_1(\sigma, r), s + \mathfrak{e}_2(\sigma, r)) \tag{2.5}$$
for any $(\sigma, r, \varsigma, s) \in \varphi_\dagger^{-1}(U_L) \subset T^*(\Sigma \times (R_1, \infty))$, where Φ_C is the map given by Proposition 2.3, φ is the map in Definition 2.4, and $\mathfrak{e} = \mathfrak{e}_1 + \mathfrak{e}_2 \mathrm{d}r \in \Omega^1(\Sigma \times (R_1, \infty))$ is explained in (2.3)*.*
- *The 1-form $\mathfrak{e}$ on $\Sigma \times (R_1, \infty)$ is exact, $\mathfrak{e} = \mathrm{d}\mathfrak{E}$, and thus $\varphi = \Phi_C \circ \mathrm{d}\mathfrak{E}$. Moreover, the potential function $\mathfrak{E}$ can taken to obey that $|\nabla^\ell \mathfrak{E}| = O(r^{\gamma - \ell})$ as $r \to \infty$, for every $\ell \geqslant 0$.*
- *The function α_L is unique up to adding a constant. Moreover, there are constants c_a, associated with the connected components L_a of $L \backslash K$, and on each $\varphi_\dagger^{-1}(T^* L_a)$:*
$$\begin{aligned} (\alpha_L \circ \varphi_\dagger)(\sigma, r, \varsigma, s) - \Big(\frac{rs}{2} + c_a\Big) &= \frac{r}{2}(\partial_r \mathfrak{E})(\sigma, r) - \mathfrak{E}(\sigma, r), \\ \Big|(\alpha_L \circ \varphi_\dagger)(\sigma, r, \varsigma, s) - \Big(\frac{rs}{2} + c_a\Big)\Big| &= O(r^\gamma) \qquad \text{as } r \to \infty. \end{aligned} \tag{2.6}$$
The restriction of $\alpha_L : U_L \to \mathbb{R}$ to the zero section $\beta_L := \alpha_L|_{\underline{0}}$ is a primitive of the Liouville form up to a minus sign, $\iota_L^* \lambda_0 = -\mathrm{d}\beta_L$.

Proof. The argument is very similar to [22, Proposition 5.3]. The proof is separated into 4 steps. The bundle projection map is denoted by π.

Step 1: Lagrangian neighbourhood near infinity. We first construct the neighbourhood of $L \backslash K$. Define an open subset $U_C - \mathfrak{e}$ of $T^*(\Sigma \times (R_1, \infty))$ by
$$U_C - \mathfrak{e} = \{(\sigma, r, \varsigma, s) : \ (\sigma, r, \varsigma + \mathfrak{e}_1(\sigma, r), s + \mathfrak{e}_2(\sigma, r)) \in U_C\} \ .$$
Its image under $\varphi_\dagger$ is an open neighbourhood of the zero section in $T^*(L \backslash K)$. Denote its image, $\varphi_\dagger(U_C - \mathfrak{e}) \subset T^*(L \backslash K)$, by $U_{L \backslash K}$. One naturally defines an embedding $\Phi_{L \backslash K} : U_{L \backslash K} \subset T^*(L \backslash K) \to \mathbb{C}^m$ by
$$(\Phi_{L \backslash K} \circ \varphi_\dagger)(\sigma, r, \varsigma, s) = \Phi_C(\sigma, r, \varsigma + \mathfrak{e}_1(\sigma, r), s + \mathfrak{e}_2(\sigma, r)) \ . \tag{2.7}$$

We prove the exactness of $\mathfrak{e}$. For the self-diffeomorphism $f_\mathfrak{e}$ on $T^*(\Sigma \times (R_1, \infty))$ defined by
$$f_\mathfrak{e}(\sigma, r, \varsigma, s) = (\sigma, r, \varsigma + \mathfrak{e}_1(\sigma, r), s + \mathfrak{e}_2(\sigma, r)) \ , \tag{2.8}$$
one has $f_\mathfrak{e}^*(\lambda_C) = \lambda_C + \mathfrak{e}$, where λ_C is the tautological 1-form on $T^*C \supset T^*(\Sigma \times (R_1, \infty))$, and $\mathfrak{e}$ is regarded as a 1-form on $T^*(\Sigma \times (R_1, \infty))$ under the pull-back of the projection.

Since $\mathrm{d}\mathfrak{e} = \mathrm{d}(\mathfrak{e}_1 + \mathfrak{e}_2 \mathrm{d}r) = 0$,
$$\mathrm{d}_\Sigma \mathfrak{e}_2 = \frac{\partial \mathfrak{e}_1}{\partial r} \quad \text{and} \quad \mathrm{d}_\Sigma \mathfrak{e}_1 = 0 \ .$$

Since $|\mathfrak{e}| = O(r^{\gamma-1})$ as $r \to \infty$, the function

$$\mathfrak{E}(\sigma, r) = -\int_r^\infty \mathfrak{e}_2(\sigma, y)\mathrm{d}y \tag{2.9}$$

is well-defined, and $|\mathfrak{E}(\sigma, r)| = O(r^\gamma)$ as $r \to \infty$. With $\mathrm{d}\mathfrak{e} = 0$, one finds that $\mathrm{d}\mathfrak{E} = \mathfrak{e}$. The rate on $|\nabla \mathfrak{E}|$ follows directly from its construction.

According to Proposition 2.3, (2.7), $\mathrm{d}\mathfrak{E} = \mathfrak{e}$, and the fact that $\varphi_\dagger$ preserves the tautological 1-form,

$$\begin{aligned}\Phi_{L\backslash K}^*\lambda_0 = \left((\varphi_\dagger^{-1})^* \circ f_\mathfrak{e}^* \circ \Phi_C^*\right)(\lambda_0) &= (\varphi_\dagger^{-1})^* f_\mathfrak{e}^*\left(\lambda_C - \mathrm{d}\left(\frac{rs}{2}\right)\right) \\ &= (\varphi_\dagger^{-1})^*\left[\lambda_C - \mathrm{d}\left(f_\mathfrak{e}^*(\frac{rs}{2}) - \mathfrak{E}\right)\right] \\ &= \lambda_L - \mathrm{d}\left[\left(\frac{rs}{2} + \frac{r\mathfrak{e}_2}{2} - \mathfrak{E}\right) \circ \varphi_\dagger^{-1}\right] \end{aligned} \tag{2.10}$$

where λ_L is the tautological 1-form on $T^*L \supset T^*(L\backslash K)$.

Step 2: constants. Since L is exact, there exists $\beta_L : L \to \mathbb{R}$ such that $\iota_L^*\lambda_0 = -\mathrm{d}\beta_L$. Due to the connectedness of L, β_L is unique up to adding a constant. Fix a choice of β_L.

On the other hand, the restriction of (2.10) on the zero section, $L\backslash K$, implies that

$$(\iota_L^*\lambda_0)|_{L\backslash K} = -\mathrm{d}\left[\left(\frac{rs}{2} + \frac{r\mathfrak{e}_2}{2} - \mathfrak{E}\right) \circ \varphi_\dagger^{-1}\right]\Big|_{L\backslash K}.$$

Therefore, one each connected component, L_a, of $L\backslash K$, there must exist a constant c_a such that

$$\beta_L|_{L_a} = \left[\left(\frac{rs}{2} + \frac{r\mathfrak{e}_2}{2} - \mathfrak{E}\right) \circ \varphi_\dagger^{-1}\right]\Big|_{L_a} + c_a.$$

With these constants c_a's, define $\alpha_{L\backslash K} : U_{L\backslash K} \to \mathbb{R}$ by

$$\alpha_{L\backslash K}\big|_{\pi^{-1}(L_a)} = \left[\left(\frac{rs}{2} + \frac{r\mathfrak{e}_2}{2} - \mathfrak{E}\right) \circ \varphi_\dagger^{-1}\right]\Big|_{\pi^{-1}(L_a)} + c_a\ . \tag{2.11}$$

Step 3: Moser's trick on T^*K. Let $\widetilde{U}_L \subset T^*L$ and $\widetilde{\Phi}_L : \widetilde{U}_L \to \mathbb{C}^m$ be smooth extensions of the open neighbourhood $U_{L\backslash K}$ and the embedding $\Phi_{L\backslash K}$ over the compact subset K. Namely, $\widetilde{U}_L$ is an open neighbourhood of the zero section in T^*L with $\widetilde{U}_L \cap \pi^{-1}(L\backslash K) = U_{L\backslash K}$, and $\widetilde{\Phi}_L|_{U_{L\backslash K}} = \Phi_{L\backslash K}$. Moreover, the embedding can be chosen so that $\widetilde{\Phi}_L|_{\underline{0}} = \iota_L$. The neighbourhood of L asserted in this theorem will be constructed by perturbing $\widetilde{U}_L$ and $\widetilde{\Phi}_L$.

Let $h \in C^\infty(L)$ be a cut-off function such that $h \equiv 1$ on $L\backslash K$, $h \equiv 0$ on $K' \subset\subset K$. Define an extension $\widetilde{\alpha}_L$ of $\alpha_{L\backslash K}$ (2.11) to $\widetilde{U}_L$ by

$$\widetilde{\alpha}_L := (h \circ \pi)\,\alpha_{L\backslash K} + (1 - h \circ \pi)(\beta_L \circ \pi)\ .$$

From step 2, the restriction of $\widetilde{\alpha}_L$ on the zero section is β_L, $\widetilde{\alpha}_L|_{\underline{0}} = \beta_L$. The goal is to construct a one-parameter family of self-diffeomorphisms, $\{\upsilon_t\}_{t\in[0,1]}$, of $\widetilde{U}_L$ with the following properties.

- $\upsilon_0 = \mathrm{id}_{\widetilde{U}_L}$ and $\upsilon_t|_{\underline{0}} \in \mathrm{Diff}_c(L)$ for all $t \in [0, 1]$.

- Let $\lambda^t = (1-t)(\lambda_L - \mathrm{d}\widetilde{\alpha}_L) + t\widetilde{\Phi}_L^*\lambda_0$. There exists a family of functions, $\{\alpha_t\}_{t\in[0,1]}$, on $\widetilde{U}_L$ with $\alpha_0 = 0$ and
$$\upsilon_t^*\lambda^t = \lambda_L - \mathrm{d}(\widetilde{\alpha}_L + \alpha_t) \tag{2.12}$$
for every $t \in [0,1]$.

Suppose $\frac{\mathrm{d}}{\mathrm{d}t}\upsilon_t = Y_t \circ \upsilon_t$. Differentiating (2.12) gives

$$\begin{aligned}
-\mathrm{d}\left(\frac{\mathrm{d}\alpha_t}{\mathrm{d}t}\right) &= \upsilon_t^*\left\{\frac{\mathrm{d}\lambda^t}{\mathrm{d}t} + \iota_{Y_t}(\mathrm{d}\lambda^t) + \mathrm{d}(\iota_{Y_t}\lambda^t)\right\} \\
&= \upsilon_t^*\left\{-(\lambda_L - \mathrm{d}\widetilde{\alpha}_L) + \widetilde{\Phi}_L^*\lambda_0 + \iota_{Y_t}\left[(1-t)(-\omega_L) - t\widetilde{\Phi}_L^*\omega_0\right] + \mathrm{d}(\iota_{Y_t}\lambda^t)\right\} \\
&= \upsilon_t^*\left\{\iota_{Y_t}\left[(1-t)(-\omega_L) - t\widetilde{\Phi}_L^*\omega_0\right] - \left[\lambda_L - \mathrm{d}\widetilde{\alpha}_L - \widetilde{\Phi}_L^*\lambda_0\right]\right\} + \mathrm{d}\left(\upsilon_t^*\iota_{Y_t}\lambda^t\right) ,
\end{aligned}$$

where $\omega_L = -\mathrm{d}\lambda_L$ is the canonical symplectic form on T^*L. By shrinking $\widetilde{U}_L$ in the fiber direction if necessary, the 2-form $(1-t)(-\omega_L) - t\widetilde{\Phi}_L^*\omega_0$ is non-degenerate for every $t \in [0,1]$. Define the one-parameter family of vector field $\{Y_t\}_{t\in[0,1]}$ by

$$\iota_{Y_t}\left[(1-t)(-\omega_L) - t\widetilde{\Phi}_L^*\omega_0\right] = \lambda_L - \mathrm{d}\widetilde{\alpha}_L - \widetilde{\Phi}_L^*\lambda_0 \ .$$

Due to (2.10), the right hand side vanishes on $\pi^{-1}(L\backslash K)$. Thus, Y_t only supports on $\pi^{-1}(K)$.

Note that the zero section is Lagrangian with respect to $(1-t)(-\omega_L) - t\widetilde{\Phi}_L^*\omega_0$, and for every V tangent to the zero section,

$$\left[(1-t)(-\omega_L) - t\widetilde{\Phi}_L^*\omega_0\right](Y_t|_{\underline{0}}, V) = -\mathrm{d}\left[\widetilde{\alpha}|_{\underline{0}} - \beta_L\right](V) = 0.$$

It follows that Y_t is tangent to the zero section. Therefore, for the diffeomorphism υ_t generated by Y_t, one has $\upsilon_t|_{\underline{0}} \in \mathrm{Diff}_c(L)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\upsilon_t^*\lambda^t = -\mathrm{d}\left(\frac{\mathrm{d}\alpha_t}{\mathrm{d}t}\right) = \mathrm{d}\left(\upsilon_t^*\iota_{Y_t}\lambda^t\right).$$

Integrating it against with t gives

$$\upsilon_t^*\lambda^t = \lambda_L - \mathrm{d}\widetilde{\alpha}_L + \mathrm{d}\left(\int_0^t \upsilon_\tau^*\iota_{Y_\tau}\lambda^\tau\,\mathrm{d}\tau\right).$$

Hence, υ_t is the desired diffeomorphism, and $\alpha_t = -\int_0^t \upsilon_\tau^*\iota_{Y_\tau}\lambda^\tau\,\mathrm{d}\tau$.

Finally, set Φ_L to be $\widetilde{\Phi}_L \circ \upsilon_1$. It follows that

$$\Phi_L^*\lambda_0 = \upsilon_1^*\widetilde{\Phi}_L^*\lambda_0 = \upsilon_1^*\lambda^{t=1} = \lambda_L - \mathrm{d}\alpha_L$$

where $\alpha_L = \widetilde{\alpha}_L + \alpha_{t=1}$.

Step 4: asymptotic behavior. It remains to verify the decay rate of α_L. Note that α_t only supports on $\pi^{-1}(K)$. By construction, we have

$$\alpha_L|_{\pi^{-1}(L_a)} = \left(\frac{rs}{2} + \frac{r\mathfrak{e}_2}{2} - \mathfrak{E}\right)\circ\varphi_\dagger^{-1} + c_a \ . \tag{2.13}$$

Thus,

$$\left|(\alpha_L\circ\varphi_\dagger)(\sigma, r, \varsigma, s) - \left(\frac{rs}{2} + c_a\right)\right| = O\left(r|\mathfrak{e}_2| + |\mathfrak{E}|\right) = O(r^\gamma) \quad \text{as } r\to\infty$$

on each $\varphi_\dagger^{-1}(T^*L_a) \subset T^*(\Sigma \times (R_1, \infty))$. This completes the proof of this theorem. □

Remark 2.7.

- In Theorem 2.6, the function β_L is harmonic if L is a special Lagrangian (calibrated by $\operatorname{Re}\Omega_0$). See [25, Lemma 6.2].
- By [22, Lemma 5.4], one finds that α_L can be expressed as follows

$$\alpha_L = \beta_L \circ \pi + \frac{1}{2}\int_0^1 \langle \mathbf{x}, \bar{\nabla}\bar{u}\rangle_{\Phi_L \circ s\mathrm{d}u}\, \mathrm{d}s\ . \tag{2.14}$$

  Here, $\mathbf{x}$ is the position vector in $\mathbb{C}^m$, $\bar{\nabla}$ is taken with respect to the standard structure of $\mathbb{C}^m$, and $\bar{u}$ is a function on $\Phi_L(U_L)$ defined to be $u \circ \pi \circ \Phi_L^{-1}$.

We will also consider the dilation of an asymptotically conical Lagrangian submanifold L by a scale $\varepsilon > 0$, which will be denoted by $\iota_{\varepsilon L} = \varepsilon \cdot \iota_L : L \to \mathbb{C}^m$. It is clear that εL is asymptotically conical with the same cone and the same rate. The following corollary describes the effect of dilation on Theorem 2.6.

Corollary 2.8. *Let $\Phi_L : U_L \subset T^*L \to \mathbb{C}^m$ be the Lagrangian neighbourhood constructed by Theorem 2.6. For any $\varepsilon > 0$, let $f_\varepsilon : T^*L \to T^*L$ be the diffeomorphism defined by $f_\varepsilon(q,p) = (q, \varepsilon^{-2}p)$. Then, the open neighbourhood $U_{\varepsilon L} := f_\varepsilon^{-1}(U_L) \subset T^*L$ of the zero section and the embedding*

$$\Phi_{\varepsilon L} = \varepsilon \cdot \Phi_L \circ f_\varepsilon : U_{\varepsilon L} \to \mathbb{C}^m$$

constitute a Lagrangian neighbourhood of $\iota_{\varepsilon L}$, and

$$\Phi_{\varepsilon L}^* \lambda_0 = \lambda_L - \mathrm{d}\left(\varepsilon^2 \cdot (\alpha_L \circ f_\varepsilon)\right) \quad \text{on } U_{\varepsilon L},$$

where α_L is the function given by Theorem 2.6.

Proof. By $f_\varepsilon^*\lambda_L = \varepsilon^{-2}\lambda_L$, $(\varepsilon\cdot)^*\lambda_0 = \varepsilon^2\lambda_0$ and Theorem 2.6,

$$\Phi_{\varepsilon L}^*\lambda_0 = f_\varepsilon^*\Phi_L^*(\varepsilon^2\lambda_0) = \varepsilon^2 f_\varepsilon^*\left(\lambda_L - \mathrm{d}\alpha_L\right) = \lambda_L - \mathrm{d}\left(\varepsilon^2 f_\varepsilon^*\alpha_L\right)\ .$$

This finishes the proof of the corollary. □

The notations f_ε and $f_{\mathfrak{e}}$ defined by (2.8) both denote self-diffeomorphisms of the cotangent bundle, whose restriction on the fibers are affine transformations. When the subscript is a real number, it is a fiberwise dilation; when the subscript is a 1-form, it is a fiberwise translation.

Remark 2.9. We remark that there are two distinct metrics on a Lagrangian neighbourhood, which we describe here. Choose local coordinates $\{q^i\}$ on L. This induces a system of local coordinates on T^*L; a point with coordinate (q^i, p_i) means the covector $p_i\mathrm{d}q^i$ at $q = (q^1, \dots, q^n)$. Given an embedding of a neighbourhood of the zero section in T^*L to $\mathbb{C}^m$, we may consider the *pullback Riemannian metric* on the total space T^*L. We can also instead consider the induced metric on L, which induces a *bundle metric* on T^*L.

- *Bundle metric.* Denote by $\underline{g}_{ij}(q)\,\mathrm{d}q^i \otimes \mathrm{d}q^j$ the induced metric on L by $\Phi_L|_{\underline{0}}$. For $p \in T_q^*L$, $||p||^2 = \underline{g}^{ij}(q)\,p_i\,p_j$. The induced metric on L by $\Phi_{\varepsilon L}|_{\underline{0}}$ is $\varepsilon^2 \underline{g}_{ij}(q)\,\mathrm{d}q^i \otimes \mathrm{d}q^j$. It follows that for $p \in T_q^*L$, $||p||_\varepsilon^2 = \varepsilon^{-2}\underline{g}^{ij}(q)\,p_i\,p_j$. Hence, $||p||_\varepsilon = \varepsilon\,||\varepsilon^{-2}p||$. In other words, imagining that U_L is of "radius 1" with respect to the bundle metric induced by Φ_L around the zero section, then $U_{\varepsilon L}$ is of "radius ε" with respect to the bundle metric induced by $\Phi_{\varepsilon L}$ around the zero section.
- *Riemannian metric.* For the Riemannian metric on T^*L, denote by $(m + j)$ for the component in p_j. Denote by $g_{\varepsilon L}$ the metric $\Phi_{\varepsilon L}^*(g_0)$. By using the chain rule,
$$\begin{aligned}(g_{\varepsilon L})_{ij}(q,p) &= \varepsilon^2 (g_L)_{ij}\big(f_\varepsilon(q,p)\big) \ ,\\ (g_{\varepsilon L})_{i(m+j)}(q,p) &= (g_L)_{i(m+j)}\big(f_\varepsilon(q,p)\big) \ ,\\ (g_{\varepsilon L})_{(m+i)(m+j)}(q,p) &= \varepsilon^{-2} (g_L)_{(m+i)(m+j)}\big(f_\varepsilon(q,p)\big)\end{aligned}$$
for any $i, j \in \{1, \ldots, m\}$.

As expected, the dilation on the embedding leads to the same effect on the bundle metric and the Riemannian metric.

Remark 2.10. Denoting $\mathfrak{r}(r) := \varepsilon r$, by (2.3) and (2.1) $\varepsilon L \backslash \varepsilon K$ is given by
$$\begin{aligned}\varepsilon \cdot \Phi_C(\sigma, r, \mathfrak{e}_1(\sigma, r), \mathfrak{e}_2(\sigma, r)) &= \Phi_C(\sigma, \varepsilon r, \varepsilon^2 \mathfrak{e}_1\,(\sigma, r), \varepsilon\,\mathfrak{e}_2(\sigma, r))\\ &= \Phi_C(\sigma, \mathfrak{r}, \varepsilon^2 \mathfrak{e}_1\,(\sigma, \varepsilon^{-1}\mathfrak{r}), \varepsilon\,\mathfrak{e}_2(\sigma, \varepsilon^{-1}\mathfrak{r}))\end{aligned}$$
This means that $\varepsilon L \backslash \varepsilon K$ is the graph of a closed 1-form on $\Sigma \times (\varepsilon R_1, \infty)$. Its potential function (for $\mathfrak{r} > \varepsilon R_1$) is given by
$$-\int_{r_\varepsilon}^{\infty} \varepsilon\,\kappa_2(\sigma, \varepsilon^{-1} y)\,\mathrm{d}y = -\varepsilon^2 \int_{\varepsilon^{-1}\mathfrak{r}} \kappa_2(\sigma, y')\mathrm{d}y' = \varepsilon^2\,\mathfrak{E}(\sigma, \varepsilon^{-1}\mathfrak{r}) \ ,$$
so that
$$\varepsilon \cdot \varphi \ = \ \Phi_C \circ d(\varepsilon^2 \mathfrak{E}(\sigma, \varepsilon^{-1} r)) \circ \mathfrak{r} \tag{2.15}$$
The above identity may be extended to a similar identity on the cotangent bundle:
$$\varepsilon \cdot \Phi_L \circ \varphi_\dagger \ = \ \Phi_C \circ f_{d(\varepsilon^2 \mathfrak{E}(\sigma, \varepsilon^{-1} r))} \circ f_\varepsilon^{-1} \circ \mathfrak{r}_\dagger, \tag{2.16}$$
where f_ε, $f_{dA} : T^*L \to T^*L$ are the diffeomorphisms defined by $f_\varepsilon(q,p) = (q, \varepsilon^{-2}p)$ and $f_{dA}(q,p) := (q, p + dA)$ respectively.

When an asymptotically conical Lagrangian L is also a special Lagrangian, Joyce in [13, section 4.1] defines two cohomological invariants. We will require one of them.

Definition 2.11. Let L be an asymptotically conical, special Lagrangian submanifold with cone $C = \Sigma \times (0, \infty)$. It follows from $\operatorname{Im}\Omega_0|_L = 0$ that $(\operatorname{Im}\Omega_0, 0)$ defines an element in the relative de Rham cohomology $\mathrm{H}^m(\mathbb{C}^m, L; \mathbb{R})$. Since Σ is in effect the boundary of L, there is a natural map $\mathrm{H}^{m-1}(L;\mathbb{R}) \to \mathrm{H}^{m-1}(\Sigma;\mathbb{R})$. Together with the long exact sequence
$$0 = \mathrm{H}^{m-1}(\mathbb{C}^m;\mathbb{R}) \to \mathrm{H}^{m-1}(L;\mathbb{R}) \xrightarrow{\cong} \mathrm{H}^m(\mathbb{C}^m, L;\mathbb{R}) \to \mathrm{H}^m(\mathbb{C}^m;\mathbb{R}) = 0 \ ,$$
the invariant $Z(L) \in \mathrm{H}^{m-1}(\Sigma;\mathbb{R})$ is defined to be the image of $[(\operatorname{Im}\Omega_0, 0)] \in \mathrm{H}^m(\mathbb{C}^m, L;\mathbb{R}) \cong \mathrm{H}^{m-1}(L;\mathbb{R})$ under the map $\mathrm{H}^{m-1}(L;\mathbb{R}) \to \mathrm{H}^{m-1}(\Sigma;\mathbb{R})$.

2.3. **The Lawlor Neck.** It is known that $\mathrm{SU}(m)$ acts transitively on the space of special Lagrangian m-planes in $\mathbb{C}^m$. In fact, the Grassmannian of oriented special Lagrangians is $\mathrm{SU}(m)/\mathrm{SO}(m)$. It follows that up to an $\mathrm{SU}(m)$ transformation, one may assume that a special Lagrangian m-plane is $\mathbb{R}^m \subset \mathbb{C}^m$. It turns out that given a pair of special Lagrangian m-planes, one may still put them into a *standard* form by $\mathrm{SU}(m)$.

Lemma 2.12 ([15, Proposition 9.1])**.** *Let* (Π^-, Π^+) *be a pair of transverse special Lagrangian* m*-planes in* $\mathbb{C}^m$*, namely,* $\Pi^- \cap \Pi^+ = \{0\}$*. There exist* $\mathbf{U} \in \mathrm{SU}(m)$ *and* $0 < \phi_1 \leqslant \ldots \leqslant \phi_m < \pi$ *such that* $\mathbf{U}(\Pi^-) = \Pi^0$ *and* $\mathbf{U}(\Pi^+) = \Pi^{\boldsymbol{\phi}}$*, where*

$$\Pi^0 = \{(x_1, \ldots, x_m) : \ x_j \in \mathbb{R}^m\} \quad \textit{and} \quad \Pi^{\boldsymbol{\phi}} = \{(e^{i\phi_1}x_1, \ldots, e^{i\phi_m}x_m) : \ x_j \in \mathbb{R}^m\} \ . \tag{2.17}$$

Moreover, $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_m)$ *is unique, and* $\sum_{j=1}^m \phi_j = k\pi$ *for some* $k \in \{1, \ldots, m-1\}$*.*

Definition 2.13. For a pair of transverse special Lagrangian m-planes in $\mathbb{C}^m$, (Π^-, Π^+), the integer k given by Lemma 2.12 is called the type of (Π^-, Π^+). Note that (Π^+, Π^-) is of type $m - k$.

Clearly, $(\Pi^- \cup \Pi^+)\backslash\{0\}$ is a Lagrangian cone, whose link is the disjoint union of two S^{m-1}'s. When (Π^-, Π^+) is of type 1, there are *special* Lagrangians asymptotic to $\Pi^- \cup \Pi^+$. They are constructed by Lawlor in [17], and are usually referred as *Lawlor necks.* The explanation below is based on [15, Example 6.11] and [19, section 1].

For positive numbers $a_1, \ldots, a_m$, introduce the functions

$$\mathrm{P}_{\mathbf{a}}(y) = \frac{-1 + \prod_{j=1}^m (1 + a_j y^2)}{y^2} \quad \text{and}$$
$$z_j(s) = \exp\left(i\, a_j \int_{-\infty}^s \frac{\mathrm{d}y}{(1 + a_j y^2)\sqrt{\mathrm{P}_{\mathbf{a}}(y)}}\right) \sqrt{a_j^{-1} + s^2}$$

for $j \in \{1, \ldots, m\}$. Define the real numbers $\phi_1, \ldots, \phi_m$ and A by

$$\phi_j = a_j \int_{-\infty}^{\infty} \frac{\mathrm{d}y}{(1 + a_j y^2)\sqrt{\mathrm{P}_{\mathbf{a}}(y)}} \quad \text{and} \quad A = \omega_m \left(\prod_{j=1}^m a_j\right)^{-\frac{1}{2}} \ , \tag{2.18}$$

where ω_m is the volume of the unit $S^{m-1} \subset \mathbb{R}^m$. With these functions and constants, the construction and the properties of Lawlor necks are summarised in the following proposition. The proof can be found in the aforementioned references.

Proposition 2.14. *For positive numbers* $a_1, \ldots, a_m$*, the followings hold true.*

(1) *The numbers defined by* (2.18) *satisfy*

$$\phi_j \in (0, \pi) \quad \textit{for all } j \ , \quad \sum_{j=1}^m \phi_j = \pi \ , \quad \textit{and} \quad A > 0 \ . \tag{2.19}$$

Moreover, (2.18) *gives a* bijection *between*

$$\{(a_1, \ldots, a_m) : \ a_j > 0 \textit{ for all } j\} \quad \textit{and} \quad \{(\phi_1, \ldots, \phi_m, A) \textit{ obeying } (2.19)\} \ .$$

With this understood, denote $(\phi_1, \ldots, \phi_m)$ *by* $\boldsymbol{\phi}$*. For* $(\boldsymbol{\phi}, A)$ *obeying* (2.19)*, let*

$$L^{\boldsymbol{\phi},A} = \left\{ (z_1(s)x_1, \ldots, z_m(s)x_m) \in \mathbb{C}^m : \ s \in \mathbb{R}, \ x_j \in \mathbb{R}, \ \sum_{j=1}^m x_j^2 = 1 \right\} , \tag{2.20}$$

They are called Lawlor necks.

(2) *The Lawlor neck* $L^{\boldsymbol{\phi},A}$ *defined by* (2.20) *is an embedded, special Lagrangian submanifold in* $\mathbb{C}^m$*. It is diffeomorphic to* $S^{m-1} \times \mathbb{R}$*, and is thus an exact Lagrangian. It is asymptotically conical to* $\Pi^0 \cup \Pi^{\boldsymbol{\phi}}$ *with rate* $\gamma = 2 - m$*, where*

$$\Pi^0 = \{(x_1, \ldots, x_m) : \ x_j \in \mathbb{R}^m\} \quad \textit{and} \quad \Pi^{\boldsymbol{\phi}} = \{(e^{i\phi_1}x_1, \ldots, e^{i\phi_m}x_m) : \ x_j \in \mathbb{R}^m\} \ .$$

(3) *The number* A *is essentially the volume of the topological* B^m *bound by the* S^{m-1} *defined by* $s = 0$*. The dilation of a Lawlor neck is still a Lawlor neck. Specifically,* $\varepsilon \cdot L^{\boldsymbol{\phi},A} = L^{\boldsymbol{\phi},\varepsilon^m A}$ *for any* $\varepsilon > 0$*.*

In item (3), one may also describe the dilation effect on the data $(a_1, \ldots, a_m)$; $\varepsilon \cdot L^{\boldsymbol{\phi},A}$ corresponds to $\varepsilon \cdot (a_1, \ldots, a_m) = (\varepsilon^{-2}a_1, \ldots, \varepsilon^{-2}a_m)$.

Because of item (2), Theorem 2.6 applies to the Lawlor necks. We would like to determine the constants c_a's described in that theorem. The Lawlor neck $L^{\boldsymbol{\phi},A}$ has two ends. One is asymptotic to Π^0, whose constant is denoted by $c_-(L^{\boldsymbol{\phi},A})$. The other is asymptotic to $\Pi^{\boldsymbol{\phi}}$, whose constant is denoted by $c_+(L^{\boldsymbol{\phi},A})$. According to step 2 of the proof of Theorem 2.6, these constants are the limit of a primitive of $-\lambda_0$. A direct computation shows that $-\lambda_0|_{L^{\boldsymbol{\phi},A}} = \frac{1}{2\sqrt{\mathrm{P}_{\mathbf{a}}(y)}}\mathrm{d}y$, and hence

$$c_+(L^{\boldsymbol{\phi},A}) - c_-(L^{\boldsymbol{\phi},A}) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\mathrm{P}_{\mathbf{a}}(y)}}\mathrm{d}y \ . \tag{2.21}$$

By a change of variable, $c_+(\varepsilon \cdot L^{\boldsymbol{\phi},A}) - c_-(\varepsilon \cdot L^{\boldsymbol{\phi},A}) = \varepsilon^2 \left[c_+(L^{\boldsymbol{\phi},A}) - c_-(L^{\boldsymbol{\phi},A})\right]$. This coincides with Corollary 2.8.

Since $\alpha_{L^{\phi,A}}$ is unique up to the addition of a constant, we may choose the asymptotic constant $c_-(L^{\phi,A}) = 0$, from which it follows that $c_+(L^{\phi,A}) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\mathrm{P}_{\mathbf{a}}(y)}}\mathrm{d}y$. We maintain this choice for the remainder of our work.

The Z-invariant (see Definition 2.11) of the Lawlor necks is computed by Joyce in [15, section 9.1]:

Lemma 2.15. *For a Lawlor neck* $L^{\boldsymbol{\phi},A}$*, let* Σ^- *be the link of* Π^0*, and* Σ^+ *be the link of* $\Pi^{\boldsymbol{\phi}}$*. The* Z*-invariant of the Lawlor neck satisfies*

$$Z(L^{\boldsymbol{\phi},A}) \cdot [\Sigma^-] = A \quad \textit{and} \quad Z(L^{\boldsymbol{\phi},A}) \cdot [\Sigma^+] = -A \ ,$$

where the notation means the evaluation on the fundamental cycles.

It is not hard to see the effect of the dilations: $Z(\varepsilon \cdot L^{\boldsymbol{\phi},A}) \cdot [\Sigma^{\mp}] = \pm\varepsilon^m A$.

Remark 2.16. The notation $\pm$ here plays the role of the index a in Theorem 2.6. The choice here matches with the s-parameter in (2.20), but is opposite to that in [15, section 9.1]. Note that this does not mean that we reverse the orientation of $L^{\phi,A}$; the orientation of a special Lagrangian is always given by $\operatorname{Re}\Omega$.

2.4. **Isolated Conical Singularities.** In [13], Proposition 2.3 is also used to describe the local behavior of isolated conical singularities of Lagrangian submanifolds. For the purpose of this paper, we focus on the case that the singularity is modeled on a pair of transverse special Lagrangian planes of type 1, whose desingularisation models are the Lawlor necks.

Definition 2.17 ([15, section 9.2])**.** Let $(M^{2m}, g, J, \omega, \Omega)$ be a Calabi–Yau manifold, and X^m be a compact manifold. Suppose that $\iota : X \to M$ is a special Lagrangian immersion. A point $x \in M$ is called a *transverse self-intersection point of X of type k* if it satisfies the following properties:

- The pre-image of x consists of exactly two points in X, x^- and x^+.
- The tangent planes $\iota_*(T_{x^-}X)$ and $\iota_*(T_{x^+}X)$ are transverse, $\iota_*(T_{x^-}X) \cap \iota_*(T_{x^+}X) = \{0\}$.
- With the identification $(T_xM, g|_x, J|_x, \omega|_x, \Omega|_x) \cong (\mathbb{C}^m, g_0, J_0, \omega_0, \Omega_0)$, $(\iota_*(T_{x^-}X), \iota_*(T_{x^+}X))$ is of type k (as defined in Definition 2.13).

As noted in Definition 2.13, the type becomes $m - k$ if one exchanges $x^\pm$. It follows from the implicit function theorem and compactness of X that if transverse self-intersection points are isolated, so are their pre-images.

To describe the structure of $\iota(X)$ near x, we utilise Darboux's theorem.

Lemma 2.18. (1) *For any x in a Calabi–Yau manifold $(M^{2m}, g, j, \omega, \Omega)$, there exist $R > 0$ and an embedding Υ from B_R, the ball of radius R in $\mathbb{C}^m$, to M, such that $\Upsilon(0) = x$, $\Upsilon^*\omega = \omega_0$, $\Upsilon^*g|_0 = g_0$ and $\Upsilon^*\Omega|_0 = \Omega_0$.*

(2) *Moreover, suppose that x is a transverse self-intersection point of type k of a special Lagrangian immersion $\iota : X \to M$. Then, there exists $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_m)$ satisfying $\phi_\ell \in (0, \pi)$ for all ℓ and $\sum_{\ell=1}^m \phi_\ell = k\pi$. Moreover, Υ can be taken to obey*

$$\Upsilon_*|_0 (\Pi^0) = \iota_*(T_{x^-}X) \quad \text{and} \quad \Upsilon_*|_0 (\Pi^{\boldsymbol{\phi}}) = \iota_*(T_{x^+}X)$$

where Π^0 and $\Pi^{\boldsymbol{\phi}}$ are given by (2.17).

Proof. The first assertion follows from the results in [4, Ch. 8 and Ch. 13]. The second assertion follows from Lemma 2.12. $\square$

In light of Lemma 2.18, we consider transverse self-intersections of Lagrangians in $\mathbb{C}^m$. Utilising Proposition 2.3, we describe the decay of the Lagrangian to the tangent cone at the self-intersection point in terms of a potential function as follows.

Lemma 2.19. *Let Π be a Lagrangian m-plane in $\mathbb{C}^m$, and apply Proposition 2.3 to $C = \Pi\backslash\{0\}$ to produce an equivariant Lagrangian neighbourhood Φ_C. Suppose that L is a Lagrangian in $\mathbb{C}^m$ with $0 \in L$ and $T_0L = \Pi$. Then, there exist $R_2 > 0$ and a smooth function $\mathfrak{A} : \Sigma \times (0, R_2) \to \mathbb{R}$ such that*

$$L \cap B_{R_2} = \{(\Phi_C \circ \mathrm{d}\mathfrak{A})(\sigma, r) = \Phi_C\left(\sigma, r, (\mathrm{d}_\Sigma\mathfrak{A})(\sigma, r), (\partial_r\mathfrak{A})(\sigma, r)\right) : \ \sigma \in \Sigma, \ 0 < r < R_2\} \ ,$$

and for $k \in \{0,1,2\}$, $|\nabla^k \mathfrak{A}| = O(r^{3-k})$ as $r \to 0$. As before, ∇ and $|\cdot|$ are computed using the cone metric $g_C = \mathrm{d}r^2 + r^2 g_\Sigma$, which is simply the flat metric on Π.

Proof. Similar to (2.3), L near the origin can be expressed as $\Phi_C(\sigma, r, \mathfrak{a}_1(\sigma,r), \mathfrak{a}_2(\sigma,r))$, where $\mathfrak{a}_1(\sigma,r) + \mathfrak{a}_2(\sigma,r)\mathrm{d}r$ is a closed 1-form on $\Sigma \times (0, R_2)$. Since $0 \in L$, $|\mathfrak{a}| = O(r)$ as $r \to 0$. Since $T_0 L = \Pi$, $|\nabla \mathfrak{a}| = O(r)$, and thus $|\mathfrak{a}| = O(r^2)$. As in the proof for Theorem 2.6, let

$$\mathfrak{A}(\sigma, r) = \int_0^r \mathfrak{a}_2(\sigma, y)\mathrm{d}y \ .$$

By a similar argument, $\mathrm{d}\mathfrak{A} = \mathfrak{a}_1 + \mathfrak{a}_2 \mathrm{d}r$. The decay rate on $\mathfrak{A}$ follows directly from its construction. $\square$

Remark 2.20. In this case, 0 is a smooth point of the Lagrangian submanifold L, i.e. a *fake* conical singularity point. In terms of the terminology in [13, section 3.2], the rate of the conical singularity is 3. We note that in the main theorem of [13, section 6.1], the rate of the conical singularity must belong to $(2,3)$.

3. Desingularisations of Special Lagrangians with Transverse Self-Intersections

Given a special Lagrangian with isolated conical singularities $\iota : X \to M$, such that there exist suitable local models for the desingularisations, Joyce in [13, Definition 6.2] shows how to construct desingularisations of the special Lagrangian. The main purpose of this section is to give an exposition of Joyce's construction, in the particular case of Lagrangians with transverse self-intersections of type I. We make this restriction so that there exist suitable local models for the desingularisation process (the Lawlor necks of section 2.3).

We will modify the construction to allow for diffeomorphisms between the desingularisations of different sizes of necks; this will allow us to set up and solve the Lagrangian mean curvature flow equation. In particular, we construct a family of embeddings ι^ε from a fixed manifold $\underline{N}$ to the Calabi–Yau manifold M, satisfying $\iota^\varepsilon \to \iota$ as $\varepsilon \to 0$.

3.1. The Static Manifold.

We first construct the underlying topological manifold for the desingularisations.

Definition 3.1. Let X be a compact manifold. Let $\iota : X \to M$ be a special Lagrangian immersion with only transverse self-intersection points of type 1, $\{x_1, \ldots, x_n\}$. Denote $\iota^{-1}(x_j)$ by x_j^- and x_j^+ for $j \in \{1, \ldots, n\}$. There are three positive numbers in the construction, $\hbar$, R_1 and R_2, with $(1+2\hbar)R_1 \leqslant (1-\hbar)R_2$. The number $\hbar$ is no greater than 1/100, and plays no significant role. The radii R_1 and R_2 may have to be taken smaller in each step if necessary.

Step 1. For each self-intersection point x_j, apply Lemma 2.18 to find a Darboux chart $\Upsilon_j : B_{R_2} \to M$. Denote by $\boldsymbol{\phi}_j$ the output of (2) of that lemma. We may assume $\iota^{-1}(\bigcup_{j=1}^n \Upsilon_j(B_{R_2}))$ is the disjoint union of $2n$ topological m-dimensional balls.

Step 2. Let C_j be the special Lagrangian cone $\Pi^0 \cup \Pi^{\boldsymbol{\phi}_j}$, and let $\Sigma_j = C_j \cap S^{2m-1}$ be its link. This link is the union of two S^{m-1}'s, which we label Σ_j^- and Σ_j^+ respectively. Apply Proposition 2.3 to find the Lagrangian neighbourhood $\Phi_{C_j} : U_{C_j} \subset T^*(\Sigma_j \times (0,\infty)) \to \mathbb{C}^m$.

Step 3. Due to Lemma 2.19, there exists a function $\mathfrak{A}_j : \Sigma_j \times (0, R_2) \to \mathbb{R}$ such that

$$\Upsilon_j^{-1}(\iota(X)\backslash\{x_j\}) = \{(\Phi_{C_j} \circ \mathrm{d}\mathfrak{A}_j)(\sigma, r) : \ \sigma \in \Sigma_j, \ 0 < r < R_2\} \ ,$$

and $|\nabla^\ell \mathfrak{A}_j| = O(r^{3-\ell})$ as $r \to 0$ for $\ell \in \{0,1,2\}$. Note that $\iota^{-1} \circ (\Upsilon_j \circ \Phi_{C_j} \circ \mathrm{d}\mathfrak{A}_j)$ induces a diffeomorphism from $\Sigma_j \times (0, R_2)$ to $\iota^{-1}(\Upsilon_j(B_{R_2})\backslash\{x_j\})$.

Step 4. By Proposition 2.14, Theorem 2.6 and Remark 2.10, there exists an $A > 0$ such that for every j, the Lawlor neck $L_j = L^{\phi_j, A}$ satisfies

$$L_j \cap (\mathbb{C}^m \backslash \overline{B_{R_1}}) = \{(\Phi_{C_j} \circ \mathrm{d}\mathfrak{E}_j)(\sigma, r) : \ \sigma \in \Sigma_j, \ r > R_1\}$$

for some function $\mathfrak{E}_j : \Sigma_j \times (R_1, \infty) \to \mathbb{R}$ with $|\nabla^\ell \mathfrak{E}_j| = O(r^{(2-m)-\ell})$ as $r \to \infty$, for any $\ell \geqslant 0$.

Step 5. The *static manifold* $\underline{N}$ for the desingularisation of X will be constructed from the following three types of pieces:

- $X^{\mathrm{o}} = X \backslash \iota^{-1}(\bigcup_{j=1}^n \Upsilon_j(\overline{B_{(1-\hbar)R_2}}))$, the *outer region*,
- $Q_j = \Sigma_j \times (R_1, R_2)$, the *intermediate region*, consisting of the connected components $Q_j^- = \Sigma_j^- \times (R_1, R_2)$ and $Q_j^+ = \Sigma_j^+ \times (R_1, R_1)$,
- $P_j = L_j \cap B_{(1+\hbar)R_1}$, the *tip region.*

For $(\sigma, r) \in \Sigma_j \times ((1-\hbar)R_2, R_2) \subset Q_j$, identify it with its image under $\iota^{-1} \circ \Upsilon_j \circ \Phi_{C_j} \circ \mathrm{d}\mathfrak{A}_j$ in X^{o}. For $(\sigma, r) \in \Sigma_j \times (R_1, (1+\hbar)R_1)$, identify it with its image under $\Phi_{C_j} \circ \mathrm{d}\mathfrak{E}_j$ in P_j. The resulting manifold is the static manifold $\underline{N}$, which is clearly a compact, smooth manifold.

3.2. **Desingularisations.** In [13, section 6.1], Joyce constructs the desingularisations as a submanifold in M. Here, we instead construct an embedding $\iota^{\boldsymbol{\varepsilon}} : \underline{N} \to M$.

Definition 3.2. For a special Lagrangian immersion, $\iota : X \to M$, with only transverse self-intersection points of type 1, $\{x_j\}_{j=1}^n$, let $\underline{N}$ be the static manifold constructed by Definition 3.1. Fix a $\tau \in (0, \frac{1}{2})$, whose precise value will be determined later. Given $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ with

$$0 < \varepsilon_j < \min\left\{1, ((1+\hbar)R_1)^{\frac{-1}{1-\tau}}, \left(\frac{(1-\hbar)R_2}{2}\right)^{\frac{1}{\tau}}\right\} \tag{3.1}$$

for all j, define a Lagrangian embedding $\iota^{\boldsymbol{\varepsilon}} : \underline{N} \to M$ on the pieces of $\underline{N}$ as follows. Its image, $\iota^{\boldsymbol{\varepsilon}}(\underline{N})$, will be denoted by $N^{\boldsymbol{\varepsilon}}$.

Step 0. Choose a smooth, increasing function $\chi : (0, \infty) \to \mathbb{R}$ such that $\chi(y) \in [0,1]$ for all y, and

$$\chi(y) = \begin{cases} 0 & \text{when } 0 < y \leqslant 1 \ , \\ 1 & \text{when } 2 \leqslant y < \infty \ . \end{cases}$$

Step 1. For any q in the outer region X^{o}, $\iota^{\boldsymbol{\varepsilon}}$ is set to be the original immersion, $\iota^{\boldsymbol{\varepsilon}}(q) = \iota(q)$.

Step 2. For any q in the tip region P_j, set $\iota^{\boldsymbol{\varepsilon}}(q)$ to be $\Upsilon_j(\varepsilon_j q)$. Namely, in the Darboux chart, it is simply the dilation by ε_j.

Step 3. The map on the intermediate region interpolates between the above two maps (refer to Figure 2 for a diagram). The procedure is the same for all $j \in \{1, \ldots, n\}$. For notational

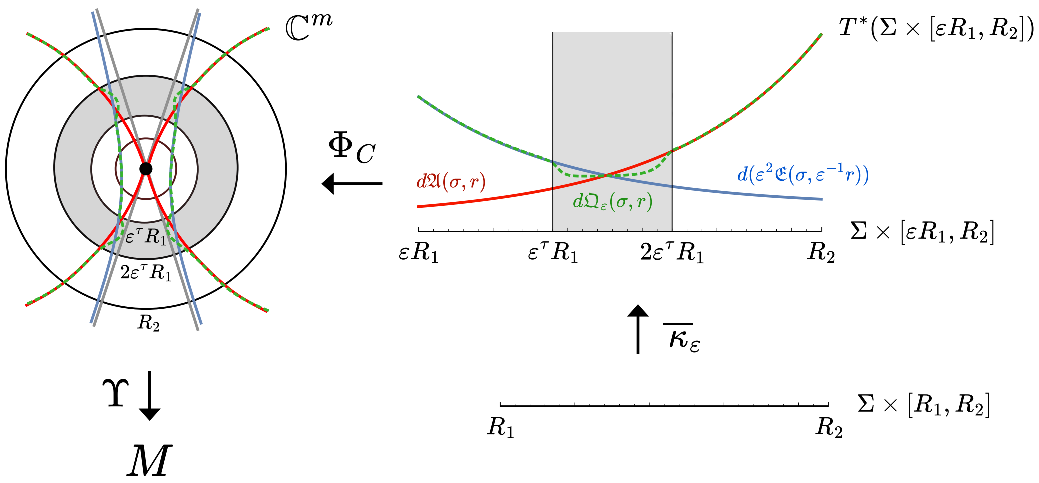

FIGURE 2. A diagram of the gluing procedure to construct the desingularisation $\iota^\varepsilon : \underline{N} \to M$ in the intermediate region, with the left side depicting the ball $B_{R_2} \subset \mathbb{C}^m$ and the right side depicting the cotangent bundle of $\Sigma_j \times [\varepsilon_j R_1, R_2]$ (for notational simplicity, j is suppressed). The interpolation region for the function $\mathfrak{Q}_{\varepsilon_j}$ is shaded.

simplicity, suppress the subscript j in ε_j, C_j etc.

$$\kappa_\varepsilon(r) = \left[1 - \chi\left(\frac{r - R_1}{\hbar R_1}\right)\right] \varepsilon\, r + \chi\left(\frac{r - R_1}{\hbar R_1}\right) r \tag{3.2}$$

for $r \in (R_1, R_2)$. It is a diffeomorphism from (R_1, R_2) to $(\varepsilon R_1, R_2)$, which will be verified momentarily. Denote the diffeomorphism $\mathrm{id}_\Sigma \times \kappa_\varepsilon : \Sigma \times (R_1, R_2) \to \Sigma \times (\varepsilon R_1, R_2)$ by $\bar{\kappa}_\varepsilon$. Next, for $(\sigma, \mathfrak{r}) \in \Sigma \times (\varepsilon R_1, R_2)$, let

$$\mathfrak{Q}_\varepsilon(\sigma, \mathfrak{r}) = \chi(\varepsilon^{-\tau}\mathfrak{r})\, \mathfrak{A}(\sigma, \mathfrak{r}) + (1 - \chi(\varepsilon^{-\tau}\mathfrak{r}))\, \varepsilon^2\, \mathfrak{E}(\sigma, \varepsilon^{-1}\mathfrak{r}) \ . \tag{3.3}$$

As noted in Remark 2.10, $\varepsilon^2\, \mathfrak{E}(\sigma, \varepsilon^{-1}\mathfrak{r})$ is the potential function of $\varepsilon\, L^{\phi,A}$. It naturally gives a Lagrangian embedding:

$$\Upsilon \circ \Phi_C \circ \mathrm{d}\mathfrak{Q}_\varepsilon : \Sigma \times (\varepsilon R_1, R_2) \to M \ .$$

Finally, ι^ε on $Q_j = \Sigma_j \times (R_1, R_2)$ is set to be $\iota^\varepsilon := \Upsilon_j \circ \Phi_{C_j} \circ \mathrm{d}\mathfrak{Q}_{\varepsilon_j} \circ \bar{\kappa}_{\varepsilon_j}$.

Remark 3.3.

- We leave it for the readers to check that N^ε is the same as the desingularisation constructed in [13, section 6.1]. This allows us to invoke the estimates established in that paper.
- For the intermediate region, $\Sigma_j \times (\varepsilon_j R_1, R_2)$ is more geometric. To be precise, the coordinate "$\mathfrak{r}$" is the (Euclidean) distance to the origin in the Darboux chart. However, in order to take the "time" derivative of a potential function, we must work on the time-independent region $\Sigma_j \times (R_1, R_2)$.

- Since $\iota(X)$ is an immersed special Lagrangian in M, and the Lawlor necks are special Lagrangians in $\mathbb{C}^m$, one can verify that $N^{\boldsymbol{\varepsilon}}$ is of zero-Maslov[2] class.

We now verify that κ_{ε_j} gives a diffeomorphism from (R_1, R_2) to $(\varepsilon_j R_1, R_2)$, and $\iota^{\boldsymbol{\varepsilon}}$ is well-defined. We continue to suppress the subscript j.

Lemma 3.4. *Suppose that $(1+2\hbar)R_1 \leqslant (1-\hbar)R_2$, then the function $\kappa_\varepsilon(r)$ defined by (3.2) is increasing for $r \in (R_1, R_2)$. Indeed, $\frac{\mathrm{d}}{\mathrm{d}r}\kappa_\varepsilon(r) \geqslant \varepsilon$. There exists $c_\ell > 0$ for all $\ell \in \mathbb{N}$, depending on $\hbar$, R_1, R_2 and χ, such that $\left|\frac{\mathrm{d}^\ell}{\mathrm{d}r^\ell}\kappa_\varepsilon(r)\right| \leqslant c_\ell$ for $r \in (R_1, R_2)$. Moreover, $\kappa_\varepsilon(r) = \varepsilon r$ when $R_1 < r < (1+\hbar)R_1$, and $\kappa_\varepsilon(r) = r$ when $(1-\hbar)R_2 < r < R_2$.*

Proof. The derivative of κ_ε is

$$\frac{\mathrm{d}}{\mathrm{d}r}\kappa_\varepsilon(r) = \varepsilon + (1-\varepsilon)\,\chi\left(\frac{r-R_1}{\hbar R_1}\right) + \frac{(1-\varepsilon)r}{\hbar R_1}\chi'\left(\frac{r-R_1}{\hbar R_1}\right) ,$$

which is clearly no less than ε, and is bounded from above. It is not hard to see that the higher order derivatives of $\kappa_\varepsilon(r)$ are uniformly bounded on (R_1, R_2).

When $r < (1+\hbar)R_1$, $\frac{r-R}{\hbar R_1} < 1$, and hence $\kappa_\varepsilon(r) = \varepsilon r$. It follows from $(1+2\hbar)R_1 \leqslant (1-\hbar)R_2$ that $\frac{r-R_1}{\hbar R_1} > 2$ when $r > (1-\hbar)R_2$. Hence, $\kappa_\varepsilon(r) = r$ when $r > (1-\hbar)R_2$. □

Lemma 3.5. *The map $\iota^{\boldsymbol{\varepsilon}}$ introduced in Definition 3.2 is well-defined.*

Proof. It follows from (3.1) that $(1+\hbar)R_1\varepsilon < \varepsilon^\tau < 2\varepsilon^\tau < (1-\hbar)R_2$.

Intermediate-Tip region. When $R_1 < r < (1+\hbar)R_1$, it follows from Lemma 3.4 that $\kappa_\varepsilon(r) = \varepsilon r$, and $\mathfrak{r} = \kappa_\varepsilon(r) < (1+\hbar)R_1\varepsilon$. Thus, $\mathfrak{Q}_\varepsilon(\sigma, \mathfrak{r}) = \varepsilon^2\,\mathfrak{E}(\sigma, \varepsilon^{-1}\mathfrak{r})$, and $(\mathfrak{Q}_\varepsilon \circ \bar{\kappa}_\varepsilon)(\sigma, r) = \varepsilon^2\,\mathfrak{E}(\sigma, r)$. Denote $\mathrm{d}_\Sigma\mathfrak{E}$ by $\mathfrak{e}_1$, and $\frac{\partial}{\partial r}\mathfrak{E}$ by $\mathfrak{e}_2$. By (1.1) and using the coordinate system introduced around (2.1),

$$\begin{aligned}((\mathrm{d}\mathfrak{Q}_\varepsilon) \circ \bar{\kappa}_\varepsilon)(\sigma, r) &= ((\bar{\kappa}_\varepsilon)_\dagger \circ \mathrm{d}(\mathfrak{Q}_\varepsilon \circ \bar{\kappa}_\varepsilon))(\sigma, r) \\ &= (\bar{\kappa}_\varepsilon)_\dagger\left(\sigma, r, \varepsilon^2\mathfrak{e}_1(\sigma, r), \varepsilon^2\mathfrak{e}_2(\sigma, r)\right) = \left(\sigma, \varepsilon r, \varepsilon^2\mathfrak{e}_1(\sigma, r), \varepsilon\mathfrak{e}_2(\sigma, r)\right) .\end{aligned}$$

This coincides with the right hand side of the first equation in Remark (2.10). It follows that $\iota^{\boldsymbol{\varepsilon}}$ is well-defined in this region.

Intermediate-Outer region. When $(1-\hbar)R_2 < r < R_2$, it follows from Lemma 3.4 that $\kappa_\varepsilon(r) = r$, and $\mathfrak{r} = \kappa_\varepsilon(r) > (1-\hbar)R_2$. In other words, $\bar{\kappa}_\varepsilon$ is the identity map on this region. One can also find that $\mathfrak{Q}_\varepsilon(\sigma, \mathfrak{r}) = \mathfrak{A}(\sigma, \mathfrak{r})$. Hence, $\iota^{\boldsymbol{\varepsilon}}$ coincides with the original map ι. □

3.3. **Weight Function.** Later on, the equations on $N^{\boldsymbol{\varepsilon}}$ will be analysed on some weighted Hölder spaces. The weight captures the geometry of the self-intersection points and the Lawlor necks. Here is the definition of the weight.

Definition 3.6. For a special Lagrangian immersion, $\iota : X \to M$, with only transverse self-intersection points of type 1, $\{x_j\}_{j=1}^n$, let $\underline{N}$ be the static manifold constructed by Definition 3.1. Given $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ satisfying (3.1), let $\iota^{\boldsymbol{\varepsilon}}$ be the Lagrangian embedding constructed by Definition 3.2. Define a smooth function $\rho_{\boldsymbol{\varepsilon}} : \underline{N} \to \mathbb{R}_+$ as follows.

[2]The argument of the complex valued function $\frac{\Omega|_{N^{\boldsymbol{\varepsilon}}}}{\mathrm{d}V_{N^{\boldsymbol{\varepsilon}}}}$ is a well-defined function on $N^{\boldsymbol{\varepsilon}}$.

- *Tip region.* For each Lawlor neck L_j, choose a smooth function $\hat{\rho}_j : L_j \to [1,\infty)$ such that $\hat{\rho}_j(\mathbf{x})$ depends only on $|\mathbf{x}|$, and $\hat{\rho}_j(\mathbf{x}) = |\mathbf{x}|$ when $|\mathbf{x}| > R_1$. For $\mathbf{x} \in P_j = L_j \cap B_{(1+\hbar)R_1}$, $\rho_{\boldsymbol{\varepsilon}}(\mathbf{x})$ is defined to be $\varepsilon_j \cdot \hat{\rho}_j(\mathbf{x})$.
- *Intermediate region.* On each $Q_j = \Sigma_j \times (R_1, R_2)$,
$$\rho_{\boldsymbol{\varepsilon}}(\sigma, r) = \kappa_{\varepsilon_j}(r) + (R_2 - r)\left[1 - \chi\left(\frac{R_2 - r}{\hbar R_2}\right)\right]$$
where $\kappa_{\varepsilon_j}(r)$ is given by (3.2). Note that $\rho_{\boldsymbol{\varepsilon}}(\sigma, r) = \kappa_{\varepsilon_j}(r)$ when $R_1 < r \leqslant (1-2\hbar)R_2$. By the proof of Lemma 3.5, $\rho_{\boldsymbol{\varepsilon}}(\sigma, r) = R_2$ when $r \geqslant (1-\hbar)R_2$. Also, note that when $(1-2\hbar)R_2 \leqslant r < R_2$, $\rho_{\boldsymbol{\varepsilon}}(\sigma, r)$ is independent of ε_j, and is increasing in r.
- *Outer region.* On the outer region $X^{\mathrm{o}} = X \backslash \iota^{-1}(\bigcup_{j=1}^n \Upsilon_j(\overline{B_{(1-\hbar)R_2}}))$, extend the function from the intermediate region by the constant R_2.

3.4. **Lagrangian Neighbourhoods.** In [13, section 6.3]., Joyce constructs the Lagrangian neighbourhood $\Psi_{N^{\boldsymbol{\varepsilon}}} : U_{N^{\boldsymbol{\varepsilon}}} \subset T^*\underline{N} \to M$ of $\iota^{\boldsymbol{\varepsilon}}(\underline{N})$ as follows.

Definition 3.7. For a special Lagrangian immersion, $\iota : X \to M$, with only transverse self-intersection points of type 1, $\{x_j\}_{j=1}^n$, let $\underline{N}$ be the static manifold constructed by Definition 3.1. Given $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ satisfying (3.1), let $\iota^{\boldsymbol{\varepsilon}}$ be the Lagrangian embedding constructed by Definition 3.2. Define a Lagrangian neighbourhood $\Psi_{N^{\boldsymbol{\varepsilon}}} : U_{N^{\boldsymbol{\varepsilon}}} \subset T^*\underline{N} \to M$ for $\iota^{\boldsymbol{\varepsilon}}$ as follows.

Step 1: Tip region. Remember that the tip region P_j is a subset of the Lawlor neck L_j. Denote by π the bundle projection of the cotangent bundle. Apply Corollary 2.8 to the Lawlor neck L_j and ε to find an open set $U_{\varepsilon_j L_j} \subset T^*L_j$ and an embedding $\Phi_{\varepsilon_j L_j} : U_{\varepsilon_j L_j} \to \mathbb{C}^m$. Define
$$U_{N^{\boldsymbol{\varepsilon}}} \cap \pi^{-1}(P_j) = U_{\varepsilon_j L_j} \cap \pi^{-1}(P_j) \ ,$$
$$\Psi_{N^{\boldsymbol{\varepsilon}}}|_{U_{N^{\boldsymbol{\varepsilon}}} \cap \pi^{-1}(P_j)} = \Upsilon_j \circ \Phi_{\varepsilon_j L_j} \ .$$

Step 2: Intermediate region. As in step 2 of Definition 3.1, apply Proposition 2.3 to the cone $C_j = \Sigma_j \times (0,\infty)$ to find an $\mathbb{R}_+$-invariant open set $U_{C_j} \subset T^*C_j$ and equivariant embedding $\Phi_{C_j} : U_{C_j} \to \mathbb{C}^m$. The map $\bar{\kappa}_{\varepsilon_j}$ given by step 3 of Definition 3.2 induces a diffeomorphism
$$(\bar{\kappa}_{\varepsilon_j})_\dagger : T^*Q_j = T^*(\Sigma_j \times (R_1, R_2)) \to T^*(\Sigma_j \times (\varepsilon_j R_1, R_2)) \ .$$
Similar to (2.8), let $f_{\mathrm{d}\mathfrak{Q}_{\varepsilon_j}}(q,p) = \big(q, p + (\mathrm{d}\mathfrak{Q}_{\varepsilon_j})(q)\big)$ be the self-diffeomorphism[3] of $T^*(\Sigma_j \times (\varepsilon_j R_1, R_2))$. Define $U_{N^{\boldsymbol{\varepsilon}}} \cap \pi^{-1}(Q_j)$ to be
$$\big((\bar{\kappa}_{\varepsilon_j})_\dagger\big)^{-1}\left\{(\sigma, \mathfrak{r}, \varsigma, \mathfrak{s}) \in T^*(\Sigma_j \times (\varepsilon_j R_1, R_2)) : \ f_{\mathrm{d}\mathfrak{Q}_{\varepsilon_j}}((\sigma,\mathfrak{r}), \varsigma + \mathfrak{s}\mathrm{d}\mathfrak{r}) \in U_C\right\} \ ,$$
and define the map to be
$$\Psi_{N^{\boldsymbol{\varepsilon}}}|_{U_{N^{\boldsymbol{\varepsilon}}} \cap \pi^{-1}(Q_j)} = \Upsilon_j \circ \Phi_{C_j} \circ f_{\mathrm{d}\mathfrak{Q}_{\varepsilon_j}} \circ (\bar{\kappa}_{\varepsilon_j})_\dagger \ .$$

Step 3: Outer region. With the help of Lemma 3.4 and Lemma 3.5, $U_{N^{\boldsymbol{\varepsilon}}}$ and $\Psi_{N^{\boldsymbol{\varepsilon}}}$ are independent of $\boldsymbol{\varepsilon}$ on the overlap between the intermediate and outer region. One use the same Moser's trick argument as that in step 3 in the proof of Theorem 2.6 to extend $U_{N^{\boldsymbol{\varepsilon}}}$ and $\Psi_{N^{\boldsymbol{\varepsilon}}}$ over X^{o}. The extensions are also independent of $\boldsymbol{\varepsilon}$.

[3] Or equivalently, $f_{\mathrm{d}\mathfrak{Q}_\varepsilon}$ sends $(\sigma, \mathfrak{r}, \varsigma, \mathfrak{s})$ to $\big(\sigma, \mathfrak{r}, \varsigma + (\mathrm{d}_\Sigma \mathfrak{Q}_\varepsilon)(\sigma, \mathfrak{r}), \mathfrak{s} + \frac{\partial \mathfrak{Q}_\varepsilon}{\partial \mathfrak{r}}(\sigma, \mathfrak{r})\big)$.

We leave it for the readers to check the well-definedness of the open set and the embedding, or one may consult [13, Definition 6.7].

4. The Lagrangian Mean Curvature Flow Equation

Given a special Lagrangian immersion $\iota : X \to M$, with only transverse self-intersection points of type 1, our goal for the remainder of this work is to construct $u : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$, and $\boldsymbol{\varepsilon}(t) = (\varepsilon_1(t), \ldots, \varepsilon_n(t))$ such that $\mathrm{d}u \in U_{N^{\boldsymbol{\varepsilon}(t)}}$ for all t, and $\Psi_{N^{\boldsymbol{\varepsilon}(t)}} \circ \mathrm{d}u$ is the solution to the mean curvature flow (where the notation $\mathrm{d}u$ denotes the spatial exterior derivative at time t). The equation reads

$$\left(\frac{\partial}{\partial t}\Psi_{N^{\boldsymbol{\varepsilon}(t)}} \circ \mathrm{d}u\right)^{\perp} = H(t) \ , \tag{4.1}$$

where $H(t)$ is the mean curvature vector of $(\Psi_{N^{\boldsymbol{\varepsilon}(t)}} \circ \mathrm{d}u)(\underline{N})$, and $\perp$ denotes the orthogonal projection onto its normal bundle. In this section, we take our first step towards this goal, by rewriting (4.1) as a differential equation involving the potential function u. In particular, with a suitable assumption on the topology of $\iota : X \to M$ (Proposition 4.5), we are able to rewrite (4.1) as (4.11).

We will require the following basic facts about the geometry of Lagrangian submanifolds. Suppose that $F : L \to (M, g, J, \omega)$ is a Lagrangian immersion, and denote by $T^{\perp}L$ the normal bundle of $F(L)$. Then there is a bundle isomorphism

$$\begin{array}{ccc} T^{\perp}L & \to & T^*L \\ \eta & \mapsto & F^*\left(\omega(\eta, \cdot)\right) \ . \end{array}$$

In particular, suppose that Y is a section of $F^*(TM)$, then

$$F^*\left(\omega(Y^{\perp}, \cdot)\right) = F^*\left(\omega(Y - Y^{\top} \cdot)\right) = F^*\left(\omega(Y, \cdot)\right) \ , \tag{4.2}$$

using the fact that $F^*\omega$ vanishes. Furthermore, suppose that M is a Calabi–Yau manifold, and let Ω be its holomorphic volume form. One has the (multi-valued) function $\theta_L = \arg(\frac{F^*(\Omega)}{\mathrm{d}V_L})$, the *Lagrangian angle*, whose exterior derivative is a well-defined 1-form on L. According to [9, section III.2.D], $\mathrm{d}\theta_L$ is the image of the mean curvature vector of L under the above isomorphism,

$$F^*\left(\omega(H_L, \cdot)\right) = -\mathrm{d}\theta_L \ . \tag{4.3}$$

4.1. **The Equation.**

The main purpose of this subsection is to rewrite (4.1) as a differential equation in the exterior derivative of the potential $u : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$. Denote by F_t the embedding $\Psi_{N^{\boldsymbol{\varepsilon}(t)}} \circ \mathrm{d}u$, and by $\theta(\mathrm{d}u)$ the Lagrangian angle of $F_t : \underline{N} \to M$. Since $N^{\boldsymbol{\varepsilon}(t)}$ is of zero-Maslov class, we may choose $\theta(\mathrm{d}u)$ to be a single-valued function. By (4.3) and (4.2), (4.1) reads

$$\mathrm{d}\left[\theta(\mathrm{d}u)\right] = -F_t^*\left(\omega\Big(\frac{\partial F_t}{\partial t}, \cdot\Big)\right) \ . \tag{4.4}$$

The right hand side will be computed on different pieces.

4.1.1. *Outer Region.* On the outer region X^{o}, $\Psi_{N^{\varepsilon(t)}}$ is independent of $\varepsilon(t)$. In this case, the right hand side of (4.4) was computed by Behrndt in his thesis [2, Lemma 4.11]. The proof is included for completeness.

Lemma 4.1. *Let $\iota_L : L \to (M, g, J, \omega)$ be a Lagrangian embedding, and $\Psi_L : U_L \subset T^*L \to M$ be a Lagrangian neighbourhood. Then, given a one-parameter family of closed 1-forms η_t on L whose image belongs to U_L,*

$$(\Psi_L \circ \eta_t)^* \left(\omega\Big(\Big(\frac{\partial (\Psi_L \circ \eta_t)}{\partial t} \Big)^{\perp}, \cdot \Big) \right) = -\frac{\partial \eta_t}{\partial t} \ .$$

Proof. We work on U_L equipped with the induced Kähler triple $(\Psi_L^*(g), \Psi_L^*(J), \omega_L)$. Denote by $\tilde{F}_t : L \to U_L$ the embedding given by η_t, i.e. $\tilde{F}_t := \Psi_L \circ \eta_t$. Since η_t is closed, $\tilde{F}_t$ is a Lagrangian embedding. By (4.2), computing the left hand side is equivalent to computing $\tilde{F}_t^*(\omega_L(\frac{\partial \tilde{F}_t}{\partial t}, \cdot))$.

Choose a local coordinate system $\{q_i\}$ on L. Let $\{p^i\}$ be the coordinate induced by $\{\mathrm{d}q_i\}$ for the fibers of T^*L. The canonical symplectic form is $\sum_i \mathrm{d}q_i \wedge \mathrm{d}p^i$. Write η_t as $\sum_i \eta_t^i(q)\mathrm{d}q_i$. In terms of the (q,p) coordinate, $\tilde{F}_t(q_1, \ldots, q_m) = (q_1, \ldots, q_m, \eta_t^1(q), \ldots, \eta_t^m(q))$, and $\frac{\partial \tilde{F}_t}{\partial t} = \sum_i \frac{\partial \eta_t^i}{\partial t} \frac{\partial}{\partial p^i}$. It follows that

$$\tilde{F}_t^* \Big(\omega_L \Big(\frac{\partial \tilde{F}_t}{\partial t}, \cdot \Big) \Big) = -\tilde{F}_t^* \Big(\sum_i \frac{\partial \eta_t^i}{\partial t} \mathrm{d}q_i \Big) = -\frac{\partial \eta_t}{\partial t} \ .$$

It finishes the proof of this lemma. □

It follows that on the outer region, (4.4) becomes

$$\mathrm{d}\left[\frac{\partial u}{\partial t} \right] = \mathrm{d}\left[\theta(\mathrm{d}u) \right] \ . \tag{4.5}$$

4.1.2. *Tip Region.* On the tip region $P_j = L_j \cap B_{(1+\hbar)R_1}$, the image of ι^{ε} belongs to $\Upsilon_j(B_{R_2})$. Computing the right-hand side of (4.4) is equivalent to computing $\tilde{F}_t^*(\omega_0(\frac{\partial \tilde{F}_t}{\partial t}, \cdot))$ for

$$\tilde{F}_t = \Phi_{\varepsilon_j(t)L_j} \circ \mathrm{d}u : P_j \subset L_j \to B_{R_2} \subset \mathbb{C}^m \ ,$$

where $\Phi_{\varepsilon_j(t)L_j}$ is defined in Corollary 2.8. We suppress the subscript j in the following calculations. By using the chain rule on $\tilde{F}_t(q) = \varepsilon(t) \cdot \Phi_{L_j}(q, \varepsilon(t)^{-2}\mathrm{d}u(q))$,

$$\frac{\partial \tilde{F}_t}{\partial t} = \frac{\varepsilon'(t)}{\varepsilon(t)} \tilde{F}_t + \varepsilon(t) \frac{\partial (\Phi_{L_j} \circ \eta_t)}{\partial t} \tag{4.6}$$

where $\eta_t = \varepsilon(t)^{-2}\mathrm{d}u$.

For the first term on the right hand side of (4.6), note that $\tilde{F}_t$ is the position vector, and thus $\tilde{F}_t^*(\omega_0(\tilde{F}_t, \cdot)) = \tilde{F}_t^*(-2\lambda_0)$. According to Corollary 2.8,

$$\begin{aligned}
\tilde{F}_t^* \Big(\omega_0 \Big(\frac{\varepsilon'(t)}{\varepsilon(t)} \tilde{F}_t, \cdot \Big) \Big) &= -2 \frac{\varepsilon'(t)}{\varepsilon(t)} (\mathrm{d}u)^* \Phi_{\varepsilon(t)L_j}^*(\lambda_0) \\
&= -2 \frac{\varepsilon'(t)}{\varepsilon(t)} \cdot (\mathrm{d}u)^* \left[\lambda_{L_j} - \mathrm{d}\left(\varepsilon(t)^2 \cdot (\alpha_{L_j} \circ f_{\varepsilon(t)}) \right) \right] \\
&= -2 \frac{\varepsilon'(t)}{\varepsilon(t)} \mathrm{d}u + 2\varepsilon(t)\varepsilon'(t)\, \mathrm{d}\left[\alpha_{L_j} \circ f_{\varepsilon(t)} \circ \mathrm{d}u \right] \ .
\end{aligned}$$

The map $f_{\varepsilon(t)} \circ \mathrm{d}u : L_j \to T^*L_j$ is $\varepsilon(t)^{-2}\mathrm{d}u$.

For the second term on the right hand side of (4.6), apply Lemma 4.1 to the map Φ_{L_j} and the family of 1-forms η_t. One finds that

$$\begin{aligned}\tilde{F}_t^*\Big(\omega_0\Big(\varepsilon(t)\frac{\partial(\Phi_L\circ\eta_t)}{\partial t},\,\cdot\,\Big)\Big) &= -\varepsilon(t)^2\frac{\partial(\varepsilon(t)^{-2}\mathrm{d}u)}{\partial t} \\ &= 2\frac{\varepsilon'(t)}{\varepsilon(t)}\mathrm{d}u - \mathrm{d}\left[\frac{\partial u}{\partial t}\right]\end{aligned}$$

To sum up, on the tip region P_j, (4.4) becomes

$$\mathrm{d}\left[\frac{\partial u}{\partial t}\right] = \mathrm{d}\left[\theta(\mathrm{d}u) + (\varepsilon_j(t)^2)'\cdot\alpha_{L_j}\circ(\varepsilon_j(t)^{-2}\mathrm{d}u)\right]\ . \tag{4.7}$$

4.1.3. *Intermediate Region.* On the intermediate region $Q_j = \Sigma_j \times (R_1, R_2)$, the image belongs to $(\Upsilon_j \circ \Phi_{C_j})(\Sigma_j \times (0, R_2))$, where $C_j = \Sigma_j \times (0,\infty) = \Pi^0 \cup \Pi^{\phi_j}$, and Φ_{C_j} is as in Definition 3.1. Computing the right hand side of (4.4) is therefore equivalent to computing $\tilde{F}_t^*(\omega_{C_j}(\frac{\partial\tilde{F}_t}{\partial t},\,\cdot\,))$ for

$$\tilde{F}_t = f_{\mathrm{d}\mathfrak{Q}_{\varepsilon_j(t)}}\circ(\bar{\kappa}_{\varepsilon_j(t)})_\dagger\circ\mathrm{d}u : Q_j \to U_{C_j} \subset T^*C_j\ .$$

The canonical symplectic form ω_{C_j} on $T^*C_j = T^*(\Sigma_j \times (0,\infty))$ is $\omega_{\Sigma_j} + \mathrm{d}\mathfrak{r}\wedge\mathrm{d}\mathfrak{s}$.

In the following calculation, we suppress the subscript j, and use the equivariant coordinates introduced around (2.1). By a direct computation,

$$\begin{aligned}(\mathrm{d}u)(\sigma,r) &= (\sigma, r, (\mathrm{d}_\Sigma u)(\sigma,r), (\partial_r u)(\sigma,r))\ , \\ (\bar{\kappa}_{\varepsilon(t)})_\dagger(\sigma,r,\varsigma,s) &= \Big(\sigma, \kappa_{\varepsilon(t)}(r), \varsigma, \big((\partial_r\kappa_{\varepsilon(t)})(r)\big)^{-1}s\Big)\ , \\ f_{\mathrm{d}\mathfrak{Q}_{\varepsilon(t)}}(\sigma,\mathfrak{r},\varsigma,\mathfrak{s}) &= \Big(\sigma,\mathfrak{r},\varsigma + (\mathrm{d}_\Sigma\mathfrak{Q}_{\varepsilon(t)})(\sigma,\mathfrak{r}), \mathfrak{s} + (\partial_{\mathfrak{r}}\mathfrak{Q}_{\varepsilon(t)})(\sigma,\mathfrak{r})\Big)\ .\end{aligned}$$

In this setting, we will also take the partial derivative of $\mathfrak{Q}_\varepsilon(\sigma,\mathfrak{r})$ (3.3) and $\kappa_\varepsilon(r)$ (3.2) in ε. Let

$$\begin{aligned}\hat{F}_t(\sigma,r) &= \big(\sigma, (\mathrm{d}_\Sigma u)(\sigma,r) + (\mathrm{d}_\Sigma\mathfrak{Q}_{\varepsilon(t)})(\sigma,\kappa_{\varepsilon(t)}(r))\big) : \Sigma\times(R_1,R_2)\to T^*\Sigma\ , \\ \check{F}_t(\sigma,r) &= \left(\kappa_{\varepsilon(t)}(r), \frac{(\partial_r u)(\sigma,r)}{(\partial_r\kappa_{\varepsilon(t)})(r)} + (\partial_{\mathfrak{r}}\mathfrak{Q}_{\varepsilon(t)})(\sigma,\kappa_{\varepsilon(t)}(r))\right) : \Sigma\times(R_1,R_2)\to T^*(0,\infty)\ .\end{aligned}$$

The above computation means that $\tilde{F}_t = (\hat{F}_t, \check{F}_t)$, and

$$\tilde{F}_t^*\Big(\omega_C\Big(\frac{\partial\tilde{F}_t}{\partial t},\,\cdot\,\Big)\Big) = \hat{F}_t^*\Big(\omega_\Sigma\Big(\frac{\partial\hat{F}_t}{\partial t},\,\cdot\,\Big)\Big) + \check{F}_t^*\Big((\mathrm{d}\mathfrak{r}\wedge\mathrm{d}\mathfrak{s})\Big(\frac{\partial\check{F}_t}{\partial t},\,\cdot\,\Big)\Big)\ . \tag{4.8}$$

For the first term on the right hand side of (4.8), the same argument as that in the proof of Lemma 4.1 shows that

$$\begin{aligned}\hat{F}_t^*\Big(\omega_\Sigma\Big(\frac{\partial\hat{F}_t}{\partial t},\,\cdot\,\Big)\Big) &= -\frac{\partial}{\partial t}\left[(\mathrm{d}_\Sigma u)(\sigma,r) + (\mathrm{d}_\Sigma\mathfrak{Q}_{\varepsilon(t)})(\sigma,\kappa_{\varepsilon(t)}(r))\right] \\ &= -\left(\mathrm{d}_\Sigma\left[\frac{\partial u}{\partial t}\right]\right)(\sigma,r) - \varepsilon'(t)\cdot\big(\mathrm{d}_\Sigma(\partial_\varepsilon\mathfrak{Q}_{\varepsilon(t)})\big)(\sigma,\kappa_{\varepsilon(t)}(r)) \\ &\quad - \varepsilon'(t)\cdot(\partial_\varepsilon\kappa_{\varepsilon(t)})(r)\cdot\big(\mathrm{d}_\Sigma(\partial_{\mathfrak{r}}\mathfrak{Q}_{\varepsilon(t)})\big)(\sigma,\kappa_{\varepsilon(t)}(r))\ .\end{aligned}$$

By a direct computation,

$$\begin{aligned}\check{F}_t^*\Big((\mathrm{d}\mathfrak{r}\wedge\mathrm{d}\mathfrak{s})\Big(\frac{\partial \check{F}_t}{\partial t},\cdot\Big)\Big) &= \varepsilon'(t)\cdot(\partial_\varepsilon\kappa_{\varepsilon(t)})(r)\cdot\mathrm{d}\left[\frac{(\partial_r u)(\sigma,r)}{(\partial_r\kappa_{\varepsilon(t)})(r)} + (\partial_{\mathfrak{r}}\mathfrak{Q}_{\varepsilon(t)})(\sigma,\kappa_{\varepsilon(t)}(r))\right] \\ &\quad -\frac{\partial}{\partial t}\left[\frac{(\partial_r u)(\sigma,r)}{(\partial_r\kappa_{\varepsilon(t)})(r)} + (\partial_{\mathfrak{r}}\mathfrak{Q}_{\varepsilon(t)})(\sigma,\kappa_{\varepsilon(t)}(r))\right]\cdot(\partial_r\kappa_{\varepsilon(t)})(r)\,\mathrm{d}r \\ &= -\left(\partial_r\left[\frac{\partial u}{\partial t}\right]\right)(\sigma,r)\,\mathrm{d}r + \mathrm{d}\left[\varepsilon'(t)\cdot(\partial_\varepsilon\kappa_{\varepsilon(t)})(r)\cdot\frac{(\partial_r u)(\sigma,r)}{(\partial_r\kappa_{\varepsilon(t)})(r)}\right] \\ &\quad + \varepsilon'(t)\cdot(\partial_\varepsilon\kappa_{\varepsilon(t)})(r)\cdot\left(\mathrm{d}_\Sigma(\partial_{\mathfrak{r}}\mathfrak{Q}_{\varepsilon(t)})\right)(\sigma,\kappa_{\varepsilon(t)}(r)) \\ &\quad - \varepsilon'(t)\cdot\left(\partial_{\mathfrak{r}}(\partial_\varepsilon\mathfrak{Q}_{\varepsilon(t)})\right)(\sigma,\kappa_{\varepsilon(t)}(r))\cdot(\partial_r\kappa_{\varepsilon(t)})(r)\,\mathrm{d}r\ .\end{aligned}$$

Putting these into (4.8) gives that

$$\tilde{F}_t^*(\omega_C(\frac{\partial\tilde{F}_t}{\partial t},\cdot)) = \mathrm{d}\left[-\frac{\partial u}{\partial t} + \varepsilon'(t)\cdot\frac{\partial_\varepsilon\kappa_{\varepsilon(t)}}{\partial_r\kappa_{\varepsilon(t)}}\cdot\partial_r u - \varepsilon'(t)\cdot(\partial_\varepsilon\mathfrak{Q}_{\varepsilon(t)})\circ\bar{\kappa}_{\varepsilon(t)}\right]\ .$$

It follows that, on the intermediate region Q_j, (4.4) becomes

$$\mathrm{d}\left[\frac{\partial u}{\partial t}\right] = \mathrm{d}\left[\theta(\mathrm{d}u) + \varepsilon_j'(t)\cdot\frac{\partial_\varepsilon\kappa_{\varepsilon_j(t)}}{\partial_r\kappa_{\varepsilon_j(t)}}\cdot\partial_r u - \varepsilon_j'(t)\cdot(\partial_\varepsilon\mathfrak{Q}_{\varepsilon_j(t)})\circ\bar{\kappa}_{\varepsilon_j(t)}\right]\ . \tag{4.9}$$

4.1.4. *Conclusion.* Equations (4.5), (4.7) and (4.9) are summarised in the following proposition:

Proposition 4.2. *Given a special Lagrangian immersion, $\iota : X \to M$, with only transverse self-intersection points of type* 1*, suppose that there is* 1*-parameter family of functions on $\underline{N}$, u, $\Lambda\in\mathbb{R}$ and $\boldsymbol{\varepsilon}(t) = (\varepsilon_1(t),\ldots,\varepsilon_n(t))$ such that each $\varepsilon_j(t)$ satisfies* (3.1) *for $t\in[\Lambda,\infty)$. Let $\Psi_{N^{\boldsymbol{\varepsilon}(t)}} : U_{N^{\boldsymbol{\varepsilon}(t)}} \to M$ be the Lagrangian neighbourhood constructed by Definition 3.7. Let $u : \underline{N}\times[\Lambda,\infty)\to\mathbb{R}$ be a function such that* $\mathrm{d}u$ *belongs to the open set $U_{N^{\boldsymbol{\varepsilon}(t)}}$ for $t\in[\Lambda,\infty)$.*

Then the one-parameter family of immersions $\Psi_{N^{\boldsymbol{\varepsilon}(t)}}\circ\mathrm{d}u : \underline{N}\to M$ is a solution to mean curvature flow if and only if

$$\mathrm{d}\left[\frac{\partial u}{\partial t}\right] = \begin{cases}\mathrm{d}\left[\theta(\mathrm{d}u) + (\varepsilon_j(t)^2)'\cdot\alpha_{L_j}\circ\mathrm{d}(\varepsilon_j(t)^{-2}u)\right] & \text{on } P_j\ , \\ \mathrm{d}\left[\theta(\mathrm{d}u) + \varepsilon_j'(t)\cdot\frac{\partial_\varepsilon\kappa_{\varepsilon_j(t)}}{\partial_r\kappa_{\varepsilon_j(t)}}\cdot\partial_r u - \varepsilon_j'(t)\cdot(\partial_\varepsilon\mathfrak{Q}_{\varepsilon_j(t)})\circ\bar{\kappa}_{\varepsilon_j(t)}\right] & \text{on } Q_j\ , \\ \mathrm{d}\left[\theta(\mathrm{d}u)\right] & \text{on } X^{\circ}\ .\end{cases} \tag{4.10}$$

Here, $\kappa_{\varepsilon_j(t)}$ and $\mathfrak{Q}_{\varepsilon_j(t)}$ are defined in Definition 3.2; α_{L_j} is the output of Theorem 2.6 on the Lawlor neck L_j.

4.2. **On the Potential.** The right hand side of (4.10) is locally exact. It is natural to ask when the (4.10) can be integrated to the level of potentials. The Lagrangian angle $\theta(\mathrm{d}u_t)$ is globally defined. We fix the branch by requiring that $\theta_{N^{\boldsymbol{\varepsilon}}}|_{X^\circ} = \theta[\mathrm{d}0]|_{X^\circ} = 0$.

Each intermediate region Q_j has two connected component, Q_j^- and Q_j^+, corresponding to Π^0 and Π^{ϕ_j} respectively. Let $\{X_b^\circ\}_{b=1}^{n'}$ be the connected components of the outer region X°. It follows that if there exist time-dependent constants, $C_{P_j}(t)$, $C_{Q_j^-}(t)$, $C_{Q_j^+(t)}$ and $C_{X_b^\circ}(t)$ such that

$$\frac{\partial u}{\partial t} = \begin{cases} \theta(\mathrm{d}u) + (\varepsilon_j(t)^2)' \cdot \alpha_{L_j} \circ \mathrm{d}(\varepsilon_j(t)^{-2}u) + C_{P_j}(t) & \text{on } P_j \ , \\ \theta(\mathrm{d}u) + \varepsilon_j'(t) \cdot \frac{\partial_\varepsilon \kappa_{\varepsilon_j(t)}}{\partial_r \kappa_{\varepsilon_j(t)}} \cdot \partial_r u - \varepsilon_j'(t) \cdot (\partial_\varepsilon \mathfrak{Q}_{\varepsilon_j(t)}) \circ \bar{\kappa}_{\varepsilon_j(t)} + C_{Q_j^\pm}(t) & \text{on } Q_j^\pm \ , \\ \theta(\mathrm{d}u) + C_{X_b^\circ}(t) & \text{on } X_b^\circ \ , \end{cases} \tag{4.11}$$

then 4.10 holds. We now investigate the necessary conditions for the existence of such constants.

4.2.1. *Intermediate-Outer region.* The overlap between the intermediate and outer region corresponds to $\Sigma_j \times ((1-\hbar)R_2, R_2) \subset Q_j$. From Lemma 3.4, 3.5, $\kappa_{\varepsilon_j(t)}(r) = r$, and $\mathfrak{Q}_{\varepsilon_j(t)}(\sigma, \mathfrak{r}) = \mathfrak{A}_j(\sigma, \mathfrak{r})$. Both functions are independent of $\varepsilon_j(t)$. It follows that the matching condition of (4.11) on the intermediate-outer region is

$$C_{X_b^\circ}(t) = C_{Q_j^\pm}(t) \qquad \text{if } \ X_b^\circ \cap Q_j^\pm \neq \varnothing \ . \tag{4.12}$$

4.2.2. *Intermediate-Tip region.* The overlap between the intermediate and tip region corresponds to $\Sigma_j \times (R_1, (1+\hbar)R_1) \subset Q_j$. The expression on the right hand side of (4.11) is based on the coordinate of each piece. To compare the equation, we have to use the same parametrization. Parametrize the overlap part of P_j by the transition map:

$$\Phi_{C_j} \circ \mathrm{d}\mathfrak{E}_j : \Sigma_j \times (R_1, (1+\hbar)R_1) \subset Q_j \to P_j \ .$$

Denote $\Phi_{C_j} \circ \mathrm{d}\mathfrak{E}_j$ by φ_j. For $u : Q_j \to \mathbb{R}$, one has to plug $u \circ \varphi_j^{-1}$ for u into (4.11) on P_j, and compose with φ_j. That is to say, (4.11) on P_j transforms into the following expression on $\Sigma_j \times (R_1, (1+\hbar)R_1) \subset Q_j$:

$$\begin{aligned} &\theta(\mathrm{d}u) + (\varepsilon_j(t)^2)' \cdot \alpha_{L_j} \circ \mathrm{d}(\varepsilon_j(t)^{-2} u \circ \varphi_j^{-1}) \circ \varphi_j + C_{P_j}(t) \\ &= \theta(\mathrm{d}u) + (\varepsilon_j(t)^2)' \cdot \alpha_{L_j} \circ (\varphi_j)_\dagger \circ \mathrm{d}(\varepsilon_j(t)^{-2}u) + C_{P_j}(t) && \text{by (1.1)} \\ &= \theta(\mathrm{d}u) + (\varepsilon_j(t)^2)' \left[\frac{r}{2} \frac{\partial_r u}{\varepsilon_j(t)^2} + \frac{r}{2} (\partial_r \mathfrak{E}_j)(\sigma, r) - \mathfrak{E}(_j\sigma, r) + c_\pm(L_j) \right] + C_{P_j}(t) && \text{by (2.6)} \ . \end{aligned}$$

Since L_j has two ends, $c_\pm(L_j)$ are the corresponding constants produced by Theorem 2.6 for the Lawlor neck $L_j = L^{\phi_j, A}$. Recall that we choose α_{L_j} such that $c_-(L_j) = 0$, and so $c_+(L_j)$ is given by the right-hand side of (2.21). For convenience, we denote $c_j := c_+(L_j)$.

For (4.11) on $Q_j \supset \Sigma_j \times (R_1, (1+\hbar)R_1)$, it follows from Lemma 3.4 that $\kappa_{\varepsilon_j(t)}(r) = \varepsilon_j(t)\, r$, and $\mathfrak{Q}_{\varepsilon_j(t)}(\sigma, \mathfrak{r}) = \varepsilon_j(t)^2\, \mathfrak{E}_j(\sigma, \varepsilon_j(t)^{-1}\mathfrak{r})$. A direct computation shows that (4.11) on $Q_j^\pm$ becomes

$$\theta(\mathrm{d}u) + \varepsilon_j'(t) \frac{r}{\varepsilon_j(t)} \partial_r u - \varepsilon_j'(t) \left[2\varepsilon_j(t) \mathfrak{E}_j(\sigma, r) - \varepsilon_j(t)\, r\, (\partial_r \mathfrak{E}_j)(\sigma, r) \right] + C_{Q_j^\pm}(t) \ .$$

Therefore, the matching condition of (4.11) on the intermediate-tip region is $(\varepsilon_j(t)^2)'\, c_j + C_{P_j}(t) = C_{Q_j^+}(t)$ and $C_{P_j}(t) = C_{Q_j^-}(t)$. The matching condition therefore reduces to

$$C_{Q_j^+}(t) - C_{Q_j^-}(t) = c_j \cdot (\varepsilon_j(t)^2)' \ . \tag{4.13}$$

4.2.3. *Conclusion.* Putting (4.12) and (4.13) together gives the following proposition. In what follows, we write $b = \underleftarrow{j}$ if Q_j^- connects to X_b°, and similarly we write $b' = \underrightarrow{j}$ if Q_j^+ connects to $X_{b'}^\circ$.

Proposition 4.3. *Assume that there exist time-dependent constants $C_b(t)$ associated with the transverse intersection points $\{x_1, \ldots, x_n\}$ of $\iota : X \to M$ such that for each $j \in \{1, \ldots, n\}$,*

$$C_{\underleftarrow{j}}(t) + c_j \cdot (\varepsilon_j(t)^2)' \;=\; C_{\underrightarrow{j}}(t), \tag{4.14}$$

where by convention $c_-(L_j) = 0$ and $c_+(L_j) =: c_j$. Then the equation (4.10) given by Proposition 4.2 can be integrated to the level of potentials (4.11), by choosing $C_{X_b^\circ}(t) := C_b(t)$, $C_{P_j} = C_{Q_j^-} := C_b(t)$ when $b = \underleftarrow{j}$, and $C_{Q_j^+} := C_{b'}(t)$ where $b' = \underrightarrow{j}$.

Since $c_j > 0$, condition (4.14) implies that every tip region must connect to two *different* components of the outer region.

4.3. **Some Graph Theory.** The purpose of this subsection is to "visualise" the conditions given by Proposition 4.3, and where possible convert the conditions (4.14) to conditions on the neck parameters $\boldsymbol{\varepsilon}$. It will be convenient to borrow some concepts from graph theory.

Firstly, we introduce a combinatorial representation of the topology of $\underline{N}$, which will be a *directed graph*, $(\mathcal{V}, \mathcal{E})$. The vertex set $\mathcal{V}$ consists of a vertex for each connected component of the outer region X_b°:

$$\mathcal{V} = \{X_1^\circ, \ldots, X_{n'}^\circ\} \ .$$

The edge set $\mathcal{E}$ consists of an edge for each tip region P_j, or equivalently, each $x_j = \iota(x_j^\pm) \in M$:

$$\mathcal{E} = \{x_1, \ldots, x_n\} \ .$$

Note that for each j, the edge associated with x_j goes from $X_{\underleftarrow{j}}^\circ$ to $X_{\underrightarrow{j}}^\circ$ by our notation convention. See Figure 3 for an example.

The *incidence* matrix for the directed graph $(\mathcal{V}, \mathcal{E})$ is the $n' \times n = |\mathcal{V}| \times |\mathcal{E}|$ matrix that contains the relationship between the edges and vertices. It is defined to be

$$\mathbf{B}_{bj} = \begin{cases} 1 & \text{if } \underrightarrow{j} = b \ , \\ -1 & \text{if } \underleftarrow{j} = b \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

For $b \in \{1, \ldots, n' = |\mathcal{V}|\}$, let X_b be the connected component of X containing X_b°. Denote by $\mathbf{V}$ the $n' \times n'$ diagonal matrix whose (b,b)-entry is the volume of $\iota(X_b)$ in M, denoted V_b.

The proof of the following lemma is elementary linear algebra, which is left as an exercise for the reader.

Lemma 4.4. *Suppose that $(\mathcal{V}, \mathcal{E})$ is a* tree. *Then, $\mathbf{B}^T \mathbf{V}^{-1} \mathbf{B}$ is a positive definite and symmetric matrix.*

In this graph-theoretic language, Proposition 4.3 can be thought of as a condition on a *vertex weighting*, i.e. a function on $\mathcal{V}$. In particular, we consider a time-dependent vertex weighting mapping $\mathcal{V} \to C^\infty([\Lambda, \infty))$, where X_b° is mapped to $C_b(t)$. The condition (4.14) may therefore be written as

$$\mathbf{B}^T \begin{bmatrix} C_1^\circ(t) \\ \vdots \\ C_{n'}^\circ(t) \end{bmatrix} = \begin{bmatrix} c_1 \cdot (\varepsilon_1(t)^2)' \\ \vdots \\ c_n \cdot (\varepsilon_n(t)^2)' \end{bmatrix} . \tag{4.15}$$

In the case where $\mathbf{B}^T\mathbf{V}^{-1}\mathbf{B}$ is invertible, (e.g. if $(\mathcal{V}, \mathcal{E})$ is a tree), then (4.15) is implied by

$$\begin{bmatrix} C_1^\circ(t) \\ \vdots \\ C_{n'}^\circ(t) \end{bmatrix} = \mathbf{V}^{-1}\mathbf{B}(\mathbf{B}^T\mathbf{V}^{-1}\mathbf{B})^{-1} \begin{bmatrix} c_1 \cdot (\varepsilon_1(t)^2)' \\ \vdots \\ c_n \cdot (\varepsilon_n(t)^2)' \end{bmatrix} . \tag{4.16}$$

We may then define $C_b^\circ(t)$ by this equation, from which it follows that (4.11) holds, and so the MCF equation may be integrated to the level of potentials. In particular, we have the following theorem for the tree case:

Proposition 4.5. *Suppose that* $(\mathcal{V}, \mathcal{E})$ *is a tree. Then the equation* (4.10) *given by Proposition 4.2 can be integrated to the level of potentials* (4.11).

Note that one can add the same (time-dependent) constant to the components of the left hand side of (4.15), and they still obey (4.14).

5. The Linear Operator and its Approximate Kernel

In this section, we derive the linearisation of the LMCF equation (Proposition 5.2), and construct an approximate kernel for this operator (5.12).

From now on, we will assume the conditions in Proposition 4.3 are satisfied, so that the Lagrangian mean curvature flow can be integrated to the level of potentials. Given any smooth function $u : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$ such that $\mathrm{d}u(x,t) \in U_{N^\varepsilon(t)}$ for all $(x,t) \in \underline{N} \times [\Lambda, \infty)$, define a function $\xi(\mathrm{d}u) : \underline{N} \to \mathbb{R}$ by

$$\xi(\mathrm{d}u) = \begin{cases} (\varepsilon_j(t)^2)' \cdot \alpha_{L_j} \circ \mathrm{d}(\varepsilon_j(t)^{-2}u) + C_{P_j}(t) & \text{on } P_j \ , \\ \varepsilon_j'(t) \cdot \frac{\partial_\varepsilon \kappa_{\varepsilon_j(t)}}{\partial_r \kappa_{\varepsilon_j(t)}} \cdot \partial_r u - \varepsilon_j'(t) \cdot (\partial_\varepsilon \mathfrak{Q}_{\varepsilon_j(t)}) \circ \bar{\kappa}_{\varepsilon_j(t)} + C_{Q_j^\pm}(t) & \text{on } Q_j^\pm \ , \\ C_{X_b^\circ}(t) & \text{on } X_b^\circ \ , \end{cases} \tag{5.1}$$

(it follows from the assumption that there exist time-dependent constants $C_{P_j}(t)$, $C_{Q_j^\pm}(t)$, and $C_{X_b^\circ}(t)$ such that $\xi(\mathrm{d}u)$ is well-defined). The LMCF on the level of the potential u near N^ε is now given by the following scalar nonlinear equation on $\underline{N} \times [\Lambda, \infty)$:

$$\partial_t u = \theta(\mathrm{d}u) + \xi(\mathrm{d}u). \tag{5.2}$$

The function $\xi(0)$ has geometric significance. In fact, it is not hard to see that $\xi(0)$ is the potential of the velocity of N^{ε}, namely,

$$(\iota^{\varepsilon})^*(\omega(\frac{\mathrm{d}\iota^{\varepsilon}}{\mathrm{d}t}, \cdot)) = \mathrm{d}[\xi(0)]. \tag{5.3}$$

For the remainder of the paper, we will linearise the right hand side of (5.2) at the zero section $\underline{0}$ and split it into zeroth order, linear and higher order parts, denoted as follows:

$$\partial_t u \;=\; \theta_{N^{\varepsilon}} + \xi(0) + \mathcal{L}^{\varepsilon}_{\underline{0}}[u] + Q^{\varepsilon}[u]. \tag{5.4}$$

5.1. **Linearised LMCF.** Denote the embedding of the zero section by $\iota^{\varepsilon} := \Phi_{N^{\varepsilon}} \circ \underline{0} : \underline{N} \times [\Lambda, \infty) \to M$. Let $u : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$ be a smooth function such that $s\,\mathrm{d}u(x,t) \in U_{N^{\varepsilon(t)}}$ for all $(x,t) \in \underline{N} \times [\Lambda, \infty)$ and small $s \in \mathbb{R}$.

We first employ the following result by Behrndt [2].

Lemma 5.1. *The deformation vector field of ι^{ε} in the direction of* $\mathrm{d}u$ *is given by*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_{N^{\varepsilon}} \circ \mathrm{d}(su) = J(\iota^{\varepsilon})_* \nabla u + (\iota^{\varepsilon})_* \widehat{V}(\mathrm{d}u) \tag{5.5}$$

for some $\widehat{V}(\mathrm{d}u) \in \Gamma(T\underline{N})$, where the gradient ∇u is computed using the induced metric $g^{\varepsilon} := (\iota^{\varepsilon})^* g$ on $\underline{N}$. Moreover, the linearisation of the Lagrangian angle is given by

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \theta(\mathrm{d}(su)) = \Delta_{g^{\varepsilon}} u \;-\; \langle \nabla\theta, \widehat{V}(\mathrm{d}u)\rangle_{g^{\varepsilon}} \;. \tag{5.6}$$

Next, we linearise $\xi(\mathrm{d}u)$. In the tip region, we have

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \xi(s\mathrm{d}u) &= (\varepsilon_j(t)^2)' \cdot \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \alpha_{L_j} \circ \mathrm{d}(\varepsilon_j(t)^{-2} su) \\ &= (2\log\varepsilon_j(t))'(\mathrm{d}\beta_{L_j})(\mathrm{d}u^V),\end{aligned} \tag{5.7}$$

where, in standard local coordinates $\{x^i, p_i\}_{i=1}^m$ of $T^*\underline{N}$, $\mathrm{d}u^V$ is the vertical vector field $\mathrm{d}u^V := \frac{\partial u}{\partial x^j}\frac{\partial}{\partial p_j}$ on $U_{N^{\varepsilon}}$. In the intermediate region, it is clear that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \xi(s\mathrm{d}u) = \varepsilon_j'(t) \cdot \frac{\partial_{\varepsilon}\kappa_{\varepsilon_j(t)}}{\partial_r \kappa_{\varepsilon_j(t)}} \cdot \partial_r u. \tag{5.8}$$

Finally, in the outer region, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \xi(s\mathrm{d}u) = 0. \tag{5.9}$$

The above computations are summarised by the following proposition.

Proposition 5.2. *The linearisation of (5.2) at the zero section is given by*

$$\begin{aligned}\partial_t u - \mathcal{L}^{\varepsilon}_{\underline{0}}[u] &:= \partial_t u - \frac{\mathrm{d}}{\mathrm{d}s}_{|_{s=0}} (\theta + \xi)(s\mathrm{d}u) \\ &= \partial_t u - \Delta_{g^{\varepsilon}} u + \langle \nabla\theta, \widehat{V}(\mathrm{d}u)\rangle_{g^{\varepsilon}} - S^{\varepsilon}[u].\end{aligned} \tag{5.10}$$

where $S^{\boldsymbol{\varepsilon}}[u]$ is a first order linear differential operator defined by

$$S^{\boldsymbol{\varepsilon}}[u] = \begin{cases} (2\log\varepsilon_j(t))'(\mathrm{d}\beta_{L_j})(\mathrm{d}u^V) & \text{on } P_j\ , \\ \varepsilon_j'(t)\cdot\frac{\partial_\varepsilon\kappa_{\varepsilon_j(t)}}{\partial_r\kappa_{\varepsilon_j(t)}}\cdot\partial_r u & \text{on } Q_j^{\pm}\ , \\ 0 & \text{on } X_b^{\mathrm{o}}\ . \end{cases} \tag{5.11}$$

5.2. **Approximate Kernels.** Define the function $\underline{\alpha}_j : P_j \cup Q_j \to \mathbb{R}$ by

$$\begin{aligned} \underline{\alpha}_j(p) &:= \alpha_{L_j}\big|_{\underline{0}}(p) && \text{for } p \in P_j, \\ \underline{\alpha}_j(\sigma, r) &:= \alpha_{L_j}\big|_{\underline{0}}\left(\varphi_j(\sigma, \varepsilon_j^{-1}\kappa_{\varepsilon_j}(r))\right) && \text{for } (\sigma, r) \in Q_j, \end{aligned}$$

where φ_j is as in Definition 2.4 for $L = L_j$. By interpolating this function with constants on the exterior region, we construct the 'approximate kernel' of our linearised operator.

Explicitly, given $\mathbf{d} := (d_1, \ldots, d_{n'}) \in \mathbb{R}^{n'}$, we define the function $w^{\boldsymbol{\varepsilon}}_{\mathbf{d}}$ to be:

$$w^{\boldsymbol{\varepsilon}}_{\mathbf{d}} := \begin{cases} d_b & \text{on } X_b^{\mathrm{o}} \\ d_{\underleftarrow{j}} + \frac{1}{c_j}(d_{\underrightarrow{j}} - d_{\underleftarrow{j}})\,\underline{\alpha}_j & \text{on } P_j \\ d_{\underleftarrow{j}}\,\chi\big(2\varepsilon_j^{-\tau}\kappa_{\varepsilon_j}(r)\big) + \left(d_{\underleftarrow{j}} + \frac{1}{c_j}(d_{\underrightarrow{j}} - d_{\underleftarrow{j}})\underline{\alpha}_j\right)\left(1 - \chi\big(2\varepsilon_j^{-\tau}\kappa_{\varepsilon_j}(r)\big)\right) & \text{on } Q_j^- \\ d_{\underrightarrow{j}}\,\chi\big(2\varepsilon_j^{-\tau}\kappa_{\varepsilon_j}(r)\big) + \left(d_{\underleftarrow{j}} + \frac{1}{c_j}(d_{\underrightarrow{j}} - d_{\underleftarrow{j}})\underline{\alpha}_j\right)\left(1 - \chi\big(2\varepsilon_j^{-\tau}\kappa_{\varepsilon_j}(r)\big)\right) & \text{on } Q_j^+ \end{cases} \tag{5.12}$$

Remark 5.3. The functions $w^{\boldsymbol{\varepsilon}}_{\mathbf{d}}$ form the 'approximate kernel' for the following reason. It is shown in [13, p 49] (see also [18, Lemma 11]) that $w^{\boldsymbol{\varepsilon}}_{\mathbf{d}}$ approximate the *small eigenfunctions* of the Laplacian $\Delta_{g^{\boldsymbol{\varepsilon}}}$, with eigenvalues of the order $O(|\boldsymbol{\varepsilon}|^{m-2})$. It follows that if $|\boldsymbol{\varepsilon}|$ is small, $w^{\boldsymbol{\varepsilon}}_{\mathbf{d}}$ approximate *harmonic functions* on $N^{\boldsymbol{\varepsilon}}$. Since our linearised operator $\mathcal{L}^{\boldsymbol{\varepsilon}}_{\underline{0}}$ will turn out to be a small perturbation of the Laplacian (see Lemma 7.11), we have $\mathcal{L}^{\boldsymbol{\varepsilon}}_{\underline{0}} w^{\boldsymbol{\varepsilon}}_{\mathbf{d}} \approx 0$. These functions are the obstructions to the uniform invertibility of the linearised operator.

6. A Priori Estimates for the Linear Operator

In this section, we prove a uniform injectivity estimate for solutions to the inhomogeneous heat equation which are orthogonal to the approximate kernel (Theorem 6.7).

We will from now on assume that $\boldsymbol{\varepsilon}(t)$ satisfies the following estimates. Ultimately (after we restrict attention to the particular case of special Lagrangian tori inside the complex torus), we will choose the neck parameter $\varepsilon(t)$ so that N^ε is close to a Lagrangian mean curvature flow. In particular, ε will solve an ODE that appears as the dominant term in an integral error, and such a solution will automatically satisfy these estimates (c.f. Remark 8.6).

Assumption 6.1. There exist $\Lambda > 0$ and $C, C' > 1$ such that:

$$\begin{aligned} C^{-1}t^{-\frac{1}{m-2}} \leqslant |\boldsymbol{\varepsilon}(t)| &\leqslant Ct^{-\frac{1}{m-2}}, \quad |\boldsymbol{\varepsilon}'(t)| \leqslant Ct^{\frac{1-m}{m-2}} \leqslant C'\boldsymbol{\varepsilon}(t)^{m-1}, \\ \text{and} \quad \frac{|\boldsymbol{\varepsilon}'(t_1) - \boldsymbol{\varepsilon}'(t_2)|}{|t_1 - t_2|^\alpha} &\leqslant Ct^{\frac{1-m}{m-2} + \frac{2\alpha}{m-2}} \leqslant C'\boldsymbol{\varepsilon}(t)^{m-1-2\alpha}. \end{aligned} \tag{6.1}$$

for all $t \in [\Lambda, \infty)$, $t_1, t_2 \in [t, 2t]$, $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$.

Note that Assumption 6.1 also provides the following bound on the weight function given in Definition 3.6, for some constant $C' > 1$:

$$(C')^{-1}\rho_{t^{-\frac{1}{m-2}}} \leqslant \rho_{\varepsilon(t)} \leqslant C'\rho_{t^{-\frac{1}{m-2}}}. \tag{6.2}$$

6.1. **Liouville Theorems.** The proof of Theorem 6.7 is based on a blow-up argument which ultimately reduces the question to Liouville-type theorems on various model spaces. We start by establishing these theorems.

6.1.1. *Lawlor Necks.* The corresponding Liouville theorem on the Lawlor neck is obtained by adapting the scheme of Lockhart and McOwen [20, 21]. The main machinery in the current setting is established by Joyce in [11, section 7.3], which is summarised here for the reader's convenience. Note that Δ in [11] is the Hodge Laplacian, which differs from the Laplacian in this paper by a minus sign.

Let $L \subset \mathbb{C}^m$ be a Lawlor neck described by Proposition 2.14. Let $\hat{\rho}(\mathbf{x}) : L \to [1, \infty)$ be the smooth function defined in Definition 3.6. Given $k \in \mathbb{N} \cup \{0\}$ and $\nu \in \mathbb{R}$, define the spaces $C^k_\nu(L)$ to be the set of locally C^k functions whose weighted norm

$$\|u\|_{C^k_\nu} = \sum_{j=0}^{k} \sup_L \left|\hat{\rho}^{j+\nu}\, \nabla^j u\right|_g$$

is finite. The covariant derivative and the norm are computed using the induced metric $g = \iota_L^* g_0$, where $\iota_L : L \to \mathbb{C}^m$ is the inclusion map. The C^∞_ν space is defined to be the intersection of all C^k_ν spaces: $C^\infty_\nu(L) = \bigcap_{k=0}^\infty C^k_\nu(L)$.

Similarly, the weighted Sobolev spaces $W^{k,p}_\delta$ is defined by the norm

$$\|u\|_{W^{k,p}_\nu} = \left(\sum_{j=0}^{k} \int_L \left|\hat{\rho}^{j+\nu}\, \nabla^j u\right|^p\, \hat{\rho}^{-m}\, \mathrm{d}V_g\right)^{1/p} .$$

As usual, denote $W^{0,p}_\nu(L)$ by $L^p_\nu(L)$. For any $\nu \in \mathbb{R}$, $p > 1$, and $k \geqslant 2$, the Laplace operator $\Delta_g : C^\infty_{\mathrm{cpt}}(L) \to C^\infty_{\mathrm{cpt}}(L)$ extends to a continuous operator

$$\Delta^{k,p}_\nu : W^{k,p}_\nu(L) \longrightarrow W^{k-2,p}_{\nu+2}(L) .$$

The operator $\Delta^{k,p}_\nu$ is Fredholm for generic ν. Here is the complete characterization. The Lawlor neck L is asymptotic to $\Pi^0 \cup \Pi^\phi$ near infinity. The link Σ of $\Pi^0 \cup \Pi^\phi$ is the disjoint union of two round spheres. Let

$$\mathcal{D}_\Sigma = \{\nu \in \mathbb{R} \,|\, \nu(-\nu + m - 2) \text{ is an eigenvalue of } \Delta_\Sigma\} .$$

It is not hard to verify that $\mathcal{D}_\Sigma$ is a discrete subset of $\mathbb{R}$ satisfying $m - 2 \in \mathcal{D}_\Sigma$, $0 \in \mathcal{D}_\Sigma$, and $\mathcal{D}_\Sigma \cap (2 - m, 0) = \varnothing$. It turns out that $\Delta^{k,p}_\nu$ is Fredholm if and only if $\nu \in \mathbb{R}\backslash\mathcal{D}_\Sigma$.

Its Fredholm index, $\mathrm{Ind}\left(\Delta^{k,p}_\nu\right) = \dim\ker\left(\Delta^{k,p}_\nu\right) - \dim\mathrm{coker}\left(\Delta^{k,p}_\nu\right)$, depends only on the connected components of $\mathbb{R}\backslash\mathcal{D}_\Sigma \ni \nu$, and is given by

$$\mathrm{Ind}\left(\Delta^{k,p}_\nu\right) = \mathbf{N}_\Sigma(\nu) , \tag{6.3}$$

where $\mathbf{N}_\Sigma : \mathbb{R} \to \mathbb{Z}$ is defined by

$$\mathbf{N}_\Sigma(\nu) = \begin{cases} -\sum_{\gamma\in\mathcal{D}_\Sigma\cap(0,\nu)} \mathbf{m}_\Sigma(\gamma) & \text{when } \nu > 0 \ , \\ \sum_{\gamma\in\mathcal{D}_\Sigma\cap[\nu,0]} \mathbf{m}_\Sigma(\gamma) & \text{when } \nu \leqslant 0 \ , \end{cases}$$

and $\mathbf{m}_\Sigma(\gamma)$ is the multiplicity of the eigenvalue $\gamma(-\gamma + m - 2)$ of Δ_Σ.

From now on, focus on the Fredholm case, $\nu \notin \mathcal{D}_\Sigma$. According to the weighted elliptic estimate and the weighted Sobolev embedding, any $u \in \ker(\Delta^{k,p}_\nu) \subset W^{k,p}_\nu(L)$ must be smooth, $u \in C^\infty_\nu(L)$. By using the maximum principle, if $u \in \ker(\Delta^{k,p}_\nu) \subset W^{k,p}_\nu(L)$ with $\nu > 0$, then $u \equiv 0$. In other words, $\Delta^{k,p}_\nu$ is injective when $\nu > 0$.

Lemma 6.2. *Let $u \in W^{k,p}_\nu(L)$ with $\nu > 0$. Suppose $\Delta_g u = 0$ in the distributional sense, then $u \equiv 0$.*

Remark 6.3. By the duality property, the cokernel of $\Delta^{k,p}_\nu$ is isomorphic to the dual space of the kernel of $\Delta^{k,q}_{-\nu+m-2}$, where $1/p + 1/q = 1$. Thus, $\Delta^{k,p}_\nu$ is a surjective when $\nu < m - 2$. It follows that $\Delta^{k,p}_\nu$ is an isomorphism when $\nu \in (0, m - 2)$.

We now prove a Liouville theorem for the heat equation on the Lawlor neck.

Proposition 6.4. *Let $u : L \times (-\infty, 0) \to \mathbb{R}$ be a solution to the heat equation $\partial_t u = \Delta_g u$. Suppose there exist $c > 0$ and $\nu \in (0, m - 2)$ such that $|u(\,\cdot\,, t)| \leqslant C\hat{\rho}^{-\nu}$ for all $t \in (-\infty, 0)$. Then, $u \equiv 0$.*

Proof. By the the weighted Schauder estimate ([2] and [31, section 3.2]) and the bootstrapping argument, $u(\,\cdot\,, t) \in C^k_\nu(L)$ for any $k \geqslant 2$. It follows that $|(\Delta_g)^\ell u| \leqslant C_\ell \hat{\rho}^{-\nu-2\ell}$ for any $\ell \in \mathbb{N}$.

Fix ℓ with $4\ell > m - 2\nu$, and let $w = (\Delta_g)^\ell u$. Clearly, $\partial_t w = \Delta_g w$. Consider $E(t) = \frac{1}{2}\int_L w^2(\,\cdot\,, t)\,\mathrm{d}V_g$. The choice of ℓ guarantees that $E(t) < \infty$. Its derivative is

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \int_L w\Delta_g w\,\mathrm{d}V_g = -\int_L |\nabla w|^2_g\,\mathrm{d}V_g \leqslant 0 \ ,$$

and hence $E(t)$ is non-increasing in t. It follows that $\lim_{t\to-\infty} E(t)$ exists, and denote the limit by E.

We claim that $E = 0$, which implies that $w \equiv 0$. Pick a sequence $t_j \to -\infty$. Define $w^{(j)}(x,t)$ to be $w(x, t_j + t)$. After passing to a subsequence, $w^{(j)}$ converges smoothly on every compact subset of $L\times\mathbb{R}$ to an eternal solution $\widehat{w} : L\times\mathbb{R} \to \mathbb{R}$ of the heat equation. Since $|\widehat{w}| \leqslant C\hat{\rho}^{-\nu-2\ell}$, it follows from the dominated convergence theorem that

$$\int_L \widehat{w}(\,\cdot\,, t)^2\,\mathrm{d}V_g = \lim_{t_j\to-\infty}\int_L w(\,\cdot\,, t_j + t)^2\,\mathrm{d}V_g = 2E \quad \text{for all } t \in \mathbb{R} \ .$$

Taking derivative in t give

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_L \widehat{w}(\,\cdot\,, t)^2\,\mathrm{d}V_g = -\int_L |\nabla\widehat{w}(\,\cdot\,, t)|^2_g\,\mathrm{d}V_g \ .$$

It follows that $\widehat{w}(\,\cdot\,, t)$ is a time-dependent constant function. Since $\widehat{w}(\,\cdot\,, t)$ tends to zero at the end of L, the constant must be 0. Hence, $E = 0$ as claimed.

In other words, $(\Delta_g)^\ell u \equiv 0$. According to Lemma 6.2, $u \equiv 0$. □

6.1.2. *Punctured m-Planes.* The model space for the intermediate region is the cone over the link. In our setting, it is the union of two punctured $\mathbb{R}^m$'s, endowed with the standard metric on $\mathbb{R}^m$.

Proposition 6.5. *Let $u : (\mathbb{R}^m\backslash\{0\})\times(-\infty,\Lambda) \to \mathbb{R}$ be a solution to the heat equation $\partial_t u = \Delta_g u$, for some $\Lambda \in \mathbb{R}$. Suppose that there exist $C > 0$ and $0 < \nu < m-2$ such that $|\nabla^\ell u(x,t)| \leqslant C|x|^{-\nu-\ell}$ for all $t \in (-\infty,\Lambda)$ and $\ell \in \{0,1,2\}$. Then, $u \equiv 0$.*

Proof. This proof is a modification of the proof of [3, Proposition 5.3]. It follows from the rate condition that u satisfies the heat equation on $\mathbb{R}^m$ in the sense of distribution.

Fix any $t_0 \in (-\infty,\Lambda)$. For any $t > 0$ and any $x_0 \in \mathbb{R}^m\backslash\{0\}$,

$$u(x_0,t_0) = \int_{\mathbb{R}^m\backslash\{0\}} \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x-x_0|^2}{4t}} u(x,t_0-t)\,\mathrm{d}x.$$

Now, fix x_0, and suppose that $t > 9|x_0|^2$. When $|x-x_0|^2 \leqslant t$, it follows from the triangle inequality that $|x| < \frac{4}{3}\sqrt{t}$, and thus

$$\left|\int_{|x-x_0|^2\leqslant t} \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x-x_0|^2}{4t}} u(x,t_0-t)\,\mathrm{d}x\right| \leqslant C_1\, t^{-\frac{m}{2}} \int_{|x|^2<\frac{16}{9}t} |x|^{-\nu}\mathrm{d}x \leqslant C_2\, t^{-\frac{\nu}{2}}.$$

When $|x-x_0|^2 \geqslant t$, it follows from the triangle inequality that $|x| \geqslant \frac{2}{3}\sqrt{t}$ and $|x-x_0| \leqslant \frac{1}{2}|x|$. Since $e^{-s} \leqslant C_3\, s^{-\frac{m}{2}}$ for any $s > 0$,

$$\left|\frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x-x_0|^2}{4t}}\right| \leqslant C_4\,|x-x_0|^{-m} \leqslant C_5\,|x|^{-m}.$$

Therefore,

$$\left|\int_{|x-x_0|^2\geqslant t} \frac{1}{(4\pi t)^{\frac{m}{2}}} e^{-\frac{|x-x_0|^2}{4t}} u(x,t_0-t)\,\mathrm{d}x\right| \leqslant C_6 \int_{|x|^2\geqslant\frac{4}{9}t} |x|^{-\nu-m}\mathrm{d}x \leqslant C_7\, t^{-\frac{\nu}{2}}.$$

Putting the estimates together gives

$$|u(x_0,t_0)| \leqslant (C_2 + C_7)\, t^{-\frac{\nu}{2}} \tag{6.4}$$

whenever $t > 9|x_0|^2$. By taking $t \to \infty$, it implies that $u \equiv 0$. □

6.1.3. *Immersed Special Lagrangians.* We now return to our original special Lagrangian immersion $\iota : X \to M$, and recall that X splits into smooth compact connected components $X = \bigcup_{b=1}^{n'} X_b$. Using the notation of Definition 3.1 and in analogy with Definition 3.6, there exists a continuous function $\rho : X \to [0,\infty)$ such that

- $\rho\circ\iota^{-1}\circ\Upsilon_j$ on $B_{R_1}\cap\Upsilon_j^{-1}(\iota(X))$ is the distance to the origin, with respect to g_0;
- $\rho \equiv R_2$ on $X \setminus \big(\bigcup_{j=1}^{n} \iota^{-1}(\Upsilon_j(B_{R_2}))\big)$;
- the zero set of ρ is exactly $\{x_1,\ldots,x_n\}$;
- ρ is smooth on $X\backslash\{x_1,\ldots,x_n\}$.

The manifold X is endowed with the smooth metric $\iota^* g$.

Proposition 6.6. *Let $b \in \{1, \ldots, n'\}$, and let $u : X_b \backslash \iota^{-1}(\{x_1 \ldots, x_n\}) \times (-\infty, 0) \to \mathbb{R}$ be a solution to the heat equation $\partial_t u = \Delta u$. Suppose that*

$$\int_{X_j} u \, \mathrm{d}V_{\iota^* g} = 0 \quad \text{and} \quad |\nabla^\ell u(\cdot, t)| \leqslant C\rho^{-\nu-\ell}$$

for some $C > 0$, $\nu \in (0, m-2)$, and all $t \in (-\infty, 0)$ and $\ell \in \{0, 1, 2\}$. Then, u vanishes on X_b.

Proof. Note that X_b is a compact, smooth manifold, with the smooth metric $\iota^* g$. It follows from the growth rate condition that u obeys the linear heat equation on X_b in the sense of distribution. Hence, u must be a constant on X_b. It follows from the zero integration condition that $u \equiv 0$. □

6.2. **A Priori Estimate for the Heat Operator.** We now apply our Liouville theorems to prove an a priori sup estimate via a blowup argument. In what follows, we recall the weight function ρ_ε of Definition 3.6, we denote the induced metric on $\underline{N}$ by $g^\varepsilon := (\iota^\varepsilon)^* g$, and define the following weighted norm for tensors on $\underline{N}$:

$$\|T\|_{\mu,\nu,\Lambda} := \sup_{(x,t) \in \underline{N} \times [\Lambda, \infty)} t^\mu \rho^\nu_{t^{\frac{-1}{m-2}}}(x) |T|_{g^\varepsilon}$$

Theorem 6.7. *Let $\boldsymbol{\varepsilon} : [\Lambda, \infty) \to \mathbb{R}^n_+$, be a smooth function satisfying (3.1) and (6.1), and fix $(\mu, \nu) \in (\frac{\nu+2}{m-2}, \infty) \times (0, m-2)$. Then there exists a constant $C > 0$ with the following significance.*

Suppose $u, \psi : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$ satisfy $\|\psi\|_{\mu,\nu,\Lambda} < \infty$ and solve the Cauchy problem:

$$\begin{cases} \partial_t u(x,t) = \Delta_{g^\varepsilon}[u](x,t) + \psi(x,t), & (x,t) \in \underline{N} \times [\Lambda, \infty), \\ u(x, \Lambda) = 0, & x \in \underline{N} \end{cases} \tag{6.5}$$

and u satisfies the orthogonality conditions

$$\int_{\underline{N}} u(x,t) w^\varepsilon_b(x) \, \mathrm{d}V_{g^\varepsilon}(x) \; = \; 0, \quad t \in [\Lambda, \infty) \tag{6.6}$$

for all $b \in \{1, \ldots, n'\}$. Then

$$\sup_{\underline{N} \times [\Lambda, \infty)} t^\mu \rho^\nu_{t^{-\frac{1}{m-2}}} |u| \leqslant C \sup_{\underline{N} \times [\Lambda, \infty)} t^\mu \rho^{\nu+2}_{t^{-\frac{1}{m-2}}} |\psi|. \tag{6.7}$$

Proof. Assume that the estimate does not hold. Then there exist sequences $u^{(j)} : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$, $\psi^{(j)} : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$, and $\boldsymbol{\varepsilon}^{(j)} : [\Lambda, \infty) \to \mathbb{R}^n_+$, satisfying the following properties:

- $u^{(j)}$, $\psi^{(j)}$ solve the Cauchy problem (6.5) for each $j \in \mathbb{N}$.
- $u^{(j)}$ satisfies the orthogonality conditions (6.6) for each $j \in \mathbb{N}$.
- $\sup_{\underline{N} \times [\Lambda, \infty)} t^\mu \rho^\nu_{t^{-\frac{1}{m-2}}} |u^{(j)}| > j \cdot \sup_{\underline{N} \times [\Lambda, \infty)} t^\mu \rho^{\nu+2}_{t^{-\frac{1}{m-2}}} |\psi^{(j)}|$ for all $j \in \mathbb{N}$.

Thus, for each j, we can pick $(x'_j, t_j) \in \underline{N} \times [\Lambda, \infty)$ such that

$$\sup_{\underline{N} \times [\Lambda, t_j]} t^\mu \rho^\nu_{t^{-\frac{1}{m-2}}} |u^{(j)}| \; = \; t^\mu_j \rho^\nu_{t_j^{-\frac{1}{m-2}}}(x'_j) |u^{(j)}(x'_j, t_j)| \; \geqslant \; \frac{j}{2} \cdot \sup_{\underline{N} \times [\Lambda, t_j]} t^\mu \rho^{\nu+2}_{t^{-\frac{1}{m-2}}} |\psi^{(j)}|. \tag{6.8}$$

By interior parabolic Schauder estimate, we must have $t_j \to \infty$. By passing to a subsequence we assume that $\Lambda < \frac{1}{2}t_j$, so that in particular $\Lambda - t_j < -\frac{1}{2}t_j < 0$. Defining

$$\|u\|_{\mu,\nu,\Lambda,j} := \sup_{\underline{N}\times[\Lambda,t_j]} t^\mu \rho^\nu_{t^{-\frac{1}{m-2}}} |u|,$$

we rescale $u^{(j)}$, $\psi^{(j)}$ by $\|u^{(j)}\|^{-1}_{\mu,\nu,\Lambda,j}$ so that in addition to the three properties above,

$$t_j^\mu \rho^\nu_{t_j^{-\frac{1}{m-2}}}(x'_j)|u^{(j)}(x'_j,t_j)| = \|u^{(j)}\|_{\mu,\nu,\Lambda,j} = 1, \quad \|\psi^{(j)}\|_{\mu,\nu+2,\Lambda,j} \to 0. \tag{6.9}$$

By passing to a subsequence, we assume that x'_j converges on $\underline{N}$, to x_∞. According to where the limit point is, we have the following 3 cases. For convenience, we denote the pullback metric by $g_j := (\iota^{\varepsilon(j)})^* g$.

Case 1. $\lim_{j\to\infty} t_j^{\frac{1}{m-2}} \rho_{t_j^{-\frac{1}{m-2}}}(x'_j) < \infty$

By the definition of ρ_ε, on passing to a subsequence it follows that there exists k such that for sufficiently large j, $x'_j \in P_k \cup Q_k^\pm$. We therefore will work in this region, suppressing the index k (e.g. L_k will be written L, $\varepsilon_k^{(j)}$ will be written $\varepsilon^{(j)}$). We define the scaling factors λ_j, the region $P_s^{(j)}$, the map $S_t^{(j)}$ and the rescaled functions $\widetilde{u}^{(j)}$, $\widetilde{\psi}^{(j)}$ as follows:

$$\begin{aligned}
&\lambda_j := \varepsilon^{(j)}(t_j) \to 0, \quad t(s) := t_j + \lambda_j^2 s,\\
&P_s^{(j)} := P \cup \Sigma \times (R_1, \varepsilon^{(j)}(t(s))^{\tau-1}), \quad P^{(j)} := \{(y,s) \,:\, s \in [-\tfrac{1}{2}\lambda_j^{-2}t_j, 0), y \in P_s^{(j)}\}\\
&S_s^{(j)} : \Sigma \times (R_1, \varepsilon^{(j)}(t(s))^{\tau-1}) \to \Sigma \times (\varepsilon^{(j)}(t(s))R_1, R_2), \quad S_s^{(j)}(p,r) := (p, \varepsilon^{(j)}(t(s))r)\\
&\widetilde{u}^{(j)} : P^{(j)} \to \mathbb{R}, \quad \widetilde{u}^{(j)}(y,s) := \begin{cases} t_j^\mu \lambda_j^\nu u^{(j)}(y, t(s)) & \text{for } y \in P,\\ t_j^\mu \lambda_j^\nu u^{(j)}\left((\overline{\kappa}_{\varepsilon^{(j)}(t(s))})^{-1} \circ S_s^{(j)}(y), t(s)\right) & \text{for } y \in P_s^{(j)}\backslash P,\end{cases}\\
&\widetilde{\psi}^{(j)} : P^{(j)} \to \mathbb{R}, \quad \widetilde{\psi}^{(j)}(y,s) := \begin{cases} t_j^\mu \lambda_j^{\nu+2} \psi^{(j)}(y, t(s)) & \text{for } y \in P,\\ t_j^\mu \lambda_j^{\nu+2} \psi^{(j)}\left((\overline{\kappa}_{\varepsilon^{(j)}(t(s))})^{-1} \circ S_s^{(j)}(y), t(s)\right) & \text{for } y \in P_s^{(j)}\backslash P.\end{cases}
\end{aligned}$$

Define $x : P^{(j)} \to \underline{N}$, $x(y,s) := (\overline{\kappa}_{\varepsilon^{(j)}(t(s))})^{-1} \circ S_s^{(j)}(y)$, and $y_j := (S_0^{(j)})^{-1} \circ \overline{\kappa}_{\varepsilon^{(j)}(t_j)}(x'_j)$, and define the time-independent weight function $\hat{\rho}^{(j)} : P^{(j)} \to \mathbb{R}_+$ as in Definition 3.6 (without the outer region interpolation) using the standard embedding of the Lawlor neck L_k into $\mathbb{C}^m$. We also endow $P_s^{(j)}$ with the rescaled metric (writing $\varepsilon^{(j)}$ for $\varepsilon^{(j)}(t(s))$ for simplicity):

$$g'_j(y,s) := \begin{cases} \lambda_j^{-2} \iota^*_{\varepsilon^{(j)}L} g_0 & \text{for } y \in P\\ \lambda_j^{-2} \varphi^* \iota^*_{\varepsilon^{(j)}L} g_0 & \text{for } y \in P_s^{(j)}\backslash P.\end{cases}$$

Up to pullback, this metric is simply $g'_j(\cdot,s) = \lambda_j^{-2} g_j(\cdot, t(s))$. Note that $\varepsilon^{(j)}(t_j)^{-2}(\times\varepsilon^{(j)})^* g_0 \to g_0$ locally uniformly in C^∞. We therefore see that the metric g'_j converges in C^∞_{loc} to $\iota_L^* g_0 = g_L$:

$$\lim_{j\to\infty} \lambda_j^{-2} \iota^*_{\varepsilon^{(j)}L} g_0 = (\iota_L)^* g_0, \qquad \lim_{j\to\infty} \lambda_j^{-2} \varphi^* \iota^*_{\varepsilon^{(j)}L} g_0 = \varphi^* g_0.$$

Now, the Case 1 assumption along with (6.2) imply that, after passing to a subsequence, there exists a constant C such that for all $j \in \mathbb{N}$ with $x'_j \in Q^{\pm}$:

$$\begin{aligned} &\varepsilon^{(j)}(t_j)^{-1}\,\rho_{\varepsilon^{(j)}(t_j)}(x'_j) < C \\ &\implies \kappa_{\varepsilon^{(j)}(t_j)}(r(x'_j)) < C\varepsilon^{(j)}(t_j) < \varepsilon^{(j)}(t_j)^{\tau}, \end{aligned}$$

implying that $y_j \in P_0^{(j)}$ lies in a compact region independent of j. Defining the time-independent weight function $\hat{\rho} : P_s^{(j)} \to \mathbb{R}^+$ as in Definition 3.6, we may use (6.9) and Assumption 6.1 to derive bounds on $\widetilde{u}^{(j)}$, $\widetilde{\psi}^{(j)}$ as follows:

- $t^{\mu}\rho_{t^{-\frac{1}{m-2}}}(x)^{\nu}|u^{(j)}(x,t)| \leqslant 1$ for $x \in \underline{N},\ t \in [\Lambda, t_j)$

$$\begin{aligned} &\implies |\widetilde{u}^{(j)}(y,s)| \leqslant \left(\frac{t_j}{t_j + \lambda_j^2 s}\right)^{\mu}\left(\frac{\varepsilon^{(j)}(t_j)}{\rho_{t(s)^{-\frac{1}{m-2}}}(x(y))}\right)^{\nu} && \text{for } s \in [\lambda_j^{-2}(\Lambda - t_j), \infty),\ y \in P_s^{(j)}, \\ &\implies |\widetilde{u}^{(j)}(y,s)| \leqslant C\hat{\rho}(y)^{-\nu} && \text{for } s \in [-\tfrac{1}{2}\lambda_j^{-2}t_j, 0),\ y \in P_s^{(j)}. \end{aligned}$$

- $\parallel \psi^{(j)} \parallel_{\mu,\nu+2,j} \to 0$

$$\begin{aligned} &\implies |\widetilde{\psi}^{(j)}(y,s)|\left(\frac{t_j + \lambda_j^2 s}{t_j}\right)^{\mu}\left(\frac{\rho_{t(s)^{-\frac{1}{m-2}}}(x(y,s))}{\varepsilon^{(j)}(t_j)}\right)^{\nu} \to 0 && \text{uniformly on } P^{(j)} \\ &\implies |\widetilde{\psi}^{(j)}(y,s)| \to 0 && \text{uniformly on } P^{(j)}. \end{aligned}$$

Now we calculate the PDE that is satisfied by $\widetilde{u}^{(j)}$ and $\widetilde{\psi}^{(j)}$. We consider the tip and intermediate regions of $P^{(j)}$ seperately.

In the tip region, $\overline{\kappa}_{\varepsilon^{(j)}(t)}^{-1} \circ S_t^{(j)} = \mathrm{Id}$. Then, by (5.10) and (6.5):

$$\partial_s\widetilde{u}^{(j)}(y,s) = t_j^{\mu}\lambda_j^{\nu+2}\partial_s u^{(j)}(x(y,s),t(s)) = t_j^{\mu}\lambda_j^{\nu+2}\left(\Delta_{g_j(t)}u^{(j)} + \psi^{(j)}\right) = \Delta_{g'_j(s))}\widetilde{u}^{(j)} + \widetilde{\psi}^{(j)}.$$

In the intermediate region,

$$\begin{aligned} \partial_s\widetilde{u}^{(j)}(y,s) &= t_j^{\mu}\lambda_j^{\nu+2}\left[\partial_t u^{(j)}(x,t) + \left(\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}r}(x)\right)^{-1}\frac{\mathrm{d}u^{(j)}}{\mathrm{d}r}(x,t)\frac{\mathrm{d}\varepsilon^{(j)}}{\mathrm{d}t}\left(-\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}\varepsilon^{(j)}}(x) + r(x)\right)\right] \\ &= t_j^{\mu}\lambda_j^{\nu+2}\left[\Delta_{g_j(t)}u^{(j)} + \psi^{(j)} - \left(\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}r}(x)\right)^{-1}\frac{\mathrm{d}u^{(j)}}{\mathrm{d}r}(x,t)\frac{\mathrm{d}\varepsilon^{(j)}}{\mathrm{d}t}\left(-\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}\varepsilon^{(j)}}(x) + r(x)\right)\right] \\ &= \Delta_{g'_j(s)}\widetilde{u}^{(j)} + \widetilde{\psi}^{(j)} - \lambda_j^2\left(\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}r}(x)\right)^{-1}\frac{\mathrm{d}\varepsilon^{(j)}}{\mathrm{d}t}\left(-\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}\varepsilon^{(j)}}(x) + r(x)\right)\frac{\mathrm{d}\widetilde{u}^{(j)}}{\mathrm{d}r}(y,s). \end{aligned}$$

We note that $\left|\lambda_j^2\left(\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}r}(x)\right)^{-1}\frac{\mathrm{d}\varepsilon^{(j)}}{\mathrm{d}t}\left(-\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}\varepsilon^{(j)}}(x) + r(x)\right)\right| \leqslant C \cdot t(s)^{\frac{1}{m-2}} \cdot t_j^{-\frac{2}{m-2}}$, so that the coefficient of $\frac{\mathrm{d}\widetilde{u}^{(j)}}{\mathrm{d}r}(y,s)$ converges to 0 on compact spacetime regions as $j \to \infty$. Therefore, passing to a subsequence, we have the convergences $y_j \to y_\infty$, $(P^{(j)}, g'_j) \to (L \times (-\infty,0), g_L)$ locally smoothly, $\widetilde{u}^{(j)} \to \overline{u}$ in $C^{1,2}_{\mathrm{loc}}$, $\widetilde{\psi}^{(j)} \to 0$ in C^0_{loc}, where $\overline{u}$ is an ancient solution to the heat equation

$$\partial_s\overline{u}(y,s) = \Delta\overline{u}(y,s), \quad (y,s) \in L \times (-\infty, 0),$$

satisfying $|\overline{u}(\cdot,\tau)| \leqslant c\hat{\rho}^{-\nu}$. Finally, $\overline{u}$ is nontrivial since $\overline{u}(y_\infty, 0) = 1$ by (6.9). This contradicts Proposition 6.4.

Case 2. $\lim_{j\to\infty} t_j^{\frac{1}{m-2}}\rho_{t_j^{-\frac{1}{m-2}}}(x_j') = \infty$ and $\lim_{j\to\infty}\rho_{t_j^{-\frac{1}{m-2}}}(x_j') = 0$.

In this case,

$$\rho_{t_j^{-\frac{1}{m-2}}}(x) = \begin{cases} t_j^{-\frac{1}{m-2}} r(x) & \text{near } R_1 \\ R_2 & \text{near } R_2, \end{cases}$$

and so the Case 2 assumption implies that there exists k such that $x_j' \in \Sigma_k\times(R_1,(1-\hbar)R_2)$, after passing to a subsequence. Suppresing the index k as before, we define the rescaled intermediate region $Q_s^{(j)}$ and rescalings of the functions $u^{(j)}$ and $\psi^{(j)}$:

$\lambda_j := \kappa_{\varepsilon^{(j)}(t_j)}(r(x_j')) = \rho_{\varepsilon^{(j)}(t_j)}(x_j') \to 0.$

$Q_s^{(j)} := \Sigma\times(\lambda_j^{-1}\varepsilon^{(j)}(t_j+\lambda_j^2 s)R_1,\ \lambda_j^{-1}(1-\hbar)R_2), \quad Q^{(j)} := \left\{(y,s)\ :\ y\in Q_s^{(j)},\ s\in[-\tfrac{1}{2}\lambda_j^{-2}t_j,0)\right\}$

$S^{(j)} : Q^{(j)} \to \Sigma\times(0,R_2), \quad S^{(j)}(\sigma,r) := (\sigma,\lambda_j r)$

$\widetilde{u}^{(j)} : Q^{(j)}\to\mathbb{R}, \quad \widetilde{u}^{(j)}(y,s) := t_j^{\mu}\lambda_j^{\nu}u^{(j)}\left(\overline{\kappa}_{\varepsilon^{(j)}}^{-1}\circ S^{(j)}(y), t_j+\lambda_j^2 s\right)$

$\widetilde{\psi}^{(j)} : Q^{(j)}\to\mathbb{R}, \quad \widetilde{\psi}^{(j)}(y,s) := t_j^{\mu}\lambda_j^{\nu+2}\psi^{(j)}(\overline{\kappa}_{\varepsilon^{(j)}}^{-1}\circ S^{(j)}(y), t_j+\lambda_j^2 s).$

Define $t(s)$, $x(y,s)$, y_j and g_j' as in Case 1. Using (6.9) we may derive bounds for $\widetilde{u}^{(j)}$:

$$\begin{aligned} &t^{\mu}\rho_{t^{-\frac{1}{m-2}}}(x)^{\nu}|u^{(j)}(x,t)| \leqslant 1 && \text{for } t\in[\Lambda,t_j], \\ \Longrightarrow\ &|\widetilde{u}^{(j)}(y,s)| \leqslant \left(\frac{t_j}{t_j+\lambda_j^2 s}\right)^{\mu} r(y)^{-\nu} && \text{for } s\in[\lambda_j^{-2}(\Lambda-t_j),0], \\ \Longrightarrow\ &|\widetilde{u}^{(j)}(y,s)| \leqslant C\, r(y)^{-\nu} && \text{for } s\in[-\tfrac{1}{2}\lambda_j^{-2}t_j,0]. \end{aligned}$$

The linear PDE satisfied by $\widetilde{u}^{(j)}$ is given by (5.10) and (6.5) in the same way as for Case 1 in the intermediate region:

$$\partial_s u^{(j)} \ = \ \Delta_{g_j'(s)}\widetilde{u}^{(j)} + \widetilde{\psi}^{(j)} + \lambda_j^2\left(\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}r}(x)\right)^{-1}\frac{\mathrm{d}\varepsilon^{(j)}}{\mathrm{d}t}\frac{\mathrm{d}\kappa_{\varepsilon^{(j)}}}{\mathrm{d}\varepsilon^{(j)}}(x)\frac{\mathrm{d}\widetilde{u}^{(j)}}{\mathrm{d}r}(y,s).$$

As in Case 1, after passing to a subsequence we have the convergences $\widetilde{u}^{(j)}\to\overline{u}$, $\widetilde{\psi}^{(j)}\to 0$, $Q^{(j)}\to(\Sigma\times(0,\infty))\times(\infty,0)$, $g_j'\to\overline{g}$, where $\overline{g}$ is the metric on $\Sigma\times(0,\infty)$ corresponding to the flat metric on two punctured copies of $\mathbb{R}^m$, and $\overline{u}$ satisfies

$$|\overline{u}(y,s)| \leqslant c\, r(y)^{-\nu}, \quad \frac{\partial\overline{u}}{\partial s} = \Delta_{\overline{g}}\overline{u} \quad \text{for } s\in(-\infty,0).$$

Furthermore, by interior parabolic Schauder estimate, $\overline{u}$ satisfies $|\nabla^k\overline{u}(y,s)| \leqslant C|y|^{-\nu-k}$, for $k\in\{0,1,2\}$. Finally, to show that $\overline{u}$ is nontrivial, note that

$$r(y_j) = \lambda_j^{-1}\rho_{t_j^{-\frac{1}{m-2}}}(x_j') = \frac{\rho_{t_j^{-\frac{1}{m-2}}}(x_j')}{\rho_{\varepsilon^{(j)}(t_j)}(x_j')} \in I := \left(\frac{1}{C_\varepsilon}, C_\varepsilon\right).$$

So passing to a subsequence if necessary, $y_j\to\overline{y}\in\Sigma\times I$, and by (6.9), $\overline{u}(\overline{y},0)\geqslant\frac{1}{2}$. This contradicts Proposition 6.5.

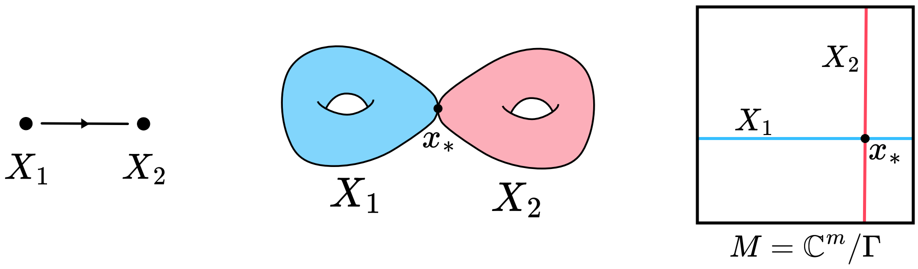

FIGURE 3. Three figures of two Lagrangian tori X_1, X_2 inside a complex torus M with a type 1 transverse intersection. In order, the figures depict: the graphical representation of section 4.3, a topological representation, a diagrammatic representation of the containment $X_1 \cup X_2 \subset M$.

Case 3. $\lim_{j\to\infty} \rho_{t_j^{-\frac{1}{m-2}}}(x'_j) > 0$.

In this case, taking C as in Assumption 6.1, we make the definitions:

$$Q_k^{(j)} := \left\{(y,s) \;:\; s \in [-\tfrac{1}{2}t_j, \infty),\; y \in \Sigma_k \times [C(t_j+s)^{-\frac{1}{m-2}}R_1, R_2]\right\}$$

$$X^{(j)} := \bigcup_{k=1}^{n} Q_k^{(j)} \cup \left(X^{\mathrm{o}} \times [-\tfrac{1}{2}t_j, \infty)\right)$$

$$\tilde{u}^{(j)} : X^{(j)} \to \mathbb{R}, \qquad \tilde{u}^{(j)}(y,s) := \begin{cases} t_j^{\mu} u^{(j)}(\overline{\kappa}_{\varepsilon(j)}^{-1}(y), t_j+s) & \text{on } Q_k^{(j)} \\ t_j^{\mu} u^{(j)}(y,\, t_j+s) & \text{on } X^{\mathrm{o}} \times [\Lambda - t_j, \infty), \end{cases}$$

$$\tilde{\psi}^{(j)} : X^{(j)} \to \mathbb{R}, \qquad \tilde{\psi}^{(j)}(y,s) := \begin{cases} t_j^{\mu} \psi^{(j)}(\overline{\kappa}_{\varepsilon(j)}^{-1}(y), t_j+s) & \text{on } Q_k^{(j)} \\ t_j^{\mu} \psi^{(j)}(y,\, t_j+s) & \text{on } X^{\mathrm{o}} \times [\Lambda - t_j, \infty), \end{cases}$$

We equip $X^{(j)}$ with the metric $g'_j(\cdot,s) := g_j(\cdot, t_j+s)$, so that we have the convergence $(X^{(j)}, g'_j) \to (X \backslash \iota^{-1}(\{x_1,\ldots,x_n\}), \iota^* g)$ in C^∞_{loc}. We may derive bounds on $\tilde{u}^{(j)}$ and $\tilde{\psi}^{(j)}$ as in Cases 1 and 2:

$$|\tilde{u}^{(j)}(y,s)| \leqslant C\,\rho_{t_j^{-\frac{1}{m-2}}}(\overline{\kappa}_{\varepsilon(j)}^{-1}(y))^{-\nu} \qquad \text{for } s \in [-\tfrac{1}{2}t_j, 0),$$

$$|\tilde{\psi}^{(j)}(y,s)| \to 0 \qquad \text{uniformly on compact subsets.}$$

After passing to a subsequence, we have convergences $\tilde{u}^{(j)} \to \overline{u}$, $\tilde{\psi}^{(j)} \to \overline{\psi}$, where $\overline{u}$ satisfies $\overline{u}(y,s) \leqslant c\rho(y)$ for $\rho : X \to [0,\infty)$ as in section 6.1.3. The Case 3 assumption implies that $\rho_{t_j^{-\frac{1}{m-2}}}(x'_j) \to P > 0$ so that $x_\infty \in X \backslash \iota^{-1}(\{x_1,\ldots,x_n\})$, and (6.9) implies that $\overline{u}(x_\infty, 0) \neq 0$. It follows from (6.5) that $\partial_s \overline{u} = \Delta_{\iota^* g}\overline{u}$, and (6.6) implies that $\int_{X_b \backslash \iota^{-1}(\{x_1,\ldots,x_n\})} \overline{u}\, dV_{\iota^* g} = 0$ for all $b \in \{1,\ldots,n'\}$. Finally, by interior parabolic Schauder estimate (see for example, [2, Proposition 7.3]), $\overline{u}$ satisfies $|\nabla^k \overline{u}(y,\tau)| \leqslant C\rho(y)^{-\nu-k}$, for $k \in \{0,1,2\}$. Proposition 6.6 now gives a contradiction. $\square$

7. Existence Theory for the Torus Case

For the remainder of this work, we will focus on a particular case of the preceding theory - flat special Lagrangians in complex tori.

Assumption 7.1. The Calabi–Yau manifold M and immersed special Lagrangian $\iota : X \to M$ take the following form:

- M is a complex torus, i.e. $M := \mathbb{C}^m/\Gamma$ for a lattice Γ (where $m \geqslant 3$), and the Calabi–Yau structure is induced from the standard one on $\mathbb{C}^m$.
- The underlying manifold X of the special Lagrangian immersion is the disjoint union of two m-tori, $X = X_1 \cup X_2$, and $\iota : X_b \to M$ is a special Lagrangian embedding for $b = 1, 2$. The map $\iota : X \to M$ has *only one* transverse self-intersection point of type 1 (as defined in Definition 2.17), which is denoted by $x_\star \in M$, and it may assumed that $\iota^{-1}(x_\star) = \{x_\star^-, x_\star^+\}$, where $x_\star^- \in X_1$ and $x_\star^+ \in X_2$.

Remark 7.2. It is fairly easy to construct examples. Consider the two planes Π^0 and Π^ϕ as (2.17) with $k = 1$. Choose a basis for Π^0 and a basis for Π^ϕ. Set the lattice to be generated by them.

Working under Assumption 7.1, the preceding theory has the following simplifications:

- The map $\Upsilon : B_R \to M$ of Lemma 2.18 may be taken to be the composition of an affine isometry $A \in \mathrm{SU}(m) \ltimes \mathbb{C}^m$ and the torus projection $\pi_\Gamma : \mathbb{C}^m \to M$; $\Upsilon = \pi_\Gamma \circ A$. It therefore satisfies the following strengthened properties: $\Upsilon^* g = g_0$, $\Upsilon^*\Omega = \Omega_0$.
- For the desingularisation of $\iota : X \to M$ constructed in section 3, only *one* Lawlor neck is needed. It is denoted by L. The constant in (2.21) is denoted by $c_L := c_+(L)$ (and $c_-(L)$ is set to be 0).
- The vector function $\boldsymbol{\varepsilon}(t)$ becomes a scalar function $\varepsilon(t)$.
- The desingularisation is denoted $\iota^\varepsilon : \underline{N} \to N^\varepsilon$. N^ε consists of one tip region, P, one intermediate region with two connected components, $Q^\pm$, and two connected components of the outer region, $X^{\mathrm{o}} = X_1^{\mathrm{o}} \cup X_2^{\mathrm{o}}$. The graph representation of X as in section 4.3 consists of two vertices, $\mathcal{V} = \{X_1^{\mathrm{o}}, X_2^{\mathrm{o}}\}$ with a single directed edge between them, $\mathcal{E} = \{P\}$. Namely, $n = 1$, $n' = 2$.
- The asymptotic cone C matches precisely with the outer region X^o, i.e. $\mathcal{A} = 0$.
- The approximate kernel defined in section 5.2 is spanned by two functions $\{1, w^\varepsilon_{(0,1)}\}$. To simplify the computations, we work instead with the basis $\{1, w^\varepsilon\}$, where w^ε is normalised to be orthogonal to 1:

$$w^\varepsilon := w^\varepsilon_{(0,1)} - \frac{1}{\mathrm{Vol}(N^\varepsilon)} \int_{\underline{N}} w^\varepsilon_{(0,1)} \mathrm{d}V_{g^\varepsilon}. \tag{7.1}$$

Since the underlying graph is a tree, Proposition 4.5 implies that the Lagrangian mean curvature flow equation may be expressed on the level of potential functions by (5.2). In particular, we define $\xi(\mathrm{d}u)$ by (5.1) using the following specific time-dependent constants:

$$\begin{aligned} C_P(t) = C_{Q^-}(t) = C_{X_1^o}(t) &:= -\frac{c_L V_2}{V_1+V_2}\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t}, \\ C_{Q^+}(t) = C_{X_2^o}(t) &:= \frac{c_L V_1}{V_1+V_2}\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t}, \end{aligned} \tag{7.2}$$

where as before, V_j denotes the volume of $\iota(X_j)$ for $j \in \{1,2\}$.

Our goal is now to find Λ, ε and u solving (5.2) under Assumption 7.1.

7.1. **Estimates in the Torus Case.** We will require the following estimates for the induced metric g^ε, its volume form $\mathrm{d}V_{g^\varepsilon}$, the nontrivial approximate kernel element w^ε, and the Lagrangian angle. Throughout we use Assumption 6.1 for estimating the time derivative and Hölder derivative of $\varepsilon(t)$, and for convenience use the notation $\partial\!\!\!/^{\alpha}_{t_1,t_2} f := \frac{f(t_1)-f(t_2)}{|t_1-t_2|^\alpha}$ for the Hölder quotient.

Lemma 7.3. *Let g_0 be the Euclidean metric on $\mathbb{R}^m$, and g_C be the cone metric on $\Sigma \times (0,\infty)$. Under Assumptions 6.1 and 7.1, the induced metric g^ε on $\underline{N}$ satisfies*

$$g^\varepsilon = g_0 \quad \text{on } X_b^o \cup (\Sigma \times [2\varepsilon^\tau, R_2)),\ b = 1,2, \tag{7.3}$$

$$|\nabla^k(g^\varepsilon - g_C)|_{g_C}(\sigma,\mathfrak{r}) = \begin{cases} O(\varepsilon^{2(1-\tau)m-\tau k}), & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon^\tau, 2\varepsilon^\tau), \\ O(\mathfrak{r}^{-2m-k}\varepsilon^{2m}), & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon R_1, \varepsilon^\tau), \end{cases},\ k = 0,1,2, \tag{7.4}$$

$$g^\varepsilon = \varepsilon^2 g_L \quad \text{on } P. \tag{7.5}$$

The volume form $\mathrm{d}V_{g^\varepsilon}$ on the tip region P satisfies (for $t_1, t_2 \in [t, 2t]$, $0 < |t_1 - t_2| < t^{-\frac{1}{m-2}}$):

$$\mathrm{d}V_{g^\varepsilon} = \varepsilon^m\,\mathrm{d}V_L, \quad \partial_\varepsilon\,\mathrm{d}V_{g^\varepsilon} = O(\varepsilon^{m-1})\,\mathrm{d}V_L, \quad \partial_t\,\mathrm{d}V_{g^\varepsilon} = O(\varepsilon(t)^{2m-2})\,\mathrm{d}V_L, \tag{7.6}$$

$$|\partial\!\!\!/^{\alpha}_{t_1,t_2}\,\mathrm{d}V_{g^\varepsilon}(t)| = O(\varepsilon(t)^{2m-2\alpha})\,\mathrm{d}V_L, \tag{7.7}$$

$$|\partial\!\!\!/^{\alpha}_{t_1,t_2}\partial_t\,\mathrm{d}V_{g^\varepsilon}(t)| = O(\varepsilon(t)^{2m-2-2\alpha})\,\mathrm{d}V_L. \tag{7.8}$$

The volume form $\mathrm{d}V_{g^\varepsilon}$ on $\Sigma \times (\varepsilon R_1, 2\varepsilon^\tau)$ satisfies (for $t_1, t_2 \in [t, 2t]$, $|t_1 - t_2| < t^{-\frac{2}{m-2}}$):

$$\mathrm{d}V_{g^\varepsilon} = \begin{cases} (1 + O(\varepsilon^{m(1-\tau)}))\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon^\tau, 2\varepsilon^\tau), \\ (1 + O(\mathfrak{r}^{-m}\varepsilon^m))\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon R_1, \varepsilon^\tau), \end{cases} \tag{7.9}$$

$$\partial_\varepsilon \mathrm{d}V_{g^\varepsilon} = \begin{cases} O(\varepsilon^{m(1-\tau)-1})\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon^\tau, 2\varepsilon^\tau), \\ O(\mathfrak{r}^{-m}\varepsilon^{m-1})\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon R_1, \varepsilon^\tau), \end{cases} \tag{7.10}$$

$$|\partial\!\!\!/^{\alpha}_{t_1,t_2}\,\mathrm{d}V_{g^\varepsilon}(t)| = \begin{cases} O(\varepsilon^{m(2-\tau)-2\alpha})\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon^\tau, 2\varepsilon^\tau), \\ O(\mathfrak{r}^{-m}\varepsilon^{2m-2\alpha})\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon R_1, \varepsilon^\tau), \end{cases} \tag{7.11}$$

$$|\partial\!\!\!/^{\alpha}_{t_1,t_2}\partial_t \mathrm{d}V_{g^\varepsilon}(t)| = \begin{cases} O(\varepsilon^{m(2-\tau)-2-2\alpha})\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon^\tau, 2\varepsilon^\tau), \\ O(\mathfrak{r}^{-m}\varepsilon^{2m-2-2\alpha})\,\mathrm{d}V_C, & (\sigma,\mathfrak{r}) \in \Sigma \times (\varepsilon R_1, \varepsilon^\tau). \end{cases} \tag{7.12}$$

Proof. We prove only the estimates for the tip region $P \subset \underline{N}$; the arguments for the other regions are analogous. Choose local coordinates for P, so that the induced metric is $(g^\varepsilon)_{ij} = g\left(\frac{\partial \iota^\varepsilon}{\partial x_i}, \frac{\partial \iota^\varepsilon}{\partial x_j}\right)$. It follows that

$$(g^\varepsilon)_{ij} = O(\varepsilon^2), \quad \partial_\varepsilon (g^\varepsilon)_{ij} = g\left(\frac{\partial^2 \iota^\varepsilon}{\partial x_i \partial \varepsilon}, \frac{\partial \iota^\varepsilon}{\partial x_j}\right) + g\left(\frac{\partial \iota^\varepsilon}{\partial x_i}, \frac{\partial^2 \iota^\varepsilon}{\partial x_j \partial \varepsilon}\right). \tag{7.13}$$

We note that $\iota^\varepsilon = \Upsilon(\varepsilon(t)\iota_L)$, where $\iota_L : L \to \mathbb{C}^m$ is the inclusion map of the Lawlor neck, so $\frac{\mathrm{d}\iota^\varepsilon}{\partial \varepsilon} = \Upsilon_*(\iota_L) = O(1)$. Therefore, we calculate using (7.13):

$$\partial_\varepsilon (g^\varepsilon)_{ij} = O(\varepsilon), \quad \partial_\varepsilon^2 (g^\varepsilon)_{ij} = 2g\left(\frac{\partial^2 \iota^\varepsilon}{\partial x_i \partial \varepsilon}, \frac{\partial^2 \iota^\varepsilon}{\partial x_j \partial \varepsilon}\right) = O(1), \quad \partial_t (g^\varepsilon)_{ij} = \varepsilon'(t)\partial_\varepsilon (g^\varepsilon)_{ij} = O(\varepsilon^m)$$
$$\not{\partial}^\alpha_{t_1 t_2} (g^\varepsilon)_{ij} = \partial_\varepsilon (g^\varepsilon)_{ij}\big|_{t=b} \cdot \varepsilon'(a) \cdot |t_1 - t_2|^{1-\alpha} = O(\varepsilon^{m+2-2\alpha})$$
$$\not{\partial}^\alpha_{t_1 t_2} \partial_t (g^\varepsilon)_{ij} = \not{\partial}^\alpha_{t_1 t_2}(\varepsilon' \partial_\varepsilon (g^\varepsilon)_{ij}) = \not{\partial}^\alpha_{t_1 t_2}(\varepsilon')\partial_\varepsilon (g^\varepsilon)_{ij}\big|_{t=t_1} + \varepsilon'(t_2)\not{\partial}^\alpha_{t_1 t_2}\partial_\varepsilon (g^\varepsilon)_{ij} = O(\varepsilon^{m-2\alpha}),$$

where $a, b \in [t_1, t_2]$. Using these estimates, along with the fact that there exist c, C such that $c\,\varepsilon^{2m} \leqslant \det(g^\varepsilon) \leqslant C\,\varepsilon^{2m}$, we may bound the volume element and its derivatives as required. □

Lemma 7.4. *Under Assumptions 6.1 and 7.1, the function w^ε satisfies (for $t_1, t_2 \in [t, 2t]$, $|t_1 - t_2| < t^{-\frac{2}{m-2}}$):*

$$\begin{aligned}
|w^\varepsilon| &\leqslant 1, \\
|\partial_t w^\varepsilon| &= \begin{cases} O(\varepsilon^{m-2}) & \text{on } \overline{\kappa}_\varepsilon^{-1}(\Sigma \times (\varepsilon R_1, \varepsilon^\tau)), \\ 0 & \text{otherwise}, \end{cases} \\
|\not{\partial}^\alpha_{t_1,t_2} \partial_t w^\varepsilon| &= \begin{cases} O(\varepsilon^{m-2-2\alpha}) & \text{on } \overline{\kappa}_\varepsilon^{-1}(\Sigma \times (\varepsilon R_1, \varepsilon^\tau)) \\ 0 & \text{otherwise}, \end{cases} \\
|\Delta_{g^\varepsilon} w^\varepsilon| &\leqslant \begin{cases} \kappa_\varepsilon(r)^{-m}\varepsilon^{m-2} & \text{on } \overline{\kappa}_\varepsilon^{-1}(\Sigma \times (\varepsilon R_1, \varepsilon^\tau)), \\ 0 & \text{otherwise}, \end{cases} \\
|\not{\partial}^\alpha_{t_1,t_2} \Delta_{g^\varepsilon} w^\varepsilon| &\leqslant \begin{cases} \kappa_\varepsilon(r)^{-m}\varepsilon^{2m-2-2\alpha} & \text{on } \overline{\kappa}_\varepsilon^{-1}(\Sigma \times (\varepsilon R_1, \varepsilon^\tau)), \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. The spatial estimates of w^ε follow from [13, Proposition 7.3]. For the time derivative, note that $\partial_t w^\varepsilon_{\mathbf{d}} = 0$ on X°_b and P from the definition of $w^\varepsilon_{\mathbf{d}}$. On Q, we compute

$$\begin{aligned}
\partial_t \kappa_\varepsilon &= \left[1 - \chi\left(\frac{r - R_1}{\hbar R_1}\right)\right]\varepsilon' r = O(\varepsilon'), \\
\partial_t \underline{\alpha} &= (\alpha_L\big|_{\underline{0}} \circ \varphi)(\sigma, \varepsilon^{-1}\kappa_\varepsilon(r))[\varepsilon^{-2}\varepsilon'\kappa_\varepsilon(r) + \varepsilon^{-1}\partial_t \kappa_\varepsilon] \\
&= O(\varepsilon^{(1-\tau)(m-3)}\varepsilon^{-1}\varepsilon') \quad \text{for } r \in \kappa_\varepsilon^{-1}(\varepsilon R_1, \varepsilon^\tau).
\end{aligned}$$

The result now follows from a calculation. The Hölder derivative estimates follow similarly, using the fact that $|t_1 - t_2| < t^{-\frac{2}{m-2}} \implies |\not{\partial}^\alpha_{t_1,t_2} f| \leqslant C \cdot |\partial_t f(c)|\varepsilon^{2-2\alpha}$ for some $c \in [t_1, t_2]$. □

Lemma 7.5. *Under Assumption 7.1, for any* $\tau \in (0, \frac{1}{2})$ *and* $k \in \{0, 1, 2\}$*, we have*

$$|\nabla^k \theta_{N^\varepsilon}(x)| = \begin{cases} O(\varepsilon^{m(1-\tau)-k\tau}) & x = (\sigma, \mathfrak{r}),\ \mathfrak{r} \in (\varepsilon^\tau, 2\varepsilon^\tau), \\ 0, & \textit{otherwise}, \end{cases} \tag{7.14}$$

where $|\cdot|$ *is computed using the pullback metric* g^ε*.*

Proof. The proof follows as in [13, Proposition 6.4], with the improvements coming from the fact that by Assumption 7.1, $\mathcal{A} = 0$ and $\Upsilon^* g = g_0$. $\square$

7.2. **Weighted Parabolic Hölder Spaces.** We define suitable Hölder spaces for our differential operators. Given $\Lambda > 0$, $(\mu, \nu) \in \mathbb{R}^2$, $\alpha \in (0, \frac{1}{2})$, and a time-dependent tensor T on $\underline{N}$, and letting $\mathrm{inj}(g^\varepsilon)$ denote the injectivity radius of the induced metric $g^\varepsilon = (\iota^\varepsilon)^* g$, we define:

$$\|T\|_{\mu,\nu,\Lambda} := \sup_{(x,t) \in \underline{N} \times [\Lambda, \infty)} t^\mu \rho^\nu_{t^{\frac{-1}{m-2}}}(x) |T|_{g^\varepsilon}, \tag{7.15}$$

$$[T]_{\mu,\nu,\alpha,\Lambda} := \sup_{t \in [\Lambda,\infty)} \sup_{\substack{x_1, x_2 \in \underline{N} \\ d_{g^\varepsilon}(x_1,x_2) < \min\{\mathrm{inj}(g^\varepsilon),1\}}} t^\mu \min\{\rho^{\nu+2\alpha}_{t^{\frac{-1}{m-2}}}(x_1), \rho^{\nu+2\alpha}_{t^{\frac{-1}{m-2}}}(x_2)\} \frac{|T(x_1,t) - T(x_2,t)|_{g^\varepsilon}}{d_{g^\varepsilon}(x_1,x_2)^{2\alpha}}, \tag{7.16}$$

$$\langle T \rangle_{\mu,\nu,\alpha,\Lambda} := \sup_{x \in \underline{N}} \sup_{t > \Lambda} \sup_{\substack{t_1, t_2 \in [t, 2t], \\ 0 < |t_1 - t_2| < t^{\frac{-2}{m-2}}}} t^\mu \rho^{\nu+2\alpha}_{t^{\frac{-1}{m-2}}}(x) \frac{|T(x,t_1) - T(x,t_2)|_{g^\varepsilon}}{|t_1 - t_2|^\alpha}. \tag{7.17}$$

Here, the norms are computed by the induced metric on the corresponding tensor bundles, and the difference $T(x_1, t) - T(x_2, t)$ is understood using the parallel transport along the unique shortest geodesic between x_1 and x_2 to compare the values.

Definition 7.6. Define a weighted parabolic Hölder norm for tensors T on $\underline{N}$ by

$$\|T\|_{P^{0,0,\alpha}_{\mu,\nu,\Lambda}} := \|T\|_{\mu,\nu,\Lambda} + [T]_{\mu,\nu,\alpha,\Lambda} + \langle T \rangle_{\mu,\nu,\alpha,\Lambda}. \tag{7.18}$$

The weighted parabolic Hölder spaces $P^{l,k,\alpha}_{\mu,\nu,\Lambda}$ are then defined to be the space of functions $u : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$ such that the norm

$$\|u\|_{P^{l,k,\alpha}_{\mu,\nu,\Lambda}} := \sum_{i=0}^{l} \|\partial_t^i u\|_{P^{0,0,\alpha}_{\mu,\nu+2i,\Lambda}} + \sum_{j=0}^{k} \|\nabla^j u\|_{P^{0,0,\alpha}_{\mu,\nu+j,\Lambda}} \tag{7.19}$$

is finite. Analogously, we define the weighted parabolic Hölder norm $\|\cdot\|_{C^{0,\alpha}_{\zeta,\Lambda}}$ (and corresponding Banach space $C^{0,\alpha}_{\zeta,\Lambda}$) for functions $h : [\Lambda, \infty) \to \mathbb{R}$:

$$\|h\|_{C^{0,\alpha}_{\zeta,\Lambda}} := \sup_{t \in [\Lambda,\infty)} t^\zeta |h(t)| + \sup_{t \in [\Lambda,\infty)} \sup_{\substack{t_1,t_2 \in [t,2t], \\ 0 < |t_1 - t_2| < t^{\frac{-2}{m-2}}}} t^\zeta \frac{|h(t_1) - h(t_2)|}{|t_1 - t_2|^\alpha}. \tag{7.20}$$

In order to apply the Schauder fixed point theorem to solve our nonlinear PDE for functions belonging to these spaces, we will require the following compact embedding theorem.

Lemma 7.7. *For $\mu' < \mu$, $\alpha' < \alpha$, $\zeta' < \zeta$, and $\Lambda > 1$, the inclusions*

$$C^{0,\alpha}_{\zeta,\Lambda} \hookrightarrow C^{0,\alpha'}_{\zeta',\Lambda}, \quad P^{l,k,\alpha}_{\mu,\nu,\Lambda} \hookrightarrow P^{l,k,\alpha'}_{\mu',\nu,\Lambda}$$

are compact.

Proof. For the first inclusion, consider a bounded sequence $\{h_k\}_{k=1}^{\infty}$ in the unit ball of $C^{0,\alpha}_{\zeta,\Lambda}$. We aim to show that there is a subsequence which is Cauchy in $C^{0,\alpha'}_{\zeta',\Lambda}$. By Arzelà–Ascoli and a diagonal argument, we may pass to a subsequence such that $\|h_k - h\|_{C^0([\Lambda,\Lambda'])} \to 0$ for a unique $h \in C^0([\Lambda,\infty))$ and for any $\Lambda' > \Lambda$.

We now show that this subsequence is Cauchy in $C^{0,\alpha'}_{\zeta',\Lambda}$. Fix $\varepsilon > 0$ and choose $\Lambda' > \Lambda$ such that $2(\Lambda')^{\zeta'-\zeta} < \varepsilon$. We estimate separately on the intervals $[\Lambda,\Lambda']$ and $[\Lambda',\infty)$ as follows:

$$\begin{aligned}
&\|h_k - h_l\|_{C^{0,\alpha'}_{\zeta',\Lambda}([\Lambda,\Lambda'])} = \sup_{t\in[\Lambda,\Lambda']}\left(t^{\zeta'}|h_k(t)-h_l(t)| + \sup_{t_1,t_2\dots} t^{\zeta'}\frac{|(h_k-h_l)(t_1)-(h_k-h_l)(t_2)|}{|t_1-t_2|^{\alpha'}}\right)\\
&\leqslant (\Lambda')^{\zeta'}\left\{\|h_k-h_l\|_{C^0([\Lambda,\Lambda'])} + \sup_{t_1,t_2\dots}\left[\frac{|(h_k-h_l)(t_1)-(h_k-h_l)(t_2)|}{|t_1-t_2|^{\alpha}}\right]^{\frac{\alpha'}{\alpha}}\left(2\|h_k-h_l\|_{C^0([\Lambda,\Lambda'])}\right)^{\frac{\alpha}{\alpha'}}\right\}\\
&\leqslant C\left[\|h_k-h_l\|_{C^0([\Lambda,\Lambda'])} + \left(\|h_k\|_{C^{0,\alpha}([\Lambda,\Lambda'])} + \|h_l\|_{C^{0,\alpha}([\Lambda,\Lambda'])}\right)^{\frac{\alpha'}{\alpha}}\left(\|h_k-h_l\|_{C^0([\Lambda,\Lambda'])}\right)^{\frac{\alpha}{\alpha'}}\right],\\
&\|h_k-h_l\|_{C^{0,\alpha'}_{\zeta',\Lambda}([\Lambda',\infty))} \leqslant (\Lambda')^{\zeta'-\zeta}\|h_k-h_l\|_{C^{0,\alpha}_{\zeta,\Lambda}} \leqslant \varepsilon.
\end{aligned}$$

Since the first estimate tends to 0 as $k,l\to\infty$, it follows that $\{h_k\}_{k=1}^{\infty}$ is Cauchy in $C^{0,\alpha'}_{\zeta',\Lambda}$ as required.

For the second inclusion, we consider the case $(l,k)=(0,0)$ for simplicity (the general case is proven similarly). Take a bounded sequence $\{u_k\}_{k=1}^{\infty}$ in the unit ball of $P^{0,0,\alpha}_{\mu,\nu,\Lambda}$. By Arzelà–Ascoli and a diagonal argument we may pass to a subsequence such that $\|u_k - u\|_{C^0(\underline{N}\times[\Lambda,\Lambda'])} \to 0$ for a unique $u : \underline{N}\times[\Lambda,\infty) \to \mathbb{R}$ and any $\Lambda' > \Lambda$. We now show that this subsequence is Cauchy in $P^{0,0,\alpha'}_{\mu',\nu,\Lambda}$. Fix $\varepsilon > 0$, and define Λ' so that $2(\Lambda')^{\mu'-\mu} \leqslant \varepsilon$. We split the domain into $\underline{N}\times[\Lambda,\Lambda']$ and $\underline{N}\times[\Lambda',\infty)$, and estimate the norm on each separately as follows.

On $\underline{N}\times[\Lambda,\Lambda']$: Both t and ρ are uniformly bounded above and below, and so $\|u_k - u_l\|_{P^{0,0,\alpha'}_{\mu',\nu,\Lambda}(\underline{N}\times[\Lambda,\Lambda'])}$ may be proven to converge to 0 as in the previous case, by considering each constituent seminorm and estimating in terms of $\|u_k - u_l\|_{C^0([\Lambda,\Lambda'])}$.

On $\underline{N}\times[\Lambda',\infty)$:

$$\|u_k - u_l\|_{P^{0,0,\alpha'}_{\mu',\nu,\Lambda}(\underline{N}\times[\Lambda',\infty)]} \leqslant (\Lambda')^{\mu'-\mu}\|u_k-u_l\|_{P^{0,0,\alpha}_{\mu,\nu,\Lambda}} \leqslant \varepsilon.$$

We therefore have proven that $\{u_k\}_{k=1}^{\infty}$ is Cauchy in $P^{0,0,\alpha'}_{\mu',\nu,\Lambda}$, as required.

□

7.3. **A Priori Estimates and Existence Theory for the Linearised Operator.** We now proceed with the linear theory for our linearised operator, which will be viewed as a bounded operator on the weighted Hölder spaces of Definition 7.6. The main result is Theorem 7.12.

Since the linearised operator has a non-trivial kernel, we prove our estimates and existence theory on the orthogonal complement of the approximate kernel, which will be denoted by:

$$\langle 1, w^\varepsilon\rangle^\perp := \left\{ u \in C^0([\Lambda,\infty), L^2(\underline{N}, g^\varepsilon)) \;:\; \int_{\underline{N}} u \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon} = \int_{\underline{N}} u \cdot 1 \, \mathrm{d}V_{g^\varepsilon} = 0, \; \forall\, t \in [\Lambda,\infty) \right\}. \tag{7.21}$$

We will first consider the simpler case of the heat operator. It is clear from the definition that we have the following.

Lemma 7.8. *Let $\mu > 0$, $\nu \in (0, m-2)$, $\alpha \in (0, 1/2)$. The linear operator $\partial_t - \mathcal{L}^\varepsilon_{\underline{0}} : C^\infty_c(\underline{N} \times (\Lambda,\infty)) \to C^\infty_c(\underline{N} \times (\Lambda,\infty))$ extends to a bounded operator*

$$\partial_t - \Delta_{g^\varepsilon} : P^{1,2,\alpha}_{\mu,\nu,\Lambda} \longrightarrow P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}.$$

We first note that our a priori estimate for the heat operator implies the following weighted Schauder estimate.

Corollary 7.9. *Let $\mu > 0$, $\nu \in (0, m-2)$, $\alpha \in (0, \frac{1}{2})$ and $\Lambda > 0$. There exists a constant $C > 0$ such that if $u \in P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^\varepsilon\rangle^\perp$ and $\psi \in P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$ solves the Cauchy problem*

$$\begin{cases} \partial_t u - \Delta_{g^\varepsilon} u = \psi & t \in [\Lambda,\infty), \\ u(x,\Lambda) = 0, & x \in \underline{N}, \end{cases} \tag{7.22}$$

then

$$\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant C \, \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}}.$$

Proof. By the scaling property of the induced metric g^ε and the standard interior Schauder estimate, we have

$$\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant C \left(\|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} + \|u\|_{\mu,\nu,\Lambda} \right).$$

Since $u \in \langle 1, w^\varepsilon\rangle^\perp$, we may apply Theorem 6.7 to bound $\|u\|_{\mu,\nu,\Lambda}$ in terms of $\|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}}$. □

Supposing now that $u \in P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^\varepsilon\rangle^\perp$ satisfies

$$\begin{cases} \partial_t u - \Delta_{g^\varepsilon} u = \psi + a(t) + b(t) w^\varepsilon & t \in [\Lambda,\infty), \\ u(x,\Lambda) = 0, & x \in \underline{N}, \end{cases} \tag{7.23}$$

then by the Schauder estimate above we have

$$\begin{aligned} \|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} &\leqslant C \|\psi + a(t) + b(t) w^\varepsilon\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \\ &\leqslant C\|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} + C\|a(t)\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} + C\|b(t) w^\varepsilon\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}}. \end{aligned} \tag{7.24}$$

It is therefore important to estimate $a(t)$ and $b(t)$ in terms of u and ψ.

Lemma 7.10. *Consider $u \in P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^\varepsilon\rangle^\perp$ and $\psi \in P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$ satisfying (7.23). Then:*

$$a(t) = \frac{1}{\mathrm{Vol}(N^\varepsilon)} \int_{\underline{N}} (\partial_t u - \psi) \, \mathrm{d}V_{g^\varepsilon}, \quad b(t) = \frac{1}{\|w^\varepsilon\|^2_{L^2}} \int_{\underline{N}} \left(\partial_t u - \Delta_{g^\varepsilon_{\underline{0}}} u - \psi \right) \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon},$$

and $a(t)$ and $b(t)$ satisfy the estimates

$$\|a(t)\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant C\|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} + C\Lambda^{-\frac{2m-2-2\alpha-\nu\tau}{m-2}}\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}},$$
$$\|b(t)w^\varepsilon\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant C\|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} + C\Lambda^{-\frac{m-2-2\alpha\tau-\nu\tau}{m-2}}\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}.$$

Proof. The formulae for $a(t)$, $b(t)$ are obtained by integrating the differential equation against the elements of the approximate kernel $\{1, w^\varepsilon\}$, and using the orthogonality conditions.

For the estimates, recall that by Assumption 6.1,

$$|t_1 - t_2| < t^{-\frac{2}{m-2}} \implies |\not{\partial}^\alpha_{t_1,t_2} f| \leqslant C \cdot |\partial_t f(c)|\varepsilon^{2-2\alpha}$$

for some $c \in [t_1, t_2]$, where for convenience we use the notation $\not{\partial}^\alpha_{t_1,t_2} f := \frac{f(t_1)-f(t_2)}{|t_1-t_2|^\alpha}$ for the Hölder quotient. Differentiating the orthogonality condition and using the estimates on the volume form from Lemma 7.3 yields

$$\begin{aligned}
t^\mu \rho^{\nu+2}_{t^{-\frac{1}{m-2}}}(x)|a(t)| &\leqslant C \cdot \sup_{t\in[\Lambda,\infty)} \left(\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \int_{\underline{N}} \rho^{-\nu}|\partial_t(\mathrm{d}V_{g^\varepsilon})| + \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \int_{\underline{N}} \rho^{-\nu-2}\,\mathrm{d}V_{g^\varepsilon} \right) \\
&\leqslant C \cdot \sup_{t\in[\Lambda,\infty)} \Bigg\{ \|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \left(\int_P \rho^{-\nu}\varepsilon^{2m-2}\,\mathrm{d}V_L + \int_{\varepsilon R_1}^{2\varepsilon^\tau} \mathfrak{r}^{-\nu-1}\varepsilon^{2m-2}\,\mathrm{d}\mathfrak{r} \right) \\
&\qquad + \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \left(\int_P \rho^{-\nu-2}\varepsilon^m\,\mathrm{d}V_L + \int_{\varepsilon R_1}^{R_2} \mathfrak{r}^{-\nu+m-3}\,\mathrm{d}\mathfrak{r} + \int_{X^o} \mathrm{d}V_{g^\varepsilon} \right) \Bigg\} \\
&\leqslant C \cdot \sup_{t\in[\Lambda,\infty)} \left(\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \cdot \varepsilon^{2m-2-\tau\nu} + \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \right), \\
t^\mu \rho^{\nu+2+2\alpha}_{t^{-\frac{1}{m-2}}}(x)|\not{\partial}^\alpha_{t_1,t_2} a(t)| &\leqslant C \cdot \sup_{t\in[\Lambda,\infty)} \Bigg(\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \int_{\underline{N}} \rho^{-\nu-2\alpha}|\partial_t(\mathrm{d}V_{g^\varepsilon})| + \rho^{-\nu}|\not{\partial}^\alpha_{t_1,t_2}\partial_t(\mathrm{d}V_{g^\varepsilon})| \\
&\qquad + \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \int_{\underline{N}} \rho^{-\nu-2-2\alpha}\,\mathrm{d}V_{g^\varepsilon} + \rho^{-\nu-2}|\not{\partial}^\alpha_{t_1,t_2}(\mathrm{d}V_{g^\varepsilon})| \Bigg) \\
&\leqslant C \cdot \sup_{t\in[\Lambda,\infty)} \left(\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \varepsilon^{2m-2-2\alpha-\tau\nu} + \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \right),
\end{aligned}$$

which implies the estimate for $a(t)$. The estimate for $b(t)$ follows analogously, using the estimates from Lemma 7.4 and the fact that w^ε, $\|w^\varepsilon\|_{L^2}$ are uniformly bounded. □

Finally, to extend the above estimates from the heat operator to our linearised operator $\mathcal{L}^\varepsilon_{\underline{0}}$, we require the following estimate on the difference between the Laplacian and the linearised operator:

Lemma 7.11. *Given $\tau < \frac{1}{m+2}$, we have the decomposition $\mathcal{L}^\varepsilon_{\underline{0}} = \Delta_{g^\varepsilon} + \mathcal{P}^\varepsilon_{\underline{0}}$, where $\mathcal{P}^\varepsilon_{\underline{0}}$ is a first order differential operator satisfying*

$$|\mathcal{P}^\varepsilon_{\underline{0}}[u]| \leqslant \begin{cases} C\,\varepsilon(t)^{m-1}|\mathrm{d}u|_{g^\varepsilon} & \text{on } P_j \cup Q^\pm_j,\ t \geqslant \Lambda, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, there exists $C(\Lambda) > 0$ with $\lim_{\Lambda\to\infty} C(\Lambda) = 0$ such that

$$\|\mathcal{P}^\varepsilon_{\underline{0}}[u]\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant C(\Lambda)\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}.$$

As a result, $\mathcal{L}^{\varepsilon}_{\underline{0}}$ *extends to a bounded operator* $\mathcal{L}^{\varepsilon}_{\underline{0}} : P^{1,2,\alpha}_{\mu,\nu,\Lambda} \to P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$.

Proof. By Proposition 5.2 we have

$$\mathcal{L}^{\varepsilon}_{\underline{0}}[u] = \Delta_{g^{\varepsilon}} u - \langle \nabla\theta_{N^{\varepsilon}}, \widehat{V}_{\underline{0}}(\mathrm{d}u)\rangle_{g^{\varepsilon}} + S^{\varepsilon}[u],$$

where $S^{\varepsilon}[u]$ is a first order linear differential operator defined by

$$S^{\varepsilon}[u] = \begin{cases} (2\log\varepsilon_j(t))'(\mathrm{d}\beta_{L_j})(\mathrm{d}u^V) & \text{on } P_j\ , \\ \varepsilon_j'(t)\cdot\frac{\partial_{\varepsilon}\kappa_{\varepsilon_j(t)}}{\partial_r\kappa_{\varepsilon_j(t)}}\cdot\partial_r u & \text{on } Q_j^{\pm}\ , \\ 0 & \text{on } X_b^{\mathrm{o}}\ . \end{cases}$$

Hence,

$$\mathcal{P}^{\varepsilon}_{\underline{0}}[u] = -\langle \nabla\theta_{N^{\varepsilon}}, \widehat{V}_{\underline{0}}(\mathrm{d}u)\rangle_{g^{\varepsilon}} + S^{\varepsilon}[u].$$

By Lemma 7.5,

$$|\langle \nabla\theta_{N^{\varepsilon}}, \widehat{V}_{\underline{0}}(\mathrm{d}u)\rangle_{g^{\varepsilon}}| \leqslant C\varepsilon(t)^{m(1-\tau)-\tau}|\mathrm{d}u|_{g^{\varepsilon}}, \quad \text{on } Q_j^{\pm},$$

By Assumption 6.1, we have

$$|S^{\varepsilon}[u]| \leqslant C\varepsilon(t)^{m-1}|\mathrm{d}u|_{g^{\varepsilon}}, \quad \text{on } P_j \cup Q_j^{\pm}.$$

Combining these together and using the assumption $\tau < \frac{1}{m+2}$ yields

$$|\mathcal{P}^{\varepsilon}_{\underline{0}}[u]| \leqslant C\,\varepsilon(t)^{m-1}|\mathrm{d}u|_{g^{\varepsilon}}.$$

From this, we further estimate:

$$t^{\mu}\rho_{\varepsilon}^{\nu+2}|\mathcal{P}^{\varepsilon}_{\underline{0}}[u]| \leqslant C\varepsilon(t)^{m-1}\rho_{\varepsilon}\left(t^{\mu}\rho_{\varepsilon}^{\nu+1}|\mathrm{d}u|_{g^{\varepsilon}}\right) \leqslant C\Lambda^{\frac{1-m}{m-2}}\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}$$

if $\tau \in (0, \frac{1}{m+2})$. The Hölder norm estimate follows similarly, by using $|\nabla\mathrm{d}\theta_{N^{\varepsilon}}| = O(\varepsilon^{m(1-\tau)-2\tau})$ and $|\partial_t\nabla\theta_{N^{\varepsilon}}| = O(\varepsilon^{m-2+\tau})$ from Lemma 7.5. □

We now combine these estimates to deduce the estimates and existence theory for $\partial_t - \mathcal{L}^{\varepsilon}_{\underline{o}}$.

Theorem 7.12. *Given* $\mu > 0$, $\nu \in (0, m-2)$, $\alpha \in (0, \frac{1}{2})$, $\tau \in (0, \frac{1}{m+2})$, *there exists* $\Lambda \gg 1$ *with the following significance. Given* $\psi \in P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$, *there exists a unique* $u \in P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^{\varepsilon}\rangle^{\perp}$ *and* $a, b : [\Lambda, \infty) \to \mathbb{R}$ *such that*

$$\begin{cases} \partial_t u - \mathcal{L}^{\varepsilon}_{\underline{0}}[u] = \psi + a(t) + b(t)w^{\varepsilon}, & t \in [\Lambda, \infty), \\ u(x, \Lambda) = 0, \quad x \in \underline{N}, \end{cases} \tag{7.25}$$

and u *satisfies the a priori estimate*

$$\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant C\|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \tag{7.26}$$

for some $C > 0$ *independent of* t.

Proof. First, we claim that, given $\psi \in P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$, there exists $\Lambda_0 \gg 1$ such that for each $\Lambda \geqslant \Lambda_0$, there exists a unique $u : \underline{N} \times [\Lambda, \infty) \to \mathbb{R}$ solving

$$\begin{cases} \partial_t u - \Delta_{g^\varepsilon} u = \psi + a(t) + b(t) w^\varepsilon & t \in [\Lambda, \infty), \\ u(x, \Lambda) = 0, & x \in \underline{N}, \end{cases} \tag{7.27}$$

with estimate $\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant C \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}}$, where $C > 0$ is independent of Λ.

For this purpose, define a zeroth order operator

$$F^\varepsilon[u] := \frac{1}{\|w^\varepsilon\|^2_{L^2}} \left\{ \int_{\underline{N}} u \cdot (\partial_t w^\varepsilon + \Delta_{g^\varepsilon} w^\varepsilon) \, \mathrm{d}V_{g^\varepsilon} + \int_{\underline{N}} u \cdot w^\varepsilon \, \partial_t \mathrm{d}V_{g^\varepsilon} \right\} + \frac{1}{\mathrm{Vol}(N^\varepsilon)} \int_{\underline{N}} u \, \partial_t \mathrm{d}V_{g^\varepsilon}.$$

Note that $F^\varepsilon[u]$ encodes how the orthogonality condition is changed in time. Let

$$\psi^\perp := \psi - \frac{1}{\mathrm{Vol}(N^\varepsilon)} \int_{\underline{N}} \psi \, \mathrm{d}V_{g^\varepsilon} - \frac{1}{\|w^\varepsilon\|^2_{L^2}} \int_{\underline{N}} \psi \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon} \cdot w^\varepsilon.$$

By standard parabolic theory, there exists $u : \underline{N} \times [\Lambda, \Lambda + T] \to \mathbb{R}$ solving

$$\begin{cases} \partial_t u - \Delta_{g^\varepsilon} u + F^\varepsilon[u] = \psi^\perp, & t \in [\Lambda, \Lambda + T], \\ u(x, \Lambda) = 0, & x \in \underline{N}. \end{cases} \tag{7.28}$$

Letting

$$a(t) = \frac{1}{\mathrm{Vol}(N^\varepsilon)} \int_{\underline{N}} (\partial_t u - \psi) \; \mathrm{d}V_{g^\varepsilon},$$
$$b(t) = \frac{1}{\|w^\varepsilon\|^2_{L^2}} \int_{\underline{N}} (\partial_t u - \Delta_{g^\varepsilon} u - \psi) \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon},$$

it follows that the triple (u, a, b) solves

$$\partial_t u - \Delta_{g^\varepsilon} u = \psi + a(t) + b(t) w^\varepsilon,$$

and by Corollary 7.9 and Lemma 7.10, for $\Lambda > 0$ large enough, the estimate $\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant C \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}}$ holds, for any $T > 0$. It follows that the operator

$$\mathcal{A}^\varepsilon_{\underline{0}} : (u, a(t), b(t)) \longmapsto \partial_t u - \Delta_{g^\varepsilon} u - a(t) - b(t) w^\varepsilon$$

as a bounded operator from $P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^\varepsilon \rangle^\perp \times C^{0,\alpha}_{\mu,\Lambda} \times C^{0,\alpha}_{\mu,\Lambda}$ to $P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$ is a linear isomorphism whose inverse is bounded by C, which is independent of $\Lambda \geqslant \Lambda_0$. This proves the claim.

Now, our goal is to show that $\mathcal{L}^\varepsilon_{\underline{0}} = \mathcal{A}^\varepsilon_{\underline{0}} - \mathcal{P}^\varepsilon_{\underline{0}}$ is invertible. Write

$$\mathcal{A}^\varepsilon_{\underline{0}} - \mathcal{P}^\varepsilon_{\underline{0}} = \mathcal{A}^\varepsilon_{\underline{0}} \left(\mathcal{I} - (\mathcal{A}^\varepsilon_{\underline{0}})^{-1} \mathcal{P}^\varepsilon_{\underline{0}} \right),$$

where $\mathcal{I}$ is the identity operator in $P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^\varepsilon \rangle \times C^{0,\alpha}_{\mu,\Lambda} \times C^{0,\alpha}_{\mu,\Lambda}$. Since by Lemma 7.11 $\|(\mathcal{A}^\varepsilon_{\underline{0}})^{-1} \mathcal{P}^\varepsilon_{\underline{0}}\| \to 0$ as $\Lambda \to \infty$, it follows that $\mathcal{I} - (\mathcal{A}^\varepsilon_{\underline{0}})^{-1} \mathcal{P}^\varepsilon_{\underline{0}}$ is invertible for large $\Lambda > 0$. Hence, $\mathcal{A}^\varepsilon_{\underline{0}} - \mathcal{P}^\varepsilon_{\underline{0}}$ is invertible for large $\Lambda > 0$. ∎

8. Estimates for the Error Terms in the Torus Case

In this section, we provide pointwise estimates for the zeroth order term, $\theta_{N^\varepsilon} + \xi(0)$ and the quadratic term $Q^\varepsilon[\mathrm{d}u]$, which will be utilised in the iteration scheme of section 9. We also estimate the projection of the zeroth order term onto the approximate kernel, whose dominant term provides the approximate ODE that $\varepsilon(t)$ should satisfy.

8.1. **The Zeroth Order Error.** The main zeroth order error estimate is the following.

Proposition 8.1. *Assume that the constants $\mu > 0$, $\nu \in (0, m-2)$, $\alpha \in (0, \frac{1}{2})$ and $\tau \in (0, \frac{1}{2})$ satisfy the relation*

$$\tau > \frac{2\alpha}{m+1+2\alpha}, \quad \frac{\nu+2}{m-2} < \mu < \frac{1}{m-2}(\tau(\nu+2) + (1-\tau)m). \tag{8.1}$$

Then, we have

$$\lim_{\Lambda\to\infty} \|\theta_{N^\varepsilon} + \xi(0)\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} = 0. \tag{8.2}$$

Precisely, we have the following bounds in terms of Λ:

$$\|\theta_{N^\varepsilon}\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant C\Lambda^{\mu - \frac{1}{m-2}(\tau(\nu+2)+(1-\tau)m)}, \quad \|\xi(0)\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant C\Lambda^{\mu-\frac{1}{m-2}(m-2\alpha)}, \tag{8.3}$$

for some $C > 0$ independent of Λ.

Remark 8.2. Notice that

$$\tau(\nu+2) + (1-\tau)m - (\nu+2) = (1-\tau)(m-2-\nu) > 0.$$

Hence, the ranges for the constants in (8.1) are non-empty.

Proof. We first estimate the Hölder norms of θ_{N^ε}. By construction, it suffices to consider the transition region $\overline{\kappa_\varepsilon}^{-1}(\Sigma \times (\varepsilon^\tau, 2\varepsilon^\tau))$.

By Lemma 7.5, we have for $x, x' \in \overline{\kappa}_\varepsilon^{-1}(\Sigma \times (\varepsilon^\tau, 2\varepsilon^\tau))$,

$$t^\mu \rho_\varepsilon^{\nu+2}(x,t)|\theta_{N^\varepsilon}|(x,t) \leqslant C\, t^\mu \varepsilon(t)^{\tau(\nu+2)}\varepsilon(t)^{(1-\tau)m} \leqslant C\, t^{\mu - \frac{1}{m-2}(\tau(\nu+2)+(1-\tau)m)},$$

and similarly, using $|\mathrm{d}\theta_{N^\varepsilon}| \leqslant C\varepsilon(t)^{(1-\tau)m-\tau}$,

$$\begin{aligned} t^\mu \rho_\varepsilon^{\nu+2+2\alpha}(x,t)\frac{|\theta_{N^\varepsilon}(x,t) - \theta_{N^\varepsilon}(x',t)|}{d_{g^\varepsilon}(x,x')^{2\alpha}} &\leqslant C\, t^\mu \varepsilon(t)^{\tau(\nu+2+2\alpha)} \cdot \varepsilon(t)^{(1-\tau)m-\tau} \cdot \varepsilon(t)^{\tau(1-2\alpha)} \\ &\leqslant C\, t^{\mu - \frac{1}{m-2}(\tau(\nu+2)+(1-\tau)m)}. \end{aligned}$$

On the other hand, for $t_2 > t_1$, $t_1, t_2 \in [t, 2t]$, $0 < |t_1 - t_2| < t^{-\frac{2}{m-1}}$, and $x = (\sigma, r) \in \bigcup_{t_1<t<t_2} \overline{\kappa}_\varepsilon^{-1}(\Sigma \times (\varepsilon^\tau(t), 2\varepsilon^\tau(t)))$,

$$\begin{aligned} &\theta_{N^\varepsilon}(x,t_1) - \theta_{N^\varepsilon}(x,t_2) \\ &= (\theta\circ\Phi_C)((\sigma,\kappa_{\varepsilon(t_1)}(r)), \mathrm{d}\mathfrak{Q}_{\varepsilon(t_1)}(\sigma,\kappa_{\varepsilon(t_1)}(r))) - (\theta\circ\Phi_C)((\sigma,\kappa_{\varepsilon(t_2)}(r)), \mathrm{d}\mathfrak{Q}_{\varepsilon(t_2)}(\sigma,\kappa_{\varepsilon(t_2)}(r))) \\ &= \int_{s=0}^{s=1} \frac{\mathrm{d}}{\mathrm{d}s}(\theta\circ\Phi_C)((\sigma,\kappa_{\varepsilon(s)}(r)), \mathrm{d}\mathfrak{Q}_{\varepsilon(s)}(\sigma,\kappa_{\varepsilon(s)}(r)))\,\mathrm{d}s, \end{aligned} \tag{8.4}$$

where $\varepsilon(s) := \varepsilon(st_2 + (1-s)t_1)$. Using $\varepsilon'(s) \leqslant C\varepsilon(t)^{m-1}|t_1 - t_2|$ and $\partial_\varepsilon \kappa_\varepsilon(r) \leqslant \frac{\kappa_\varepsilon(r)}{\varepsilon} \leqslant \varepsilon^{\tau-1}$, we deduce that

$$\partial_s \mathrm{d}\mathfrak{Q}_{\varepsilon(s)}(\sigma, \kappa_{\varepsilon(s)}(r)) = O(\varepsilon(t)^{m+(1-\tau)(m-2)}) \cdot |t_1 - t_2|,$$

measuring by the induced metric $g^{\varepsilon}_{\underline{0}}$. Inserting this into (8.4) gives

$$|\theta_{N^\varepsilon}(x, t_1) - \theta_{N^\varepsilon}(x, t_2)| \leqslant C\, \varepsilon(t)^{m-2+\tau}|t_1 - t_2|.$$

Thus,

$$t^\mu \rho^{\nu+2+2\alpha}_{\varepsilon(t)}(x) \frac{|\theta_{N^\varepsilon}(x, t_1) - \theta_{N^\varepsilon}(x, t_2)|}{|t_1 - t_2|^\alpha} \leqslant C\, t^{\mu - \frac{1}{m-2}(\tau(\nu+2)+m+\tau-2\alpha(1-\tau))}.$$

Putting these together, we obtain

$$\|\theta_{N^\varepsilon}\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant C\, \Lambda^{\mu - \frac{\tau(\nu+2)+(1-\tau)m}{m-2}} = o(1), \quad \text{as } \Lambda \to \infty.$$

if $\tau > \frac{2\alpha}{m+1+2\alpha}$ and $\mu < \frac{\tau(\nu+2)+(1-\tau)m}{m-2}$.

We now estimate $\xi(0)$. Recalling our choice of time-dependent constants from (7.2), $\xi(0)$ is given by

$$\xi(0) = \begin{cases} (\varepsilon(t)^2)' \left(\alpha_L\big|_{\underline{0}} - \frac{c_L V_2}{V_1+V_2}\right) & \text{on } P_j\ , \\ \varepsilon'(t) \left(2\varepsilon(t)\frac{c_L V_1}{V_1+V_2} - (\partial_\varepsilon \mathfrak{Q}_{\varepsilon(t)}) \circ \bar{\kappa}_{\varepsilon(t)}\right) & \text{on } Q^+\ , \\ \varepsilon'(t) \left(-2\varepsilon(t)\frac{c_L V_2}{V_1+V_2} - (\partial_\varepsilon \mathfrak{Q}_{\varepsilon(t)}) \circ \bar{\kappa}_{\varepsilon(t)}\right) & \text{on } Q^-\ , \\ -(\varepsilon(t)^2)' \frac{c_L V_2}{V_1+V_2} & \text{on } X_1^\circ\ , \\ (\varepsilon(t)^2)' \frac{c_L V_1}{V_1+V_2} & \text{on } X_2^\circ\ . \end{cases}$$

In the transition region we have

$$\varepsilon(t)' \partial_\varepsilon \mathfrak{Q}_\varepsilon = O(\varepsilon(t)^{m+(m-2)(1-\tau)}),$$

which is smaller than $(\varepsilon(t)^2)'$. From this observation it is now easy to deduce that

$$\|\xi(0)\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant C\, \Lambda^{\mu - \frac{m-2\alpha}{m-2}}.$$

$\square$

8.2. **The Quadratic Error.** Let $Q^\varepsilon[\mathrm{d}u] := -\theta_{N^\varepsilon} - \xi(0) + \partial_t u - \mathcal{L}^\varepsilon_{\underline{0}}[u]$ be the quadratic error term. We now estimtate $Q^\varepsilon[\mathrm{d}u]$ in terms of weighted norms of u.

Proposition 8.3. *There is $C > 0$ and $\Lambda \gg 1$ such that if $u \in P^{1,2,\alpha}_{\mu,\nu,\Lambda}$, $t, t_1, t_2 \geqslant \Lambda$ with $0 < |t_1 - t_2| < t^{\frac{-2}{m-2}}$, and $x, x_1, x_2 \in \underline{N}$ with $0 < d_{g^\varepsilon}(x_1, x_2) < \rho_{\varepsilon(t)}(x_1)$, then*

$$|Q^\varepsilon[\mathrm{d}u]|(x, t) \leqslant C\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \cdot t^{-2\mu}\rho^{-2\nu-4}_{\varepsilon(t)}(x), \tag{8.5}$$

$$\frac{|Q^\varepsilon[\mathrm{d}u](x_1, t) - Q^\varepsilon[\mathrm{d}u](x_2, t)|}{d_{g^\varepsilon}(x_1, x_2)^{2\alpha}} \leqslant C\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \cdot t^{-2\mu}\rho^{-2\nu-4-2\alpha}_{\varepsilon(t)}(x_1), \tag{8.6}$$

$$\frac{|Q^\varepsilon[\mathrm{d}u](x, t_1) - Q^\varepsilon[\mathrm{d}u](x, t_2)|}{|t_1 - t_2|^\alpha} \leqslant C\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \cdot t^{-2\mu}\rho^{-2\nu-4-2\alpha}_{\varepsilon(t)}(x) + Ct^{-\frac{m+2-2\alpha}{m-1}}. \tag{8.7}$$

Proof. Write $Q^\varepsilon[\mathrm{d}u] = Q^\varepsilon_\theta[\mathrm{d}u] + Q^\varepsilon_\xi[\mathrm{d}u]$, where

$$Q^\varepsilon_\theta[\mathrm{d}u] = \theta_{N^\varepsilon}[\mathrm{d}u] - \theta_{N^\varepsilon} - \Delta_{N^\varepsilon} u + \langle \nabla\theta_{N^\varepsilon}, \hat{V}_{\underline{0}}(\mathrm{d}u)\rangle,$$
$$Q^\varepsilon_\xi[\mathrm{d}u] = \xi[\mathrm{d}u] - \xi(0) - S^\varepsilon[u].$$

We first estimate Q^ε_θ. In the tip region, the induced metric is uniformly equivalent to the metric $\varepsilon_j^2 g_{L_j}$. Using the scale-invariant property of Lagrangian angle we have

$$|Q^\varepsilon_\theta(x,t,\mathrm{d}u(x,t),\nabla\mathrm{d}u(x,t))| \leqslant C\left(\varepsilon_j^{-2}|\mathrm{d}u|^2_{g^\varepsilon} + |\nabla\mathrm{d}u|^2_{g^\varepsilon}\right), \quad x \in P_j,$$

for some $C > 0$ independent of ε. Similarly, in the intermediate region, the metric is uniformly equivalent to the cone metric, and the scale-invariant property of Lagrangian angle implies

$$|Q^\varepsilon_\theta((\mathfrak{r},\sigma),t,\mathrm{d}u((\mathfrak{r},\sigma),t),\nabla\mathrm{d}u((\mathfrak{r},\sigma),t))| \leqslant C\left(\mathfrak{r}^{-2}|\mathrm{d}u|^2_{g^\varepsilon} + |\nabla\mathrm{d}u|^2_{g^\varepsilon}\right), \quad (\mathfrak{r},\sigma)) \in \kappa_\varepsilon Q_j^\pm,$$

for some $C > 0$ independent of ε. Combining these estimates yields

$$|Q^\varepsilon_\theta(x,t,\mathrm{d}u(x,t),\nabla\mathrm{d}u(x,t))| \leqslant C\left(\rho_\varepsilon^{-2}(x,t)|\mathrm{d}u(x,t)|^2_{g^\varepsilon} + |\nabla\mathrm{d}u(x,t)|^2_{g^\varepsilon}\right), \quad (x,t) \in \underline{N} \times [\Lambda,\infty).$$

Multiply both sides by $t^\mu\rho_\varepsilon^{\nu+2}$ gives

$$\begin{aligned} t^\mu\rho_\varepsilon^{\nu+2}|Q^\varepsilon_\theta[\mathrm{d}u]| &\leqslant Ct^\mu\left(\rho_\varepsilon^\nu|\mathrm{d}u|^2_{g^\varepsilon} + \rho_\varepsilon^{\nu+2}|\nabla\mathrm{d}u|^2_{g^\varepsilon}\right) \\ &= C\left[t^{-\mu}\rho_\varepsilon^{-\nu-2}\left(t^\mu\rho_\varepsilon^{\nu+1}|\mathrm{d}u|_{g^\varepsilon}\right)^2 + t^{-\mu}\rho_\varepsilon^{-\nu-2}\left(t^\mu\rho_\varepsilon^{\nu+2}|\nabla\mathrm{d}u|_{g^\varepsilon}\right)^2\right] \\ &\leqslant 2Ct^{-\mu}\rho_\varepsilon^{-\nu-2}\|u\|^2_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}. \end{aligned}$$

The estimate for $Q^\varepsilon_\xi[\mathrm{d}u]$ follows similarly. From the explicit expression (5.1), we only need to consider the tip region. By Taylor theorem,

$$\xi_{N^\varepsilon}[\mathrm{d}u] = (\varepsilon_j^2)'\left[\alpha_L(x,0) + \partial_y\alpha_L(x,0)\cdot\varepsilon_j^{-2}\mathrm{d}u + O(\varepsilon_j^{-4}|\mathrm{d}u|^2_{g_{L_j}})\right].$$

It follows that, using $|(\varepsilon_j^2)'| \ll 1$,

$$|Q^\varepsilon_\xi[\mathrm{d}u](x,t)| \leqslant C\varepsilon_j^{-2}|\mathrm{d}u(x,t)|^2_{g^\varepsilon} \leqslant C\rho_\varepsilon^{-2}(x,t)|\mathrm{d}u(x,t)|^2_{g^\varepsilon}.$$

Multiply both sides by $t^\mu\rho_\varepsilon^{\nu+2}$ and estimate as above gives

$$t^\mu\rho_\varepsilon^{\nu+2}|Q^\varepsilon_\xi[\mathrm{d}u]| \leqslant Ct^{-\mu}\rho_\varepsilon^{-\nu-2}\|u\|^2_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}.$$

Combining everything together yields

$$|Q^\varepsilon[\mathrm{d}u]| \leqslant |Q^\varepsilon_\theta[\mathrm{d}u]| + |Q^\varepsilon_\xi[\mathrm{d}u]| \leqslant Ct^{-2\mu}\rho_\varepsilon^{-2\nu-4}.$$

This proves (8.5).

To prove (8.6), we fix $t \in [\Lambda,\infty)$, and view $\theta_{N^\varepsilon}[\mathrm{d}u](\cdot,t)$ and $\xi[\mathrm{d}u](\cdot,t)$ as coming from restricting smooth functions

$$\Theta(x,y,z), \quad \Xi(x,y), \quad x \in \underline{N},\ y \in T^*_x\underline{N},\ z \in \bigotimes^2 T^*_x\underline{N},$$

to the graph $\{(x,\mathrm{d}u(x,t),\nabla\mathrm{d}u(x,t)) : x \in \underline{N}\}$, namely, we have

$$\theta_{N^\varepsilon}[\mathrm{d}u](x,t) = \Theta(x,\mathrm{d}u(x,t),\nabla\mathrm{d}u(x,t)), \quad \xi[\mathrm{d}u](x,t) = \Xi(x,\mathrm{d}u(x,t)).$$

Note that by scale-invariant property we have

$$|\partial_x^a\partial_y^b\partial_z^c\Theta| \leqslant C\rho_\varepsilon^{-a-b}, \quad |\partial_x^a\partial_y^b\Xi| \leqslant C\rho_\varepsilon^{-a-b}, \quad a,b,c \in \mathbb{N}\cup\{0\}.$$

Then a long but straightforward computation using mean value theorem shows that, for $x_1 \neq x_2 \in \underline{N}$ with $d_{g^\varepsilon}(x_1,x_2) < \rho_\varepsilon(x_1,t)$,

$$t^\mu\rho_\varepsilon^{\nu+2+2\alpha}(x_1)\frac{|Q^\varepsilon_\theta[\mathrm{d}u](x_1,t) - Q^\varepsilon_\theta[\mathrm{d}u](x_2,t)|}{d_{g^\varepsilon}(x_1,x_2)^{2\alpha}} \leqslant Ct^{-\mu}\rho_\varepsilon^{-\nu-2}(x_1)\|u\|^2_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}.$$

Similarly,

$$t^\mu\rho_\varepsilon^{\nu+2+2\alpha}(x_1)\frac{|Q^\varepsilon_\xi[\mathrm{d}u](x_1,t) - Q^\varepsilon_\xi[\mathrm{d}u](x_2,t)|}{d_{g^\varepsilon}(x_1,x_2)^{2\alpha}} \leqslant Ct^{-\mu}\rho_\varepsilon^{-\nu-2}(x_1)\|u\|^2_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}.$$

Combining these estimates yields (8.6)

Finally, we prove (8.7). Similar computation as Joyce [13, Proposition 5.8] and Pacini [28, Proposition 5.6] shows that if α,β are small closed 1-forms on $\underline{N}$, then for each fixed t,

$$|Q^\varepsilon[\alpha] - Q^\varepsilon[\beta]| \leqslant C\left(\rho_\varepsilon^{-1}|\alpha-\beta| + |\nabla(\alpha-\beta)|\right)\left(\rho_\varepsilon^{-1}|\alpha| + \rho_\varepsilon^{-1}|\beta| + |\nabla\alpha| + |\nabla\beta|\right).$$

Letting $\alpha = \mathrm{d}u(\cdot,t_1)$, $\beta = \mathrm{d}u(\cdot,t_2)$ yields

$$t^\mu\rho_\varepsilon^{\nu+2+2\alpha}(x,t)\frac{|Q^{\varepsilon(t)}[\mathrm{d}u(x,t_1)] - Q^{\varepsilon(t)}[\mathrm{d}u(x,t_2)]|}{|t_1-t_2|^\alpha} \leqslant Ct^{-\mu}\rho_\varepsilon^{-\nu-2}(x,t)\|u\|^2_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}.$$

On the other hand, by the assumption on ε we have

$$\frac{|Q^{\varepsilon(t_1)}[\mathrm{d}u(x,t_2)] - Q^{\varepsilon(t_2)}[\mathrm{d}u(x,t_2)]|}{|t_1-t_2|} \leqslant C\sup_j|\varepsilon_j'(t)\varepsilon_j(t)| \leqslant Ct^{-\frac{m}{m-1}}.$$

Combining these estimates, we conclude that for $t_1,t_2 \geqslant \Lambda$ with $0 < |t_1-t_2| < t^{-\frac{2}{m-2}}$,

$$\frac{|Q^\varepsilon[\mathrm{d}u](x,t_1) - Q^\varepsilon[\mathrm{d}u](x,t_2)|}{|t_1-t_2|^\alpha} \leqslant Ct^{-2\mu}\rho_\varepsilon^{-2\nu-4-2\alpha}\|u\|^2_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} + Ct^{-\frac{m+2-2\alpha}{m-1}}.$$

This proves (8.7). ☐

8.3. **Projection onto the Approximate Kernel.** Finally, we will require the following integral estimates, which are the projection of the zeroth order terms onto the approximate kernel.

Lemma 8.4. *We have*

$$\int_{\Sigma^\pm\times(\varepsilon^\tau,2\varepsilon^\tau)} \theta_{N^\varepsilon}\,\mathrm{d}V_{N^\varepsilon} = \pm\varepsilon^m A + O(\varepsilon^{(1+\tau)m}). \tag{8.8}$$

Proof. This follows from the proof of [13, Proposition 7.5] by estimating θ_{N^ε} by $\sin(\theta_{N^\varepsilon})+O(\theta^3)$, and using Lemma 2.15 and Lemma 7.5. ☐

Applying this Lemma, we have the following projection formula for the zeroth order term.

Proposition 8.5. *The L^2 projection of the zeroth order error $\theta_{N^\varepsilon}+\xi(0)$ onto the approximate kernel* $\mathrm{span}_{\mathbb{R}}\{1, w^\varepsilon_{(0,1)}\}$ *is given by*

$$\int_{\underline{N}}[\theta_{N^\varepsilon}+\xi(0)]\cdot w^\varepsilon_{(0,1)}\,\mathrm{d}V_{g^\varepsilon} = c_L\frac{V_1V_2}{V_1+V_2}\left\{\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} + \frac{A}{c_L}\frac{V_1+V_2}{V_1V_2}\varepsilon^m(t)\right\} + O(\varepsilon^{(1+\tau)m}), \tag{8.9}$$

and

$$\int_{\underline{N}} [\theta_{N^\varepsilon} + \xi(0)] \cdot 1 \, \mathrm{d}V_{g^\varepsilon} = O(\varepsilon^{(1+\tau)m}). \tag{8.10}$$

Proof. It follows from these choices that

$$\begin{aligned}\int_{\underline{N}} \xi(0) \cdot w^\varepsilon_{(0,1)} \, \mathrm{d}V_{g^\varepsilon} &= \int_P \frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t}\beta_L + \int_{Q^\pm} \xi(0) \cdot w^\varepsilon_{(0,1)} + \int_{X_2^o} c_L \frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} \\ &= V_2 \cdot c_L \cdot \frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} + O(\varepsilon^{(1+\tau)m}(t)),\end{aligned}$$

where in the second line we used the assumption that $\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} = O(\varepsilon^{m-1}(t))$.

Combining with Lemma 8.4 and using the fact that the volume of the interpolating region is $O(\varepsilon^{\tau m})$ yield

$$\begin{aligned}&\int_{\underline{N}} [\theta_{N^\varepsilon} + \xi(0)] \cdot w^\varepsilon_{(0,1)} \, \mathrm{d}V_{g^\varepsilon} \\ &= \varepsilon^m(t)A + V_2 \cdot c_L \frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} - \frac{c_L V_2}{V_1 + V_2}\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t}(V_2 - O(\varepsilon^{\tau m})) + O(\varepsilon^{(\tau+1)m} + \varepsilon^{(2-\tau)m}) \\ &= c_L \frac{V_1 V_2}{V_1 + V_2}\left\{\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} + \frac{A}{c_L}\frac{V_1 + V_2}{V_1 V_2}\varepsilon^m(t)\right\} + O(\varepsilon^{(\tau+1)m}).\end{aligned}$$

This proves (8.9). Equation (8.10) follows from a similar computation. □

By (8.9), (8.10), the projection onto the normalised approximate kernel element w^ε (as defined in (7.1)) takes the same form:

$$\begin{aligned}&\int_{\underline{N}} [\theta_{N^\varepsilon} + \xi(0)] \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon} \\ &\quad = c_L \frac{V_1 V_2}{V_1 + V_2}\left\{\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} + \frac{A}{c_L}\frac{V_1 + V_2}{V_1 V_2}\varepsilon^m(t)\right\} + O(\varepsilon^{(\tau+1)m}).\end{aligned} \tag{8.11}$$

Remark 8.6. In the iteration scheme of Section 9, the above integrals will appear as error terms that we wish to minimise. We will therefore define $\varepsilon(t)$ to be a small perturbation of a solution of the following ODE:

$$\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} + \frac{A}{c_L}\left(\frac{1}{V_1} + \frac{1}{V_2}\right)\varepsilon^m(t) \;=\; 0. \tag{8.12}$$

We note that any solution $\varepsilon(t)$ to this ODE satisfies Assumption 6.1.

9. Solving the Nonlinear Equation in the Torus Case

We are now ready to state and prove our main theorem precisely. For the remainder of the paper, we make the following assumptions on $\nu, \alpha, \tau, \mu, \zeta$, which imply all previously made assumptions on these constants:

$$\nu \in \left(\max\left\{\frac{m}{2}-2,0\right\},m-2\right),\quad \alpha\in\left(0,\frac{1}{2}\right),\quad \tau\in\left(\frac{2\alpha}{m+1+2\alpha},\frac{1}{m+2}\right), \tag{9.1}$$
$$\mu\in\left(\frac{\nu+2+2\alpha}{m-2},\frac{1}{m-2}(\tau(\nu+2)+(1-\tau)m)\right),\quad \zeta\in\left(0,\min\left\{\frac{\tau m}{m-2},\mu-\frac{\nu+2+2\alpha}{m-2}\right\}\right).$$

(For example, $(\nu,\alpha,\tau,\mu)=(\frac{3m-8}{4},\frac{1}{100},\frac{1}{2(m+2)},\frac{7(1-\tau)m}{8(m-2)})$ and ζ sufficiently small).

Theorem 9.1. *Let $m\geqslant 3$, let $\iota : X\to M$ be a special Lagrangian immersion in a flat complex torus $(M^{2m},g,J,\omega,\Omega)$ satisfying Assumption 7.1, and let $\nu,\alpha,\tau,\mu,\zeta$ be real constants satisfying (9.1). Let $\underline{N}$ be the corresponding abstract manifold as defined in Definition 3.1, and let $P^{1,2,\alpha}_{\mu,\nu,\Lambda}, C^{0,\alpha}_{\zeta,\Lambda}$ be the Banach spaces on $\underline{N}\times[\Lambda,\infty)$ and $[\Lambda,\infty)$ respectively as defined in Definition 7.6.*

Then there exist $u\in P^{1,2,\alpha}_{\mu,\nu,\Lambda}$, $\varepsilon:[\Lambda,\infty)\to(0,\infty)$ satisfying Assumption 6.1, and $a:[\Lambda,\infty)\to\mathbb{R}$ such that

$$\begin{cases}\partial_t u=\theta(\mathrm{d}u)+\xi(\mathrm{d}u)+a(t) & \text{for } t>\Lambda\\ u(x,\Lambda)=0 & \text{on } \underline{N}\times\{0\},\end{cases} \tag{9.2}$$

where $\theta(\mathrm{d}u)$ is the Lagrangian angle of the Lagrangian embedding $\Psi_{N^{\varepsilon(t)}}\circ\mathrm{d}u:\underline{N}\to M$ as in section 4.1, and $\xi(\mathrm{d}u)$ is defined in (5.1) with constants C_P, $C_{Q^{\pm}}$, $C_{X_b^\circ}$ defined by (7.2).

The family $\Psi_{N^{\varepsilon(t)}}\circ\mathrm{d}u:\underline{N}\to M$ of Lagrangian submanifolds satisfies mean curvature flow, and forms an infinite-time singularity. As $t\to\infty$ we have smooth convergence $N^{\varepsilon}\to\iota(X)$ away from the transverse self-intersection point.

As shown in section 4, given a pair (u,ε) satisfying 9.2, the family of Lagrangian embeddings $\Psi_{N^\varepsilon}\circ du:\underline{N}\times[\Lambda,\infty)\to M$ is an eternal Lagrangian mean curvature flow $F_t:\underline{N}\to M$ that forms an infinite-time singularity, converging away from the singular point to the immersion $\iota:X\to M$.

To prove Theorem 9.1, we first carefully define an iteration map $\mathscr{I}$ on the Banach space $P^{1,2,\alpha}_{\mu,\nu,\Lambda}\times C^{0,\alpha}_{\zeta,\Lambda}$ for which a fixed point (u,h) corresponds to a solution (u,ε) of (9.2). We then show that $\mathscr{I}$ maps a compact subset of $P^{1,2,\alpha}_{\mu,\nu,\Lambda}\times C^{0,\alpha}_{\zeta,\Lambda}$ continuously into itself, and apply the Schauder fixed point theorem to conclude that a fixed point exists.

9.1. **Definition of the Iteration Map.** Denote the unit balls of $P^{1,2,\alpha}_{\mu,\nu,\Lambda}$, $C^{0,\alpha}_{\zeta,\Lambda}$ by

$$\mathcal{B}^{\alpha}_{\mu,\nu,\Lambda}:=\left\{u\in P^{1,2,\alpha}_{\mu,\nu,\Lambda}\ :\ \|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}\leqslant 1\right\},\quad \mathcal{I}^{k,\alpha}_{\zeta,\Lambda}:=\left\{h\in C^{k,\alpha}_{\zeta,\Lambda}\ :\ \|h\|_{P^{k,\alpha}_{\zeta,\Lambda}}\leqslant 1\right\}.$$

We now define the iteration map $\mathscr{I}:\mathcal{B}^{\alpha}_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda}\to\mathcal{B}^{\alpha}_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$. Given a pair $(u,h)\in\mathcal{B}^{\alpha}_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$, the pair $(v,k)=\mathscr{I}(u,h)\in P^{1,2,\alpha}_{\mu,\nu,\Lambda}\times C^{0,\alpha}_{\zeta,\Lambda}$ is defined as follows:

Step 1. (ansatz for $\varepsilon(t)$): First, we define

$$\varepsilon(t):=\left[\frac{m-2}{2}\frac{A}{c_L}\frac{V_1+V_2}{V_1V_2}\cdot t+\int_\Lambda^t h(s)\,\mathrm{d}s\right]^{-\frac{1}{m-2}}, \tag{9.3}$$

and use $\varepsilon(t)$ to construct the Lagrangian embedding $\iota^{\varepsilon(t)}$ and related quantities and functions that depend on $\varepsilon(t)$ as in section 3. By definition, $\varepsilon(t)$ satisfies the ODE:

$$\frac{\mathrm{d}\varepsilon^2}{\mathrm{d}t} + \frac{A}{c_L}\frac{V_1+V_2}{V_1V_2}\varepsilon^m = -\frac{2}{m-2}\varepsilon^m h(t).$$

It is easy to check that $\varepsilon(t)$ satisfies Assumption 6.1. We then construct the desingularisation $N^{\varepsilon(t)}$ using $\varepsilon(t)$.

Step 2. ($u \rightsquigarrow v$): Next, we define $v \in P^{1,2,\alpha}_{\mu,\nu,\Lambda}$. Define $\psi := \theta_{N^\varepsilon} + \xi(0) + Q^\varepsilon[\mathrm{d}u]$. By Proposition 8.1 and Proposition 8.3, we see that $\psi \in P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$. We may therefore apply Theorem 7.12, to show that there exist $v \in P^{1,2,\alpha}_{\mu,\nu,\Lambda} \cap \langle 1, w^\varepsilon\rangle^\perp$, $a : [\Lambda,\infty) \to \mathbb{R}$ and $b : [\Lambda,\infty) \to \mathbb{R}$ satisfying:

$$\begin{cases} \partial_t v - \mathcal{L}^\varepsilon_{\underline{0}}[v] = \theta_{N^\varepsilon} + \xi(0) + Q^\varepsilon[\mathrm{d}u] + a(t) + b(t)w^\varepsilon, \quad t \in [\Lambda,\infty), \\ v(x,\Lambda) = 0, \quad x \in \underline{N}, \end{cases} \tag{9.4}$$

and

$$\|v\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant C \cdot \|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}}. \tag{9.5}$$

Step 3. ($h \rightsquigarrow k$): Finally, we define $k \in C^{0,\alpha}_{\zeta,\Lambda}$. Integrating (9.4) against the functions 1 and w^ε respectively, and using the projection formulae (8.10) and (8.11), we obtain the following expressions for $a(t)$ and $b(t)$:

$$\begin{aligned} a(t) &= \frac{1}{\mathrm{Vol}(N^\varepsilon)}\int_{\underline{N}} \left(\partial_t v - \mathcal{L}^\varepsilon_{\underline{0}}[v] - \psi\right)\,\mathrm{d}V_{g^\varepsilon} \\ &= \frac{1}{\mathrm{Vol}(N^\varepsilon)}\left(\int_{\underline{N}} (\partial_t v - \mathcal{L}^\varepsilon_{\underline{0}}[v] - Q^\varepsilon[\mathrm{d}u])\,\mathrm{d}V_{g^\varepsilon} + O(\varepsilon^{(1+\tau)m})\right), \qquad (9.6) \\ b(t) &= \frac{1}{\|w^\varepsilon\|^2_{L^2}}\int_{\underline{N}} \left(\partial_t v - \mathcal{L}^\varepsilon_{\underline{0}}[v] - \psi\right) w^\varepsilon\,\mathrm{d}V_{g^\varepsilon} \\ &= \frac{1}{\|w^\varepsilon\|^2_{L^2}}\Bigg(\int_{\underline{N}} (\partial_t v - \mathcal{L}^\varepsilon_{\underline{0}}[v] - Q^\varepsilon[\mathrm{d}u])w^\varepsilon\,\mathrm{d}V_{g^\varepsilon} \\ &\qquad\qquad -c_L\frac{V_1V_2}{V_1+V_2}\left[\frac{\mathrm{d}\varepsilon^2}{\mathrm{d}t} + \frac{\varepsilon^m A}{c_L}\frac{V_1+V_2}{V_1V_2}\right] + O(\varepsilon^{(1+\tau)m})\Bigg) \\ &= \frac{1}{\|w^\varepsilon\|^2_{L^2}}\left(\int_{\underline{N}} (\partial_t v - \mathcal{L}^\varepsilon_{\underline{0}}[v] - Q^\varepsilon[\mathrm{d}u])w^\varepsilon\,\mathrm{d}V_{g^\varepsilon} + \frac{c_LV_1V_2}{V_1+V_2}\frac{2\varepsilon^m h(t)}{m-2} + O(\varepsilon^{(1+\tau)m})\right). \qquad (9.7) \end{aligned}$$

It is therefore natural to define $k(t)$ as follows, in order to cancel out the dominant term from this expansion of $b(t)$:

$$k(t) := h(t) - \frac{m-2}{2c_L}\frac{V_1+V_2}{V_1V_2}\varepsilon^{-m}\|w^\varepsilon\|^2_{L^2}\cdot b(t). \tag{9.8}$$

9.2. **Estimates for the Iteration Map.** In order to apply the Schauder fixed point theorem, we now aim to prove the following proposition regarding the iteration map $\mathscr{I}$:

Proposition 9.2. *For any $\mu' < \mu$, $\alpha' < \alpha$, $\zeta' < \zeta$, the iteration map $\mathscr{I} : \mathcal{B}^{\alpha}_{\mu,\nu,\Lambda} \times \mathcal{I}^{0,\alpha}_{\zeta,\Lambda} \to P^{1,2,\alpha}_{\mu,\nu,\Lambda} \times C^{0,\alpha}_{\zeta,\Lambda}$ defined in section 9.1 is continuous with respect to the norm on $P^{1,2,\alpha'}_{\mu',\nu,\Lambda} \times C^{0,\alpha'}_{\zeta',\Lambda}$, and has image lying in $\mathcal{B}^{\alpha}_{\mu,\nu,\Lambda} \times \mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$.*

We first estimate the projection of the inhomogeneous term ψ onto the approximate kernel.

Lemma 9.3. *Let*

$$G(t) := c_L \frac{V_1 V_2}{V_1 + V_2} \left\{ \frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} + \frac{\varepsilon^m(t) A}{c_L} \frac{V_1 + V_2}{V_1 V_2} \right\} - \int_{\underline{N}} \psi \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon}. \tag{9.9}$$

Then, if $\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant 1$, $\nu \in (\max\{\frac{m}{2} - 2, 0\}, m-2)$ and $\mu > \frac{\nu+2}{m-2}$, then for any $\overline{\zeta} > 0$ satisfying

$$\overline{\zeta} < \min\{\tfrac{\tau m}{m-2}, 2(\mu - \tfrac{\nu+2}{m-2})\}$$

and $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$, it follows that

$$|G(t)| + \frac{|G(t_1) - G(t_2)|}{|t_1 - t_2|^\alpha} \leqslant C\varepsilon(t)^m \cdot t^{-\overline{\zeta}}. \tag{9.10}$$

Proof. By projection formula (8.11) we have

$$|G(t)| \leqslant C\varepsilon(t)^{(\tau+1)m} + \int_{\nu_N} |Q^\varepsilon[\mathrm{d}u]| \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon}.$$

Since $\|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \leqslant 1$, by (8.5) we have

$$\int_{\nu_N} |Q^\varepsilon[\mathrm{d}u]| \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon} \leqslant C\, t^{-2\mu} \int_{\underline{N}} \rho_\varepsilon^{-2\nu-4} \, \mathrm{d}V_{g^\varepsilon}.$$

Note that for any small $a > 0$, by assumption we have $-2\nu - 4 + m - a < 0$. Hence,

$$\int_{\underline{N}} \rho_\varepsilon^{-2\nu-4} \, \mathrm{d}V_{g^\varepsilon} = \int_{\underline{N}} \rho_\varepsilon^{-2\nu-4+m-a} \rho_\varepsilon^{-m+a} \, \mathrm{d}V_{g^\varepsilon} \leqslant C\varepsilon(t)^{-2\nu-4+m-a} \int_{\underline{N}} \rho_\varepsilon^{-m+a} \, \mathrm{d}V_{g^\varepsilon}.$$

As $\rho_\varepsilon \to \hat{r}$ and $\rho_\varepsilon^{-m+a} \leqslant C\hat{r}^{-m+a}$ on $\underline{N}$, where $\hat{r}$ is the intrinsic distance to the intersection point on $X_1 \cup X_2$, dominated convergence theorem implies

$$\int_{\underline{N}} \rho_\varepsilon^{-m+a} \, \mathrm{d}V_{g^\varepsilon} \leqslant \int_{X_1 \cup X_2} \hat{r}^{-m+a} \, \mathrm{d}V_{X'} \leqslant C(a) < \infty$$

for all $t \geqslant \Lambda$. It follows that, by choosing a sufficiently small such that $\mu > \frac{\nu+2}{m-2} + \frac{a}{2}$,

$$\int_{\underline{N}} |Q^\varepsilon(\mathrm{d}u)| \cdot w^\varepsilon \, \mathrm{d}V_{g^\varepsilon} \leqslant C(a)\varepsilon(t)^m t^{-2\mu + \frac{2}{m-2}(\nu+2+\frac{a}{2})} \leqslant C(a)\varepsilon(t)^m t^{-2(\mu - \frac{\nu+2}{m-2} - \frac{a}{2})},$$

which shows

$$|G(t)| \leqslant C\varepsilon(t)^m \cdot t^{-\overline{\zeta}}.$$

A similar argument using (8.6) gives

$$\frac{|G(t_1) - G(t_2)|}{|t_1 - t_2|^\alpha} \leqslant C\varepsilon(t)^m \cdot t^{-\overline{\zeta}}$$

providing $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$. ∎

Lemma 9.4. *Given $\psi \in P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}$ with $\|\psi\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} \leqslant 1$, let the triple $(u, a(t), b(t))$ be the solution to the Cauchy problem*

$$\begin{cases} \partial_t u - \mathcal{L}^{\varepsilon}_{\underline{0}}[u] = \psi + a(t) + b(t) w^{\varepsilon}, & \text{on } \underline{N} \times (\Lambda, \infty), \\ u(\cdot, \Lambda) = 0, & \text{on } \underline{N}, \end{cases}$$

with orthogonality condition $u \in \langle 1, w^{\varepsilon} \rangle^{\perp}$. Define

$$E(t) := \int_{\underline{N}} (\psi + b(t) w^{\varepsilon}) \cdot w^{\varepsilon} \, \mathrm{d}V_{g^{\varepsilon}}. \tag{9.11}$$

Then if $\mu > \frac{\nu+2+2\alpha}{m-2}$, then for $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$ and $\overline{\zeta} > 0$ satisfying $\overline{\zeta} < \mu - \frac{\nu+2+2\alpha}{m-2}$,

$$|E(t)| + \frac{|E(t_1) - E(t_2)|}{|t_1 - t_2|^{\alpha}} \leqslant C \, \varepsilon(t)^m t^{-\zeta}. \tag{9.12}$$

Proof. Write $E(t) = E_0(t) - E_1(t)$, where

$$E_0(t) = \int_{\underline{N}} \partial_t u \cdot w^{\varepsilon} \, \mathrm{d}V_{g^{\varepsilon}}, \quad E_1(t) = \int_{\underline{N}} \mathcal{L}^{\varepsilon}_{\underline{0}}[u] \cdot w^{\varepsilon} \, \mathrm{d}V_{g^{\varepsilon}}.$$

We first estimate $E_0(t)$. By differentiating orthogonality condition in time, we have

$$E_0(t) = -\int_{\underline{N}} u \cdot \partial_t w^{\varepsilon} \, \mathrm{d}V_{g^{\varepsilon}} - \int_{\underline{N}} u \cdot w^{\varepsilon} \, \partial_t \mathrm{d}V_{g^{\varepsilon}}.$$

It follows from Lemma 7.3 and Lemma 7.4, and the assumption on $\varepsilon(t)$ that

$$|E_0(t)| + \frac{|E_0(t_1) - E_0(t_2)|}{|t_1 - t_2|^{\alpha}} \leqslant C \, \varepsilon(t)^m \, t^{-\mu + \frac{1}{m-2}(\nu+2+2\alpha)} \|u\|_{P^{0,0,\alpha}_{\mu,\nu,\Lambda}}$$

for $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$.

To estimate $E_1(t)$, write

$$E_1(t) = \int_{\underline{N}} \Delta_{g^{\varepsilon}} u \cdot w^{\varepsilon} \, \mathrm{d}V_{g^{\varepsilon}} + \int_{\underline{N}} \mathcal{P}^{\varepsilon}_{\underline{0}}[u] \cdot w^{\varepsilon} \, \mathrm{d}V_{g^{\varepsilon}},$$

where $\mathcal{P}^{\varepsilon}_{\underline{0}} := \mathcal{L}^{\varepsilon}_{\underline{0}} - \Delta_{g^{\varepsilon}}$. Then by Lemma 7.4 and Lemma 7.11 we obtain

$$\begin{aligned} |E_1(t)| &\leqslant \int_{\underline{N}} |u| \cdot |\Delta w^{\varepsilon}| \, \mathrm{d}V_{g^{\varepsilon}} + \int_{\underline{N}} |\mathcal{P}^{\varepsilon}_{\underline{0}}[u]| \cdot |w^{\varepsilon}| \, \mathrm{d}V_{g^{\varepsilon}} \\ &\leqslant C \, \|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \, t^{-\mu + \frac{\nu+2}{m-2}} \, \varepsilon(t)^m. \end{aligned}$$

Similar estimate using the Hölder estimates in Lemma 7.4, Lemma 7.3 and Lemma 7.11 yields

$$\frac{|E_1(t_1) - E_1(t_2)|}{|t_1 - t_2|^{\alpha}} \leqslant C \, \|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} \, t^{-\mu + \frac{\nu+2+2\alpha}{m-2}} \, \varepsilon(t)^m$$

for $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$. Hence,

$$|E_1(t)| + \frac{|E_1(t_1) - E_1(t_2)|}{|t_1 - t_2|^{\alpha}} \leqslant C \, \varepsilon(t)^m \, t^{-\mu + \frac{\nu+2+2\alpha}{m-2}} \, \|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}$$

for $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$. Combining these estimates, along with Theorem 7.12, we obtain

$$|E(t)| + \frac{|E(t_1) - E(t_2)|}{|t_1 - t_2|^{\alpha}} \leqslant C \, \varepsilon(t)^m \, t^{-\mu + \frac{\nu+2+2\alpha}{m-2}}$$

for $0 < |t_1 - t_2| < t^{-\frac{2}{m-2}}$. □

Proof of Proposition 9.2. To show that $\mathscr{I}$ is continuous with respect to the norm of $P^{1,2,\alpha'}_{\mu',\nu,\Lambda} \times C^{0,\alpha'}_{\zeta',\Lambda}$, one may use a contradiction argument as in the proof of [3, Proposition 5.3].

For the estimate on $k(t)$, note that by definition we have

$$\begin{aligned} k(t) &= h(t) - \frac{m-2}{2c_L}\frac{V_1+V_2}{V_1V_2}\varepsilon^{-m}\|w^\varepsilon\|^2_{L^2}\cdot b(t) \\ &= -\varepsilon(t)^{-m}\frac{m-2}{2}\left\{\frac{\mathrm{d}\varepsilon^2(t)}{\mathrm{d}t} + \frac{\varepsilon^m(t)A}{c_L}\frac{V_1+V_2}{V_1V_2} + \frac{1}{c_L}\frac{V_1+V_2}{V_1V_2}\|w^\varepsilon\|^2_{L^2}b(t)\right\} \\ &= -\varepsilon(t)^{-m}\frac{(m-2)(V_1+V_2)}{2c_LV_1V_2}(G(t)+E(t)). \end{aligned}$$

It follows from Lemma 9.4 and Lemma 9.3 that we may choose $\bar{\zeta} > \zeta$ such that

$$|k(t)| + \frac{|k(t_1)-k(t_2)|}{|t_1-t_2|^\alpha} \leqslant Ct^{-\bar{\zeta}}, \quad 0 < |t_1-t_2| < t^{-\frac{2}{m-2}}. \tag{9.13}$$

$$\implies \|k\|_{P^{0,\alpha}_{\zeta,\Lambda}} \leqslant C\Lambda^{-(\bar{\zeta}-\zeta)}. \tag{9.14}$$

Finally, we may estimate v using (9.5), Proposition 8.1, and Proposition 8.3:

$$\begin{aligned} \|v\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}} &\leqslant C\left(\|\theta^\varepsilon_N\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} + \|\xi(0)\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}} + \|Q^\varepsilon[\mathrm{d}u]\|_{P^{0,0,\alpha}_{\mu,\nu+2,\Lambda}}\right) \\ &\leqslant C\left(\Lambda^{\mu-\frac{1}{m-2}(\tau(\nu+2)+(1-\tau)m)} + \Lambda^{\mu-\frac{1}{m-2}(m-2\alpha)} + \Lambda^{-\mu+\frac{\nu+2}{m-2}}\right). \end{aligned}$$

Taking Λ sufficiently large therefore ensures that $\mathscr{I}$ maps $\mathcal{B}^\alpha_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$ to itself, as required. □

9.3. **Proof of Theorem 9.1.** Consider the iteration map $\mathscr{I}$ defined in section 9.1. By Proposition 9.2, it may be viewed as a function on the product of unit balls, $\mathscr{I} : \mathcal{B}^\alpha_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda} \to \mathcal{B}^\alpha_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$.

By Lemma 7.7, $\mathcal{B}^\alpha_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$ is a compact subset of $P^{1,2,\alpha'}_{\mu',\nu,\Lambda}\times C^{0,\alpha'}_{\zeta',\Lambda}$ for any $\mu' < \mu$, $\alpha' < \alpha$, $\zeta' < \zeta$. Since $\mathscr{I}$ is a continuous map by Proposition 9.2, we may therefore apply the Schauder fixed point theorem to conclude that there exist $(u,h) \in \mathcal{B}^\alpha_{\mu,\nu,\Lambda}\times\mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$ such that $(v,k) := \mathscr{I}(u,h) = (u,h)$. Define $\varepsilon(t)$ and the Lagrangian embedding ι^ε using the function h as in (9.3). Since $h \in \mathcal{I}^{0,\alpha}_{\zeta,\Lambda}$, $\varepsilon(t)$ satisfies Assumption 6.1. By (9.4) and (9.8), the fixed point (u,h) satisfies

$$\begin{aligned} h(t) = k(t) &\implies b(t) = 0\ , \\ u(t) = v(t) &\implies \partial_t u - \mathcal{L}^\varepsilon_{\underline{0}}[u] = \theta_{N^\varepsilon} + \xi(0) + Q^\varepsilon[\mathrm{d}u] + a(t) \\ &\implies \partial_t u = \theta(\mathrm{d}u)\ +\ \xi(\mathrm{d}u) + a(t), \end{aligned}$$

as required.

Finally, since $u \in \mathcal{B}^\alpha_{\mu,\nu,\Lambda}$, we have

$$|\mathrm{d}u(x,t)|_{g^\varepsilon} \leqslant \|u\|_{P^{1,2,\alpha}_{\mu,\nu,\Lambda}}\, t^{-\mu}\rho^{-\nu-1}_{t^{-\frac{1}{m-2}}}(x) \leqslant C\, t^{-\mu+\frac{\nu+2}{m-2}}\,\varepsilon(t), \qquad \text{for all } (x,t) \in \underline{N}\times[\Lambda,\infty). \tag{9.15}$$

Since $\mu > \frac{\nu+2}{m-2}$, this shows that the time-dependent 1-form $\mathrm{d}u(\cdot,t)$ is contained in the Lagrangian neighbourhood $U_{N^{\varepsilon(t)}}$ for all $t \geqslant \Lambda$ for Λ sufficiently large. We may then apply

Proposition 4.2 to obtain a solution to the mean curvature flow given by $\Psi_{N^{\varepsilon(t)}} \circ \mathrm{d}u(\cdot, t)$. The estimate (9.15) implies that $\Psi_{N^{\varepsilon(t)}} \circ \mathrm{d}u$ converges to the immersion $\iota : X \to M$.

□

References

[1] W. Ao, A. DelaTorre, M. d. M. González, and J. Wei, *A gluing approach for the fractional Yamabe problem with isolated singularities*, J. Reine Angew. Math. **763** (2020), 25–78.

[2] T. Behrndt, *Generalized Lagrangian mean curvature flow in almost Calabi–Yau manifolds*, 2011. Thesis (Ph.D.)–University of Oxford.

[3] S. Brendle and N. Kapouleas, *Gluing Eguchi-Hanson metrics and a question of Page*, Comm. Pure Appl. Math. **70** (2017), no. 7, 1366–1401.

[4] A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001.

[5] J. Chen and A. Sun, *Mean curvature flow with multiplicity 2 convergence*, preprint, available at `arXiv:2312.17457`.

[6] J. Chen and W. He, *A note on singular time of mean curvature flow*, Math. Z. **266** (2010), no. 4, 921–931.

[7] B. Choi, P. Daskalopoulos, and J. King, *Type II singularities on complete non-compact Yamabe flow*, J. Reine Angew. Math. **772** (2021), 83–119.

[8] P. Daskalopoulos, M. del Pino, and N. Sesum, *Type II ancient compact solutions to the Yamabe flow*, J. Reine Angew. Math. **738** (2018), 1–71.

[9] R. Harvey and H. B. Lawson Jr., *Calibrated geometries*, Acta Math. **148** (1982), 47–157.

[10] K. Hattori, *New examples of compact special Lagrangian submanifolds embedded in hyper-Kähler manifolds*, J. Symplectic Geom. **17** (2019), no. 2, 301–335.

[11] D. Joyce, *Special Lagrangian submanifolds with isolated conical singularities. I. Regularity*, Ann. Global Anal. Geom. **25** (2004), no. 3, 201–251.

[12] ———, *Special Lagrangian submanifolds with isolated conical singularities. II. Moduli spaces*, Ann. Global Anal. Geom. **25** (2004), no. 4, 301–352.

[13] ———, *Special Lagrangian submanifolds with isolated conical singularities. III. Desingularisation, the unobstructed case*, Ann. Global Anal. Geom. **26** (2004), no. 1, 1–58.

[14] ———, *Special Lagrangian submanifolds with isolated conical singularities. IV. Desingularisation, obstructions and families*, Ann. Global Anal. Geom. **26** (2004), no. 2, 117–174.

[15] ———, *Special Lagrangian submanifolds with isolated conical singularities. V. Survey and applications*, J. Differential Geom. **63** (2003), no. 2, 279–347.

[16] ———, *Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow*, EMS Surv. Math. Sci. **2** (2015), no. 1, 1–62.

[17] G. Lawlor, *The angle criterion*, Invent. Math. **95** (1989), no. 2, 437–446.

[18] D. A. Lee, *Connected sums of special Lagrangian submanifolds*, Comm. Anal. Geom. **12** (2004), no. 3, 553–579.

[19] Y.-I. Lee, *Embedded special Lagrangian submanifolds in Calabi-Yau manifolds*, Comm. Anal. Geom. **11** (2003), no. 3, 391–423.

[20] R. Lockhart, *Fredholm, Hodge and Liouville theorems on noncompact manifolds*, Trans. Amer. Math. Soc. **301** (1987), no. 1, 1–35.

[21] R. B. Lockhart and R. C. McOwen, *Elliptic differential operators on noncompact manifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no. 3, 409–447.

[22] J. D. Lotay and A. Neves, *Uniqueness of Langrangian self-expanders*, Geom. Topol. **17** (2013), no. 5, 2689–2729.

[23] V. V. Lyčagin, *Sufficient orbits of a group of contact diffeomorphisms*, Mat. Sb. (N.S.) **104(146)** (1977), no. 2(10), 248–270, 335 (Russian).

[24] R. C. McLean, *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), no. 4, 705–747.

[25] A. Neves, *Singularities of Lagrangian mean curvature flow: zero-Maslov class case*, Invent. Math. **168** (2007), no. 3, 449–484.

[26] ———, *Finite time singularities for Lagrangian mean curvature flow*, Ann. of Math. (2) **177** (2013), no. 3, 1029–1076.

[27] Y.-G. Oh, *Mean curvature vector and symplectic topology of Lagrangian submanifolds in Einstein-Kähler manifolds*, Math. Z. **216** (1994), no. 3, 471–482.

[28] T. Pacini, *Special Lagrangian conifolds, II: gluing constructions in $\mathbb{C}^m$*, Proc. Lond. Math. Soc. (3) **107** (2013), no. 2, 225–266.

[29] K. Smoczyk, *A canonical way to deform a Lagrangian submanifold*, preprint, available at `arXiv:dg-ga/9605005`.

[30] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259.

[31] W.-B. Su, *Mean curvature flow of asymptotically conical Lagrangian submanifolds*, Trans. Amer. Math. Soc. **373** (2020), no. 2, 1211–1242.

[32] R. P. Thomas, *Moment maps, monodromy and mirror manifolds*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 467–498.

[33] R. P. Thomas and S.-T. Yau, *Special Lagrangians, stable bundles and mean curvature flow*, Comm. Anal. Geom. **10** (2002), no. 5, 1075–1113.

[34] J. Wei, Q. Zhang, and Y. Zhou, *On the parabolic gluing method and singularity formation*, C. R. Math. Acad. Sci. Soc. R. Can. **44** (2022), no. 4, 69–121.

[35] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I*, Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

National Center for Theoretical Sciences, Mathematics Division, Taipei 106319, Taiwan

Email address: `weibo.su@ncts.tw`

Department of Mathematics, National Taiwan University, Taipei 106319, Taiwan

Email address: `cjtsai@ntu.edu.tw`

Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

Email address: `albert.j.wood@kcl.ac.uk`