Super Nested Arrays: Linear Sparse Arrays with Reduced Mutual Coupling – Part II: High-Order Extensions

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Abstract

In array processing, mutual coupling between sensors has an adverse effect on the estimation of parameters (e.g., DOA). Sparse arrays such as nested arrays, coprime arrays, and minimum redundancy arrays (MRA) have reduced mutual coupling compared to uniform linear arrays (ULAs). These arrays also have a difference coarray with $O(N^2)$ virtual elements, where N is the number of physical sensors, and can therefore resolve $O(N^2)$ uncorrelated source directions. But these well-known sparse arrays have disadvantages: MRAs do not have simple closed-form expressions for the array geometry; coprime arrays have holes in the coarray; and nested arrays contain a dense ULA in the physical array, resulting in significantly higher mutual coupling than coprime arrays and MRAs. In a companion paper, a sparse array configuration called the (second-order) super nested array was introduced, which has many of the advantages of these sparse arrays, while removing most of the disadvantages. Namely, the sensor locations are readily computed for any N (unlike MRAs), and the difference coarray is exactly that of a nested array, and therefore hole-free. At the same time, the mutual coupling is reduced significantly (unlike nested arrays). In this paper, a generalization of super nested arrays is introduced, called the Qth-order super nested array. This has all the properties of the second-order super nested array with the additional advantage that mutual coupling effects are further reduced for $Q > 2$. Many theoretical properties are proved and simulations are included to demonstrate the superior performance of these arrays.

Index Terms

Sparse arrays, nested arrays, coprime arrays, super nested arrays, mutual coupling, DOA estimation.

I. Introduction

In array processing, mutual coupling between sensors has an adverse effect on the estimation of parameters (e.g., DOA) [1]–[7]. Sparse arrays such as nested arrays [8], coprime arrays [9], and minimum redundancy arrays (MRA)

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[10] have reduced mutual coupling compared to uniform linear arrays (ULAs). These arrays also have a difference coarray with $O(N^2)$ virtual elements, where N is the number of physical sensors, and can therefore resolve $O(N^2)$ uncorrelated source directions. But these sparse arrays have shortcomings: MRAs do not have simple closed-form expressions for the array geometry [10]; coprime arrays have holes in the coarray [9]; and nested arrays contain a dense ULA in the physical array [8], resulting in significantly higher mutual coupling than coprime arrays and MRAs. For details please see the companion paper [11] and references therein.

In the companion paper [11], a sparse array configuration called the (second-order) super nested array was introduced, which has many of the advantages of these sparse arrays, while removing some of the disadvantages. Namely, the sensor locations are well-defined and readily computed for any N (unlike MRAs), and the difference coarray is exactly that of a nested array, and therefore hole-free. At the same time, the mutual coupling is reduced compared to nested arrays. Super nested arrays were designed by rearranging the dense ULA part of a nested array in such a way that the coarray remains unchanged, but mutual coupling is reduced by reducing the number of elements with small inter-element spacings. Quantitatively, this is described in terms of the weight function $w(m)$, which is equal to the number of sensor pairs whose inter-element spacing is $m\lambda/2$. It was shown in [11] that the first three weight functions of second-order super nested arrays are

$$w(1) = \begin{cases} 2, & \text{if } N_1 \text{ is even}, \\ 1, & \text{if } N_1 \text{ is odd}, \end{cases} \tag{1}$$

$$w(2) = \begin{cases} N_1 - 3, & \text{if } N_1 \text{ is even}, \\ N_1 - 1, & \text{if } N_1 \text{ is odd}, \end{cases} \tag{2}$$

$$w(3) = \begin{cases} 3, & \text{if } N_1 = 4, 6, \\ 4, & \text{if } N_1 \text{ is even}, N_1 \geq 8, \\ 1, & \text{if } N_1 \text{ is odd}, \end{cases} \tag{3}$$

Contrast this with the nested array which has $w(1) = N_1, w(2) = N_1 - 1$ and $w(3) = N_1 - 2$. While $w(1)$ and $w(3)$ are significantly better in (1) and (3), there is plenty of room for improving $w(2)$, and possibly $w(m), m > 3$.

In this paper, a generalization of super nested arrays is introduced and called the *Qth-order super nested array*. It has all the good properties of the second-order super nested array with the additional advantage that mutual coupling effects are further reduced for $Q > 2$. For a given number of physical array elements N, Qth-order super nested arrays have the following properties: (a) the sensor locations can be defined using a simple algorithm, (b) the physical array has the same aperture as the nested array, (c) the difference coarray is exactly identical to that of the nested array (hence hole free), and (d) the weight functions are further improved, compared even to second-order super nested arrays.

Like the parent nested array, the physical sensor locations of Qth-order super nested arrays are related to two integers N_1 and N_2 (Fig. 3(a) of [11]). The detailed description of Qth-order super nested arrays depends on whether N_1 is even or odd. For odd N_1, there is a simple closed-form expression for the sensor locations, but for even N_1, the locations have to be defined recursively as we shall elaborate. A MATLAB code to find the sensor locations of Qth-order super nested arrays is given in [12]. The proof that Qth-order super nested arrays have a coarray identical to the parent nested array is rather involved, and one of the main goals of the paper is to establish this very important result for both N_1 odd and N_1 even. We also analyze the weight functions $w(m)$ in great depth (again, quite involved in its detail because of the intricate definition of the array geometry). The good news is that it is possible to improve the crucial weights $w(1)$, $w(2)$, and $w(3)$, compared to nested arrays (see Theorem 2 and Theorem 4). In particular, $w(2)$ is only about half that of second-order super nested arrays.

While the results of [11] can in principle be regarded as special cases of this paper, it is very uneconomic to present them as special cases of this paper. The reason is that the proofs in this paper are very complicated – they use induction, with the proofs in [11] as the basis for induction. The clarity of [11] would have been greatly compromised if it had been presented as a special case of this.

A. *Paper Outline*

Section II introduces Qth-order super nested arrays in terms of the parent nested array. The construction is based on some recursive rules to rearrange the sensors of nested arrays through successive systematic stages. In Section III, we formally define Qth-order super nested arrays for odd N_1. Many properties of these arrays are given, the highlights being (a) the result that the difference coarray is identical to that of the parent nested array (Theorem 1 and Corollary 1), and (b) that the weight functions (hence mutual coupling effects) are significantly reduced (Theorem 2). Since the details are considerably different for even N_1, Section IV is dedicated to a presentation of this case. Detailed proofs of some of the claims of Section III and IV are relegated to Section V and VI, for ease of flow. Section VII presents simulation results and detailed comparison of performances, demonstrating clearly that Qth-order super nested arrays with $Q > 2$ outperform other arrays in the presence of mutual coupling.

II. General Guidelines for the Construction of Super Nested Arrays

Assume that sensors are defined over physical locations nd, where n belongs to some integer set $\mathbb{S}$. $d = \lambda/2$ is the minimum sensor separation among sensors, where λ is the wavelength of the incoming wave. For instance, nested arrays with N_1 and N_2 have the following integer set:

$$\begin{aligned}\mathbb{S}_{\text{nested}} = \{&1, 2, \ldots, N_1, \\ &(N_1+1), 2(N_1+1), \ldots N_2(N_1+1)\},\end{aligned} \tag{4}$$

where N_1 and N_2 are positive integers. Second-order super nested arrays [11] define the sensor locations using an integer set $\mathbb{S}^{(2)}$, which can be partitioned into five ULA portions ($\mathbb{X}_1^{(2)}$, $\mathbb{Y}_1^{(2)}$, $\mathbb{X}_2^{(2)}$, $\mathbb{Y}_2^{(2)}$, $\mathbb{Z}_1^{(2)}$) and an additional set ($\mathbb{Z}_2^{(2)}$).

Fig. 1 summarizes the hierarchy among nested arrays, second-order super nested arrays, and Qth-order super nested arrays. It has been mentioned in [11] that the sets $\mathbb{X}_1^{(2)}$, $\mathbb{Y}_1^{(2)}$, $\mathbb{X}_2^{(2)}$, and $\mathbb{Y}_2^{(2)}$ are obtained by rearranging

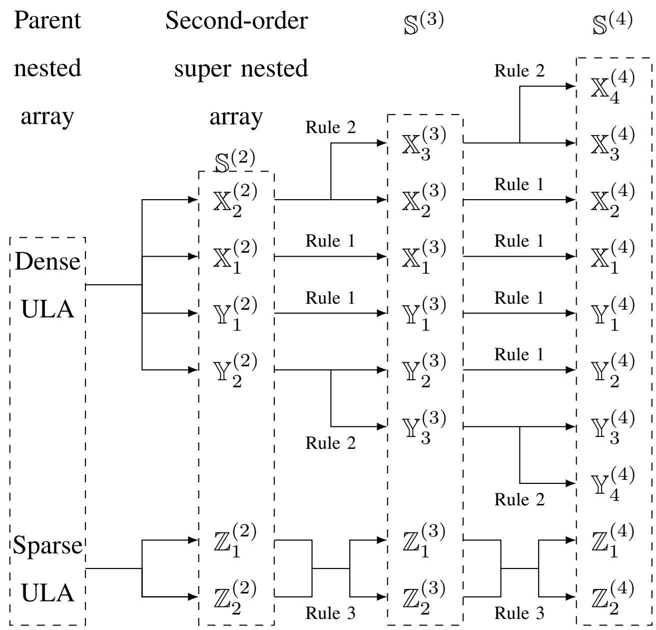


Fig. 1. Hierarchy of nested arrays, second-order super nested arrays $\mathbb{S}^{(2)}$, and Qth-order super nested arrays $\mathbb{S}^{(Q)}$. Arrows indicate the origin of the given sets. For instance, $\mathbb{X}_2^{(4)}$ originates from $\mathbb{X}_2^{(3)}$ while $\mathbb{Y}_3^{(3)}$ is split into $\mathbb{Y}_3^{(4)}$ and $\mathbb{Y}_4^{(4)}$. It can be observed that the sets $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ result from the dense ULA part of nested arrays. The sparse ULA portion of nested arrays is rearranged into the sets $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$.

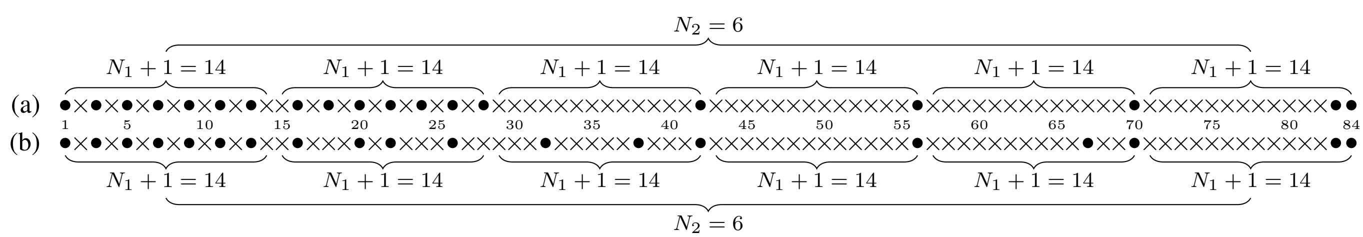


Fig. 2. 1D representations of (a) second-order super nested arrays, $\mathbb{S}^{(2)}$, and (b) third-order super nested arrays, $\mathbb{S}^{(3)}$, where $N_1 = 13$ and $N_2 = 6$. Bullets denote sensor locations while crosses indicate empty locations.

the dense ULA part of parent nested arrays, as in Lemma 1 of [11]. The sparse ULA part of parent nested arrays is reorganized into $\mathbb{Z}_1^{(2)}$ and $\mathbb{Z}_2^{(2)}$ of second-order super nested arrays [11].

The formal definition of Qth-order nested arrays will be given in the next section. To develop some feeling for it, first consider $Q = 3$. Third-order super nested arrays, as specified by the integer set $\mathbb{S}^{(3)}$, consist of eight sets as follows: $\mathbb{X}_1^{(3)}$, $\mathbb{Y}_1^{(3)}$, $\mathbb{X}_2^{(3)}$, $\mathbb{Y}_2^{(3)}$, $\mathbb{X}_3^{(3)}$, $\mathbb{Y}_3^{(3)}$, $\mathbb{Z}_1^{(3)}$, and $\mathbb{Z}_2^{(3)}$, which can be recursively generated from the sets $\mathbb{X}_1^{(2)}$, $\mathbb{Y}_1^{(2)}$, $\mathbb{X}_2^{(2)}$, $\mathbb{Y}_2^{(2)}$, $\mathbb{Z}_1^{(2)}$, $\mathbb{Z}_2^{(2)}$ in second-order super nested arrays. For instance, $\mathbb{X}_1^{(3)}$ is identical to $\mathbb{X}_1^{(2)}$. $\mathbb{X}_2^{(2)}$ is split into two sets $\mathbb{X}_2^{(3)}$ and $\mathbb{X}_3^{(3)}$. The same connections also apply to $\mathbb{Y}_1^{(2)}$, $\mathbb{Y}_2^{(2)}$, $\mathbb{Y}_1^{(3)}$, $\mathbb{Y}_2^{(3)}$, and $\mathbb{Y}_3^{(3)}$. Finally, the elements in $\mathbb{Z}_1^{(2)}$ and $\mathbb{Z}_2^{(2)}$ are rearranged into $\mathbb{Z}_1^{(3)}$ and $\mathbb{Z}_2^{(3)}$. Hence, it can be interpreted that the sets $\mathbb{X}_q^{(3)}$ and $\mathbb{Y}_q^{(3)}$ for $q = 1, 2, 3$ originate from the dense ULA of parent nested arrays while $\mathbb{Z}_1^{(3)}$ and $\mathbb{Z}_2^{(3)}$ emanate from the

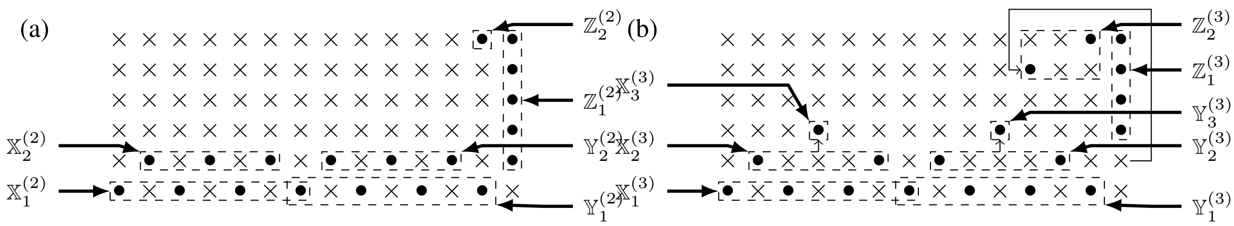


Fig. 3. 2D representations of (a) second-order super nested arrays, $\mathbb{S}^{(2)}$, and (b) third-order super nested arrays, $\mathbb{S}^{(3)}$, where $N_1 = 13$ and $N_2 = 6$. Bullets denote sensor locations while crosses indicate empty locations. The dashed rectangles mark the sets $\mathbb{X}_q^{(Q)}$, $\mathbb{Y}_q^{(Q)}$, $\mathbb{Z}_1^{(Q)}$, and $\mathbb{Z}_2^{(Q)}$ for $1 \leq q \leq Q$. Thin arrows illustrate how sensors migrate from $\mathbb{S}^{(Q-1)}$ to $\mathbb{S}^{(Q)}$.

sparse ULA of parent nested arrays.

Fourth-order super nested arrays (or super nested arrays with $Q = 4$) generalize third-order super nested arrays further. It can be deduced from Fig. 1 that $\mathbb{X}_3^{(3)}$ and $\mathbb{Y}_3^{(3)}$ are divided into $\mathbb{X}_3^{(4)}$, $\mathbb{X}_4^{(4)}$ and $\mathbb{Y}_3^{(4)}$, $\mathbb{Y}_4^{(4)}$, respectively. Similarly, $\mathbb{Z}_1^{(3)}$ and $\mathbb{Z}_2^{(3)}$ are rearranged into $\mathbb{Z}_1^{(4)}$ and $\mathbb{Z}_2^{(4)}$. The remaining sets of fourth-order super nested arrays are the same as their correspondences in third-order super nested arrays. To be more specific, the defining rules to go from $(Q-1)$th-order super nested arrays to Qth-order super nested arrays are

***Rule 1*:** $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ replicate $\mathbb{X}_q^{(Q-1)}$ and $\mathbb{Y}_q^{(Q-1)}$, respectively, for $1 \leq q \leq Q-2$. That is, we simply copy these portions from the $(Q-1)$th-order super nested array to the Qth-order super nested array.

***Rule 2*:** $\mathbb{X}_{Q-1}^{(Q-1)}$ and $\mathbb{Y}_{Q-1}^{(Q-1)}$ are split into $\mathbb{X}_{Q-1}^{(Q)}$, $\mathbb{X}_Q^{(Q)}$ and $\mathbb{Y}_{Q-1}^{(Q)}$, $\mathbb{Y}_Q^{(Q)}$, respectively, according to rules to be specified in Section III and IV.

***Rule 3*:** $\mathbb{Z}_1^{(Q-1)}$ and $\mathbb{Z}_2^{(Q-1)}$ are reorganized into $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$, using appropriate rules.

Next, we give a concrete example of how Qth-order super nested arrays are obtained from $(Q-1)$th-order super nested arrays. Fig. 2 and 3 depict the 1D/2D representations of the second-order super nested array (in parts (a)) and the third-order one (in parts (b)), respectively, where the details of 2D representations can be found in Fig. 1 of [11]. In this example, it is obvious that $\mathbb{X}_1^{(2)} = \mathbb{X}_1^{(3)}$ and $\mathbb{Y}_1^{(2)} = \mathbb{Y}_1^{(3)}$, which satisfy Rule 1. To explain Rule 2, we consider the following sets in Fig. 3:

$$\mathbb{X}_2^{(2)} = \{16, 18, 20\}, \quad \mathbb{X}_2^{(3)} = \{16, 20\}, \quad \mathbb{X}_3^{(3)} = \{32\}. \tag{5}$$

The middle element of $\mathbb{X}_2^{(2)}$, which is the element 18 in this case, is selected and relocated to the third layer of 2D representations. It becomes the element 32 in $\mathbb{X}_3^{(3)}$. The remaining elements in $\mathbb{X}_2^{(2)}$, which correspond to sensor locations 16 and 20, constitute $\mathbb{X}_2^{(3)}$. Finally, Rule 3 can also be clarified using Fig. 3. In the second-order super nested array, we consider the sensor located at $2(N_1 + 1) = 28$, which is the leftmost element of $\mathbb{Z}_1^{(2)}$. However, this sensor is removed from $\mathbb{S}^{(2)}$ and inserted to $\mathbb{S}^{(3)}$ at location 67, as indicated by a thin arrow in Fig. 3(b). This new sensor location is included in $\mathbb{Z}_2^{(3)} = \{67, 83\}$, which explains Rule 3. Furthermore, after all these operations, the first layer in 2D representations does not change while only some elements (18, 24, and 28 in Fig. 3) in the second layer are rearranged to somewhere else.

Summarizing, Qth-order super nested arrays can be recursively generated from $(Q-1)$th-order super nested arrays, as elaborated in Fig. 1. In the following two sections, based on the parameter N_1, we will give formal definitions for super nested arrays, which are consistent with Rule 1, 2, and 3. These definitions also enable us to determine the sensor locations explicitly.

III. QTH-ORDER SUPER NESTED ARRAYS, N_1 IS ODD

Here is the formal definition of Qth-order super nested arrays if N_1 is an odd number:

Definition 1 (Qth-order super nested arrays, N_1 is odd)**.** *Let N_1 be an odd number, $N_2 \geq 2Q-1$, and $Q \geq 1$. Qth-order super nested arrays are characterized by the integer set $\mathbb{S}^{(Q)}$, defined by*

$$\mathbb{S}^{(Q)} = \left(\bigcup_{q=1}^{Q} \mathbb{X}_q^{(Q)} \cup \mathbb{Y}_q^{(Q)} \right) \cup \mathbb{Z}_1^{(Q)} \cup \mathbb{Z}_2^{(Q)}.$$

For a positive integer q satisfying $1 \leq q \leq Q$, $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ are defined as

$$\begin{aligned}
\mathbb{X}_q^{(Q)} &= \left\{ (q-1)(N_1+1) + 2^{q-1} + d_q^{(Q)}\ell \mid 0 \leq \ell \leq L_q^{(Q)} \right\}, \\
\mathbb{Y}_q^{(Q)} &= \left\{ q(N_1+1) - 2^{q-1} - d_q^{(Q)}\ell \mid 0 \leq \ell \leq L_q^{(Q)} \right\}, \\
d_q^{(Q)} &= \begin{cases} 2^q, & \textit{if } q = 1, 2, \ldots, Q-1, \\ 2^{Q-1}, & \textit{if } q = Q, \end{cases} \\
L_q^{(Q)} &= \begin{cases} \left\lfloor \frac{1}{2} \left(\frac{N_1+1}{2^q} - 1 \right) \right\rfloor, & \textit{if } q = 1, 2, \ldots, Q-1, \\ \left\lfloor \frac{N_1+1}{2^Q} - 1 \right\rfloor, & \textit{if } q = Q, \end{cases}
\end{aligned}$$

where $\lfloor \cdot \rfloor$ is the floor function. $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$ are given by

$$\begin{aligned}
\mathbb{Z}_1^{(Q)} &= \left\{ \ell(N_1+1) \mid Q \leq \ell \leq N_2 \right\}, \\
\mathbb{Z}_2^{(Q)} &= \left\{ (N_2+1-q)(N_1+1) - 2^q + 1 \mid 1 \leq q \leq Q-1 \right\}.
\end{aligned}$$

For convenience of the reader, here is a MATLAB code for Qth-order super nested arrays [12]. In particular, `super_nested.m` returns the set $\mathbb{S}^{(Q)}$ given the array parameters N_1, N_2, and Q.

If $Q = 1$, the corresponding array configuration degenerates to nested arrays with parameter N_1 and N_2. Putting $Q = 2$ in Definition 1 gives us Definition 7 in [11][1]. For any pair of N_1, N_2, and Q satisfying the assumption of Definition 1, super nested arrays can be characterized in a closed-form and scalable fashion.

It can be inferred from Definition 1 that the inter-element spacing of $\mathbb{X}_q^{(Q)}$, $\mathbb{Y}_q^{(Q)}$, and $\mathbb{Z}_1^{(Q)}$ are $d_q^{(Q)}$, $d_q^{(Q)}$, N_1+1, respectively. For instance, in Fig. 3(b), it can be seen that $\mathbb{X}_1^{(3)}$ and $\mathbb{Y}_1^{(3)}$ are ULA with sensor separation 2. The sensor separation for $\mathbb{X}_2^{(3)}$ and $\mathbb{Y}_2^{(3)}$ is $d_2^{(3)} = 4$. $\mathbb{Z}_1^{(3)}$ is a ULA of sensor separation $N_1 + 1 = 14$. This property is

[1]Here $\mathbb{X}_q^{(2)}$ and $\mathbb{Y}_q^{(2)}$ for $q = 1, 2$, could be slightly different across Definition 7 of [11] and Definition 1. In Definition 7 of [11], these sets are disjoint while in Definition 1, they might not be disjoint. Even so, both definitions lead to the same $\mathbb{S}^{(2)}$ but the latter one possesses the symmetric property: $|\mathbb{X}_q^{(Q)}| = |\mathbb{Y}_q^{(Q)}|$, which will be more useful in the following development.

very similar to second-order super nested arrays. Notice from Fig. 3(a) that $\mathbb{S}^{(2)}$ consists of a set of ULAs $\mathbb{X}_1^{(2)}$, $\mathbb{X}_2^{(2)}$, $\mathbb{Y}_1^{(2)}$, and $\mathbb{Y}_2^{(2)}$, each with sensor separation 2, another ULA $\mathbb{Z}_1^{(2)}$ with sensor separation $N_1+1=14$, and finally a singleton $\mathbb{Z}_2^{(2)}$.

Now we show that if an array is constructed according to Definition 1, then it satisfies Rule 1, 2, and 3 in Section II. This statement is obviously true for Rule 1. For Rule 2 and 3, the details can be clarified by the following two lemmas:

Lemma 1. *Let N_1 be an odd number and $\mathbb{S}^{(Q)}$ be a super nested array with order Q, as defined in Definition 1. Then $\mathbb{X}_{Q-1}^{(Q)}$ is composed of even terms (related to even ℓ) of $\mathbb{X}_{Q-1}^{(Q-1)}$ and $\mathbb{X}_{Q}^{(Q)}-(N_1+1)$ consists of odd terms (related to odd ℓ) of $\mathbb{X}_{Q-1}^{(Q-1)}$. These properties also hold true for $\mathbb{Y}_{Q-1}^{(Q-1)}$, $\mathbb{Y}_{Q-1}^{(Q)}$, and $\mathbb{Y}_{Q}^{(Q)}$.*

Proof: According to Definition 1, any element in $\mathbb{X}_{Q-1}^{(Q-1)}$ can be written as $(Q-2)(N_1+1)+2^{Q-2}+2^{Q-2}\ell$ where $0\le \ell \le L_{Q-1}^{(Q-1)}$. If $\ell=2k$ is an even number, we obtain

$$\begin{aligned}&(Q-2)(N_1+1)+2^{Q-2}+2^{Q-1}k\\&=(Q-2)(N_1+1)+2^{Q-2}+d_{Q-1}^{(Q)}k,\end{aligned}$$

where $0\le k\le \frac{1}{2}L_{Q-1}^{(Q-1)}$. Since k is an integer and $\lfloor x\rfloor \le \frac{1}{2}\lfloor 2x\rfloor < \lfloor x\rfloor+1$, $0\le k\le \frac{1}{2}L_{Q-1}^{(Q-1)}$ is equivalent to $0\le k\le L_{Q-1}^{(Q)}$. That is, even terms of $\mathbb{X}_{Q-1}^{(Q-1)}$ are exactly $\mathbb{X}_{Q-1}^{(Q)}$.

If $\ell=2k+1$ is an odd number, the elements are,

$$\begin{aligned}&(Q-2)(N_1+1)+2^{Q-2}+2^{Q-2}(2k+1)\\&=\left[(Q-1)(N_1+1)+2^{Q-1}+d_{Q}^{(Q)}k\right]-(N_1+1),\end{aligned}$$

where k is a non-negative integer with $0\le 2k+1\le L_{Q-1}^{(Q-1)}$. The range of k can be rearranged to be $0\le k\le L_{Q}^{(Q)}$ because $\lfloor x\rfloor \le \frac{1}{2}\lfloor 2x\rfloor < \lfloor x\rfloor+1$. It can be deduced that odd terms in $\mathbb{X}_{Q-1}^{(Q-1)}$ are exactly $\mathbb{X}_{Q}^{(Q)}-(N_1+1)$. The proof for $\mathbb{Y}_{Q-1}^{(Q-1)}$, $\mathbb{Y}_{Q-1}^{(Q)}$, and $\mathbb{Y}_{Q}^{(Q)}$ is similar. ■

Lemma 2. *Let N_1 be an odd number. Assume that $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$ satisfy Definition 1. Then*

$$\begin{aligned}\mathbb{Z}_1^{(Q)}&=\mathbb{Z}_1^{(Q-1)}\backslash\{(Q-1)(N_1+1)\},\\ \mathbb{Z}_2^{(Q)}&=\mathbb{Z}_2^{(Q-1)}\cup\{(N_2+1-(Q-1))(N_1+1)-2^{Q-1}+1\},\end{aligned}$$

where $\mathbb{A}\backslash\mathbb{B}$ denotes the relative complement of $\mathbb{B}$ in $\mathbb{A}$.

Proof: This proof follows from Definition 1 directly. ■

We now prove that the $\mathbb{X}_q^{(Q)}$ and the $\mathbb{Y}_q^{(Q)}$ parts of the super nested array, reduced modulo N_1+1 are exactly equal to the dense part of the parent nested array:

Lemma 3 (Relation to dense ULA of nested array)**.** *Let $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ for $q=1,2,\ldots,Q$ be defined in Definition*

1. Define the sets $\mathbb{A}_q^{(Q)} = \mathbb{X}_q^{(Q)} - (q-1)(N_1+1)$ *and* $\mathbb{B}_q^{(Q)} = \mathbb{Y}_q^{(Q)} - (q-1)(N_1+1)$*. Then*

$$\bigcup_{q=1}^{Q} \mathbb{A}_q^{(Q)} \cup \mathbb{B}_q^{(Q)} = \{1, 2, \ldots, N_1\}.$$

Proof: This lemma is proved using induction on Q since when $Q = 2$, Lemma 1 in [11] holds true. Assuming Lemma 3 holds for $Q-1$, the case of Q becomes

$$\begin{aligned}
\bigcup_{q=1}^{Q} \mathbb{A}_q^{(Q)} \cup \mathbb{B}_q^{(Q)} &= \bigcup_{q=1}^{Q-2} \mathbb{A}_q^{(Q)} \cup \mathbb{B}_q^{(Q)} \\
&\cup \left((\mathbb{X}_{Q-1}^{(Q)} \cup (\mathbb{X}_Q^{(Q)} - (N_1+1))) - (Q-2)(N_1+1) \right) \\
&\cup \left((\mathbb{Y}_{Q-1}^{(Q)} \cup (\mathbb{Y}_Q^{(Q)} - (N_1+1))) - (Q-2)(N_1+1) \right) \\
&= \left(\bigcup_{q=1}^{Q-2} \mathbb{A}_q^{(Q-1)} \cup \mathbb{B}_q^{(Q-1)} \right) \cup (\mathbb{X}_{Q-1}^{(Q-1)} - (Q-2)(N_1+1)) \\
&\cup (\mathbb{Y}_{Q-1}^{(Q-1)} - (Q-2)(N_1+1)) \\
&= \bigcup_{q=1}^{Q-1} \mathbb{A}_q^{(Q-1)} \cup \mathbb{B}_q^{(Q-1)} = \{1, 2, \ldots, N_1\}.
\end{aligned} \tag{6}$$

Here Rule 1 and Lemma 1 are utilized. ■

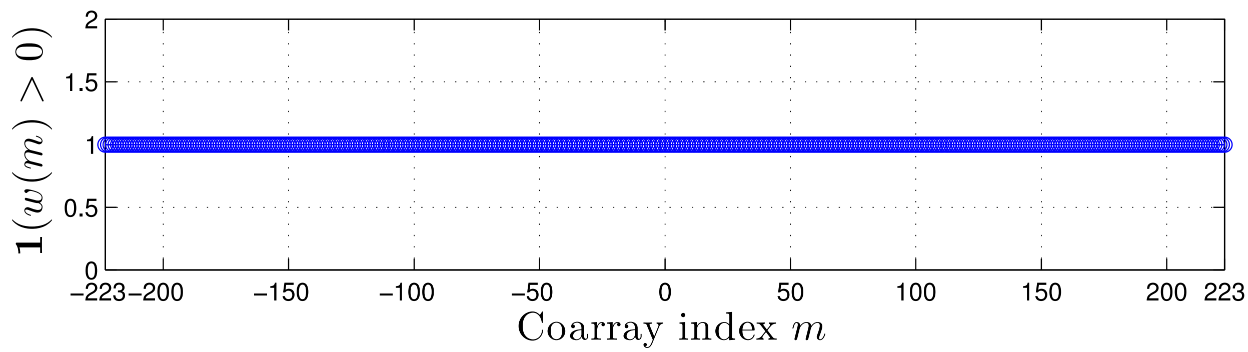
Lemma 4 (Total number of sensors)**.** *If* N_1 *is an odd number, the number of elements in* $\mathbb{S}^{(Q)}$*, as defined in Definition 1, is* $N_1 + N_2$*.*

Proof: The proof is given by induction on Q. According to Lemma 2 in [11], the cardinality of $\mathbb{S}^{(2)}$ is N_1+N_2. If $\mathbb{S}^{(Q-1)}$ has cardinality $N_1 + N_2$, we will show that $\mathbb{S}^{(Q)}$ also has cardinality $N_1 + N_2$. According to Rule 1, the number of elements in $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ for $1 \leq q \leq Q-2$ remains unchanged in $\mathbb{S}^{(Q-1)}$ and $\mathbb{S}^{(Q)}$. Lemma 1 does not alter the total number of elements since $\mathbb{X}_{Q-1}^{(Q)}$ and $\mathbb{X}_Q^{(Q)} - (N_1+1)$ correspond to the even and odd terms in $\mathbb{X}_{Q-1}^{(Q-1)}$, respectively. It is also evident that Lemma 2 preserves the total number of sensors. By induction, $\left|\mathbb{S}^{(Q)}\right| = \left|\mathbb{S}^{(Q-1)}\right| = N_1 + N_2$ for $Q \geq 2$. ■

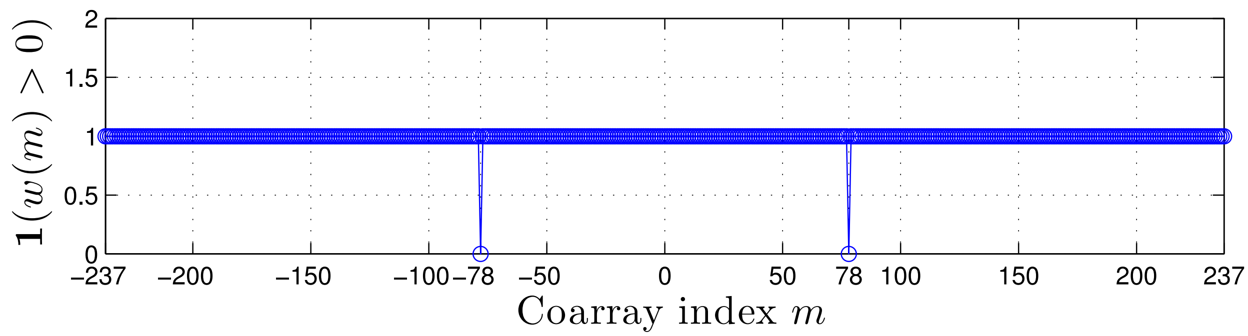
One of the most striking properties of the Qth-order super nested array is that the coarray is exactly identical to that of the parent nested array. This is proved in the following theorem and the corollary:

Theorem 1. *If* $N_1 \geq 3 \cdot 2^Q - 1$ *is an odd number,* $N_2 \geq 3Q - 4$*, and* $Q \geq 3$*, then* Q*th-order super nested arrays are restricted arrays, i.e., the difference coarray is hole-free.*

Proof: This proof is based on induction on Q. Beginning with Theorem 1 in [11], we know $\mathbb{S}^{(2)}$ are restricted arrays. If $(Q-1)$th-order super nested arrays are restricted arrays, it can be inferred that Qth-order super nested arrays are still restricted arrays. The details are quite involved, and can be found in Section V. To clarify these details, a numerical demonstration of the mechanics of the proof is also provided in Section I of the supplementary document [13]. ■



(a) $N_1 = 31, N_2 = 7, Q = 5$.



(b) $N_1 = 33, N_2 = 7, Q = 5$.

Fig. 4. An example to show that $N_1 \geq 3 \cdot 2^Q - 1$ is not necessary in order to make the coarray of $\mathbb{S}^{(Q)}$ hole free. Here we consider the indicator function of $w(m) > 0$ for the super nested array with (a) $N_1 = 31, N_2 = 7, Q = 5$ and (b) $N_1 = 33, N_2 = 7, Q = 5$. It can be inferred that (a) is a restricted array, because $w(m) > 0$ for $-223 \leq m \leq 223$. However, (b) is not a restricted array since $w(78) = w(-78) = 0$.

Corollary 1. *If $N_1 \geq 3 \cdot 2^Q - 1$ is an odd number, $N_2 \geq 3Q - 4$, and $Q \geq 3$, then Qth-order super nested arrays have the same coarray as their parent nested array.*

Proof: Due to Theorem 1, applying the chain of arguments in Corollary 1 of [11] proves this corollary. ■

The *sufficient* conditions on N_1, N_2, and Q in Theorem 1 guarantee that such array configuration is a restricted array. However, these conditions are not necessary. For instance, Fig. 4 examines the coarray of Qth-order super nested arrays if (a) $N_1 = 31, N_2 = 7, Q = 5$ and (b) $N_1 = 33, N_2 = 7, Q = 5$, where the indicator function $\mathbf{1}(P)$ is 1 if the statement P is true and 0 if P is false. Theorem 1 requires N_1 to be at least $3 \cdot 2^Q - 1 = 95$. In Fig. 4(a) ($N_1 = 31 < 95$), it can be inferred that $w(m) > 0$ for $-223 \leq m \leq 223$ so this array configuration is a restricted array. On the other hand, the array in Fig. 4(b) ($N_1 = 33 < 95$) is not a restricted array since $w(78) = w(-78) = 0$.

Recall that the weight function $w(2)$ of the second-order super nested array was as in Eq. (2). The next theorem shows that the super nested array for $Q > 2$ has significantly improved weight function $w(2)$, which is crucial to reducing the mutual coupling effects.

Theorem 2. *Assume that $N_1 \geq 3 \cdot 2^Q - 1$ is an odd number, $N_2 \geq 3Q - 4$, and $Q \geq 3$. The weight function $w(m)$*

of Qth-order super nested arrays satisfies

$$w(1) = 1, \qquad w(2) = 2\left\lfloor \frac{N_1}{4} \right\rfloor + 1, \qquad w(3) = 2.$$

Proof: For $m = 1$, the sensors located at $N_2(N_1+1)-1$ and $N_2(N_1+1)$ contribute to $w(1)$. We need to show other combinations do not result in $w(1)$. It is obvious that the self-differences among $\mathbb{X}_q^{(Q)}$, $\mathbb{Y}_q^{(Q)}$, $\mathbb{Z}_1^{(Q)}$, and $\mathbb{Z}_2^{(Q)}$ have sensor separation at least 2. Since these sets are defined in the increasing order, it suffices to show that the difference between the maximum element in one set and the minimum element in the succeeding set, is strictly greater than 1. Assume $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ satisfies $\min(\mathbb{Y}_q^{(Q)}) - \max(\mathbb{X}_q^{(Q)}) = 1$. We have $L_q^{(Q)} = (N_1 - 2^q)/(2d_q^{(Q)})$. This is a contradiction since $L_q^{(Q)}$ is an integer but $(N_1 - 2^q)/(2d_q^{(Q)})$ is not, if N_1 is an odd number. On the other hand, it is obvious that $\mathbb{Y}_q^{(Q)}$ and $\mathbb{X}_{q+1}^{(Q)}$ do not cause $w(1)$.

Next, $w(2)$ results from the self difference in $\mathbb{X}_1^{(Q)} \cup \mathbb{Y}_1^{(Q)}$. First, we check the difference between $\min(\mathbb{Y}_1^{(Q)})$ and $\max(\mathbb{X}_1^{(Q)})$:

$$\min(\mathbb{Y}_1^{(Q)}) - \max(\mathbb{X}_1^{(Q)}) = \begin{cases} 0, & \text{if } N_1 = 4r+1, \\ 2, & \text{if } N_1 = 4r+3. \end{cases} \tag{7}$$

Besides, consider the sensor pair located at $(q-1)(N_1+1) + 2L_1^{(Q)} \in \mathbb{X}_q^{(Q)}$ and $q(N_1+1) - 2L_1^{(Q)} \in \mathbb{Y}_q^{(Q)}$ for some $2 \leq q \leq Q$. The exact value of q can be uniquely solved from the definitions of $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$. Their difference becomes

$$\begin{aligned} &(q(N_1+1) - 2L_1^{(Q)}) - ((q-1)(N_1+1) + 2L_1^{(Q)}) \\ =& \begin{cases} 2, & \text{if } N_1 = 4r+1, \\ 4, & \text{if } N_1 = 4r+3. \end{cases} \end{aligned} \tag{8}$$

If $N_1 = 4r+1$, there are $L_1^{(Q)}$ pairs in $\mathbb{X}_1^{(Q)}$ and $L_1^{(Q)}$ in $\mathbb{Y}_1^{(Q)}$ with separation 2. One more pair can be found in (8). In this case, $w(2)$ becomes

$$w(2) = 2L_1^{(Q)} + 1 = 2r + 1 = 2\left\lfloor \frac{N_1}{4} \right\rfloor + 1,$$

On the other hand, according to (7), if $N_1 = 4r+3$, $w(2)$ can be written as

$$w(2) = 2L_1^{(Q)} + 1 = 2r + 1 = 2\left\lfloor \frac{N_1}{4} \right\rfloor + 1.$$

When $m = 3$, $w(3)$ results from the following two pairs:

1) $\min(\mathbb{X}_2^{(Q)})$ and $\max(\mathbb{Y}_1^{(Q)})$. The difference is

$$[(N_1+1)+2] - [(N_1+1)-1] = 3.$$

2) $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$. We obtain

$$(N_2-1)(N_1+1) - [(N_2-1)(N_1+1) - 3] = 3,$$

which completes the proof. ∎

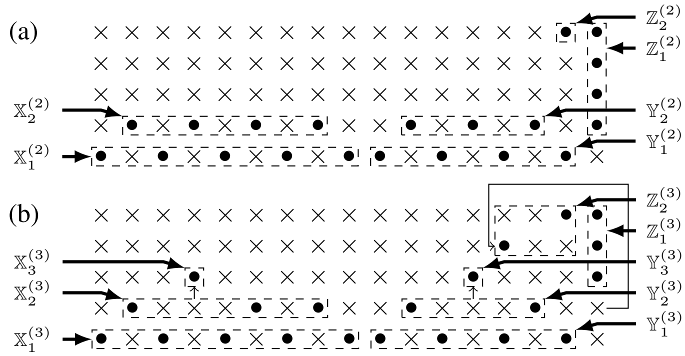


Fig. 5. 2D representations of (a) the second-order super nested array $\mathbb{S}^{(2)}$ and (b) the third-order super nested array $\mathbb{S}^{(3)}$, where $N_1 = 16$ (even) and $N_2 = 5$. Bullets represent physical sensors while crosses denote empty space. Thin arrows illustrate the recursive rules (Rule 2 and Rule 3) in Fig. 1.

IV. QTH-ORDER SUPER-NESTED ARRAYS, N_1 IS EVEN

For odd N_1, we presented three recursive rules between $\mathbb{S}^{(Q)}$ and $\mathbb{S}^{(Q-1)}$, as described in Fig. 3, Lemma 1, and Lemma 2. For even N_1, the framework in Fig. 1 still holds true but the details in Rule 2 are different from Lemma 1.

As an example, Fig. 5 displays 2D representations of super nested arrays with $N_1 = 16$ and $N_2 = 5$. The recursive rules are depicted by thin arrows in Fig. 5(b). First, the following sets are considered:

$$\mathbb{X}_2^{(2)} = \{19, 21, 23, 25\}, \ \mathbb{X}_2^{(3)} = \{19, 23, 25\}, \ \mathbb{X}_3^{(3)} = \{38\}. \tag{9}$$

It is clear that (9) justifies Rule 2 in Fig. 1. However, (9) does not satisfy Lemma 1 since $\mathbb{X}_2^{(3)}$ contains an odd term, which is the element 25 in this example. On the other hand, Fig. 5 gives $\mathbb{Z}_1^{(2)} = \{34, 51, 68, 85\}$, $\mathbb{Z}_2^{(2)} = \{84\}$, $\mathbb{Z}_1^{(3)} = \{51, 68, 85\}$, and $\mathbb{Z}_2^{(3)} = \{65, 84\}$. It can be readily shown that these sets satisfy Rule 3 and Lemma 2 precisely.

Hence, it can be inferred from Fig. 5 that for even N_1, $\mathbb{S}^{(Q)}$ can be still generated from $\mathbb{S}^{(Q-1)}$ using three recursive rules. Rule 1 and Rule 3 can be utilized directly but Rule 2 needs further development. The formal definition of super nested arrays when N_1 is even is now given in a recursive manner as follows:

Definition 2 (Qth-order super nested arrays, N_1 is even)**.** *Let N_1 be an even number, $N_2 \geq 2Q$, and $Q \geq 3$. A Qth-order super nested array is specified by the integer set $\mathbb{S}^{(Q)}$,*

$$\mathbb{S}^{(Q)} = \left(\bigcup_{q=1}^{Q} \mathbb{X}_q^{(Q)} \cup \mathbb{Y}_q^{(Q)} \right) \cup \mathbb{Z}_1^{(Q)} \cup \mathbb{Z}_2^{(Q)}.$$

These nonempty subsets $\mathbb{X}_q^{(Q)}$, $\mathbb{Y}_q^{(Q)}$, $\mathbb{Z}_1^{(Q)}$, and $\mathbb{Z}_1^{(Q)}$ satisfy

1) ***(Rule 1)*** *For $1 \leq q \leq Q-2$, $\mathbb{X}_q^{(Q)} = \mathbb{X}_q^{(Q-1)}$.*
2) ***(Rule 2)*** *$\mathbb{X}_{Q-1}^{(Q)}$ and $\mathbb{X}_Q^{(Q)}$ can be obtained from $\mathbb{X}_{Q-1}^{(Q-1)}$ by*

a) *If the cardinality of* $\mathbb{X}_{Q-1}^{(Q-1)}$ *is odd, then*

$$\mathbb{X}_{Q-1}^{(Q)} = \{\textit{Even terms of } \mathbb{X}_{Q-1}^{(Q-1)}\},$$
$$\mathbb{X}_{Q}^{(Q)} = \{(\textit{Odd terms of } \mathbb{X}_{Q-1}^{(Q-1)}) + (N_1 + 1)\},$$

where the definition of even/odd terms are consistent with Lemma 1.

b) *Otherwise, we call the last element in* $\mathbb{X}_{Q-1}^{(Q-1)}$ *as the* ***extra term****. Then*

$$\mathbb{X}_{Q-1}^{(Q)} = \{\textit{Even terms of } \mathbb{X}_{Q-1}^{(Q-1)}\} \cup \{\textit{the extra term}\},$$
$$\mathbb{X}_{Q}^{(Q)} = \{(\textit{Odd terms of } \mathbb{X}_{Q-1}^{(Q-1)}, \textit{ except the extra term}) + (N_1 + 1)\}.$$

$\mathbb{Y}_q^{(Q)}$ *share similar properties as* $\mathbb{X}_q^{(Q)}$ *in Rule 1 and 2.*

3) ***(Rule 3)*** *The sets* $\mathbb{Z}_1^{(Q)}$ *and* $\mathbb{Z}_2^{(Q)}$ *are given by*

$$\mathbb{Z}_1^{(Q)} = \{\ell(N_1{+}1) \mid Q \le \ell \le N_2\},$$
$$\mathbb{Z}_2^{(Q)} = \{(N_2{+}1{-}q)(N_1{+}1){-}2^q{+}1 \mid 1 \le q \le Q-1\},$$

which is equivalent to the recursive formula in Lemma 2.

A MATLAB code for Definition 2 is included in `super_nested.m` [12], where the input parameters are N_1, N_2, and Q and the sensor locations $\mathbb{S}^{(Q)}$ are delivered as output. This function first takes second-order super nested arrays $\mathbb{S}^{(2)}$ as an initial condition, then applies Definition 2 multiple times to obtain $\mathbb{S}^{(3)}$, $\mathbb{S}^{(4)}$, up to $\mathbb{S}^{(Q)}$.

Next, we will clarify Rule 2 in Definition 2 using Fig. 5. According to (9), the cardinality of $\mathbb{X}_2^{(2)}$ is 4 so Rule 2b is applicable. For $\mathbb{X}_2^{(2)}$, the extra term is 25, the even terms are 19 and 23, and the odd terms are 21 and 25. Using the expressions in Rule 2b of Definition 2, we obtain $\mathbb{X}_2^{(3)}$ and $\mathbb{X}_3^{(3)}$, which are identical to (9). On the other hand, if we consider $\mathbb{Y}_2^{(2)} = \{28, 30, 32\}$ in Fig. 5, then the cardinality of $\mathbb{Y}_2^{(2)}$ becomes 3, implying Rule 2a is applicable. The even terms and odd terms of $\mathbb{Y}_2^{(2)}$ are $28, 32$ and 30, respectively. As a result, $\mathbb{Y}_2^{(3)} = \{28, 32\}$ and $\mathbb{Y}_3^{(3)} = \{47\}$, which are consistent with Fig. 5.

In short, for even N_1, super nested arrays are defined in a recursive fashion (Definition 2). The only dissimilarity from the odd N_1 case is that, sometimes the extra terms need to be considered (Rule 2b of Definition 2).

Next we will prove some important properties which result from Definition 2 of the super nested array.

Lemma 5 (Relation to dense ULA of nested array)**.** *Let* $\mathbb{S}^{(Q)}$ *be a super nested array, as defined in Definition 2, when* N_1 *is an even number. Let* $\mathbb{A}_q^{(Q)} = \mathbb{X}_q^{(Q)} - (q-1)(N_1+1)$ *and* $\mathbb{B}_q^{(Q)} = \mathbb{Y}_q^{(Q)} - (q-1)(N_1+1)$ *for* $q = 1, 2, \ldots, Q$*. Then*

$$\bigcup_{q=1}^{Q} \mathbb{A}_q^{(Q)} \cup \mathbb{B}_q^{(Q)} = \{1, 2, \ldots, N_1\}.$$

Proof: First by Lemma 1 in [11], we know this lemma is true for $Q = 2$. Then we use proof by induction. Based on Rule 2 of Definition 2, $\mathbb{X}_{Q-1}^{(Q)} \cup (\mathbb{X}_{Q}^{(Q)} - (N_1 + 1)) = \mathbb{X}_{Q-1}^{(Q-1)}$ and $\mathbb{Y}_{Q-1}^{(Q)} \cup (\mathbb{Y}_{Q}^{(Q)} - (N_1 + 1)) = \mathbb{Y}_{Q-1}^{(Q-1)}$ Therefore, the argument in the proof of Lemma 3 can be applied. ■

Lemma 6 (Total number of sensors)**.** *Let* $\mathbb{S}^{(Q)}$ *be a Qth-order super nested array defined by Definition 2. Then* $\left|\mathbb{S}^{(Q)}\right| = N_1 + N_2$.

Proof: The proof is the same as that of Lemma 4. ■

The coarray of the Qth-order super nested array is identical to that of the parent nested array. This was proved earlier for odd N_1. The same is true for even N_1, as shown by the theorem and corollary below.

Theorem 3. *If* $N_1 \geq 2 \cdot 2^Q + 2$ *is an even number,* $N_2 \geq 3Q - 4$*,* $Q \geq 3$*, then Qth-order super nested arrays are restricted arrays. That is, their coarray is hole-free.*

Proof: The proof is similar to that of Theorem 1. The details are quite involved, and are presented in Section VI. As a numerical example, Section II of the supplementary document [13] illustrates these details in Section VI. ■

Corollary 2. *If* $N_1 \geq 2 \cdot 2^Q + 2$ *is an even number,* $N_2 \geq 3Q - 4$*,* $Q \geq 3$*, then Qth-order super nested arrays have the same coarray as the parent nested arrays.*

Proof: This proof is identical to that of Corollary 1. ■

The next theorem shows that the super nested array for $Q > 2$ has significantly improved weight function $w(2)$, which is crucial to reducing the mutual coupling effects.

Theorem 4. *Assume that* $N_1 \geq 2 \cdot 2^Q + 2$ *is an even number,* $N_2 \geq 3Q - 4$*, and* $Q \geq 3$*. Then, the weight function* $w(m)$ *of Qth-order super nested arrays satisfies*

$$
\begin{aligned}
w(1) &= 2, \\
w(2) &= \begin{cases} \frac{N_1}{2} + 1, & \text{if } N_1 = 8k - 2, \\ \frac{N_1}{2} - 1, & \text{if } N_1 = 8k + 2, \\ \frac{N_1}{2}, & \text{otherwise,} \end{cases} \\
w(3) &= 5,
\end{aligned}
$$

where k *is an integer.*

Proof: The proof is quite similar to that of Theorem 2 in [11] and Theorem 2 in this paper. The parameters A_1, B_1, A_2, B_2 follow the same definition as in the companion paper [11]. For $w(1)$, the two sensor pairs are identical to those in the second-order ones, which have been identified in the proof of Theorem 2 in [11].

For the weight function $w(2)$, there are some cases:

1) The self-differences of $\mathbb{X}_1^{(Q)}$ and $\mathbb{Y}_1^{(Q)}$ contribute to $(A_1)_+ + (B_1)_+$ pairs, which is $N_1/2-1$.
2) If $N_1 = 4r$, then $A_2 = r-1$ and $B_2 = r-2$. If A_2+1 is even, there is an extra term in $\mathbb{X}_2^{(Q)}$. Note that the maximum ULA element in $\mathbb{X}_2^{(Q)}$ is less than the extra term by 2, as indicated in Lemma 7-3. The similar conclusion applies to $\mathbb{Y}_2^{(Q)}$. Hence, depending on the even/odd properties of A_2 and B_2, there is exactly one pair of sensors with sensor separation 2, in $\mathbb{X}_2^{(Q)} \cup \mathbb{Y}_2^{(Q)}$ when $N_1 = 4r$.
3) When $N_1 = 4r+2$, $A_2 = r$, $B_2 = r-2$. If $r = 2k-1$ is an odd number, A_2+1 and B_2+1 are both even numbers. One extra term exists in $\mathbb{X}_2^{(Q)}$ and another one can be found in $\mathbb{Y}_2^{(Q)}$. There are two pairs of sensor separation 2. If $r = 2k$ is an even number, A_2+1 and B_2+1 are odd numbers. There is no extra term in $\mathbb{X}_2^{(Q)}$ and $\mathbb{Y}_2^{(Q)}$.

Hence, $w(2)$ is given by

$$w(2) = \begin{cases} \frac{N_1}{2}, & \text{if } N_1 = 4r, \\ \frac{N_1}{2}+1, & \text{if } N_1 = 4(2k-1)+2, \\ \frac{N_1}{2}-1, & \text{if } N_1 = 4(2k)+2, \end{cases}$$

which proves the $w(2)$ part.

$w(3)$ can be found in these sensor pairs:

1) Four sensor pairs have been identified in the proof of Theorem 2 in [11]. It is applicable because $\mathbb{X}_1^{(Q)} = \mathbb{X}_1^{(2)}$, $\mathbb{Y}_1^{(Q)} = \mathbb{Y}_1^{(2)}$, $\min(\mathbb{X}_2^{(Q)}) = \min(\mathbb{X}_2^{(2)})$, and $\max(\mathbb{Y}_2^{(Q)}) = \max(\mathbb{Y}_2^{(2)})$.
2) One more pair exists between $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$. They are $(N_2-1)(N_1+1) \in \mathbb{Z}_1^{(Q)}$ and $(N_2-1)(N_1+1)-3 \in \mathbb{Z}_2^{(Q)}$.

Then the proof is complete. ∎

Remarks based on Theorem 2 and 4

It seems that the super nested arrays with odd N_1 is superior to those with even N_1 in terms of the weight functions $w(1)$, $w(2)$, and $w(3)$. However, in some scenarios, the super nested arrays with even N_1 is preferred. For instance, suppose that we want to design super nested arrays with $N = 41$ physical sensors such that the number of identifiable sources is maximized. It was proved in [8] that the optimal N_1 is given by $N_1 = (N-1)/2 = 20$, which is an even number.

Another remark is that, $w(1)$, $w(2)$, and $w(3)$ remain unchanged for $Q \geq 3$. However, this phenomenon does *not* imply that super nested arrays for $Q \geq 3$ have the same performance in the presence of mutual coupling. Instead, super nested arrays for $Q > 3$ could reduce the mutual coupling further. It is because the overall performance depends on the mutual coupling models, which are functions of the array geometry, as mentioned in (9) and (10) of the companion paper [11]. Super nested arrays with $Q > 3$ tend to make array geometries *more sparse*, as discussed extensively in Section III and IV. It can be shown that the weight functions like $w(4)$, $w(5)$, and so on, decrease as Q increases. Hence, qualititatively, mutual coupling could be reduced for super nested arrays with $Q > 3$.

Furthermore, the judgement of the estimation error based on on the weight functions $w(1)$, $w(2)$, and $w(3)$, is *qualitative* and does not always lead to right conclusions [11]. For example, Fig. 6 shows that for source spacing

TABLE I

27 CASES IN THE PROOF OF THEOREM 1

$n_2 \quad \backslash \quad n_1$	$\mathbb{X}_Q^{(Q)}-(N_1+1)$	$\mathbb{Y}_Q^{(Q)}-(N_1+1)$	$(Q-1)(N_1+1)$
$\mathbb{X}_q^{(Q-1)}$, $1 \leq q \leq Q-2$	Case 1	Case 10	Case 19
$\mathbb{X}_{Q-1}^{(Q)}$	Case 2	Case 11	Case 20
$\mathbb{X}_Q^{(Q)}-(N_1+1)$	Case 3	Case 12	Case 21
$\mathbb{Y}_q^{(Q-1)}$, $1 \leq q \leq Q-2$	Case 4	Case 13	Case 22
$\mathbb{Y}_{Q-1}^{(Q)}$	Case 5	Case 14	Case 23
$\mathbb{Y}_Q^{(Q)}-(N_1+1)$	Case 6	Case 15	Case 24
$\mathbb{Z}_1^{(Q)}$	Case 7	Case 16	Case 25
$(Q-1)(N_1+1)$	Case 8	Case 17	Case 26
$\mathbb{Z}_2^{(Q-1)}$	Case 9	Case 18	Case 27

$\Delta\bar{\theta} = 0.001$, the super nested array with $Q = 2, N_1 = N_2 = 17$ ($w(2) = 16$) outperforms the super nested array with $Q = 3, N_1 = N_2 = 17$ ($w(2) = 9$).

V. Proof of Theorem 1

According to Theorem 1 in [11], second-order super nested arrays are restricted arrays. To prove the same for Qth-order nested arrays with $Q > 2$, we use induction. Thus, assume that $\mathbb{S}^{(Q-1)}$ are restricted arrays. We need to show that $\mathbb{S}^{(Q)}$ are also restricted arrays under certain sufficient conditions. In the following development, we use $\mathbb{D}^{(Q)}$ to denote the difference set of Qth-order super nested arrays, $\mathbb{S}^{(Q)}$.

The main concept of the proof works as follows. Let $n_1 \in \mathbb{S}^{(Q-1)}\backslash\mathbb{S}^{(Q)}$ and $n_2 \in \mathbb{S}^{(Q-1)}$. It is obvious that $n_1 - n_2$ belongs to $\mathbb{D}^{(Q-1)}$. We need to show that there exist some $n_1', n_2' \in \mathbb{S}^{(Q)}$ such that $n_1' - n_2' = n_1 - n_2$. If the above statement holds true for every $n_1 \in \mathbb{S}^{(Q-1)}\backslash\mathbb{S}^{(Q)}$ and $n_2 \in \mathbb{S}^{(Q-1)}$, it is equivalent saying that $\mathbb{S}^{(Q)}$ is a restricted array.

Table I lists 27 combinations, where $n_1 \in \mathbb{S}^{(Q-1)}\backslash\mathbb{S}^{(Q)}$ is divided into 3 subsets in each column and $n_2 \in \mathbb{S}^{(Q-1)}$ is partitioned into 9 categories in each row. In every case, given n_1 and n_2, we need to identify the associated n_1' and n_2' such that (a) $n_1', n_2' \in \mathbb{S}^{(Q)}$, which will be elaborated in detail, and (b) $n_1' - n_2' = n_1 - n_2$, which is simple to check.

(Case 1) Any n_1 and n_2 in this case can be written as

$$\begin{cases} n_1 = (Q-2)(N_1+1) + 2^{Q-1} + 2^{Q-1}\ell_1, \\ n_2 = (q-1)(N_1+1) + 2^{q-1} + 2^q\ell_2, \end{cases}$$

where $0 \leq \ell_1 \leq L_Q^{(Q)}$ and $0 \leq \ell_2 \leq L_q^{(Q)}$. According to Definition 1, $L_Q^{(Q)} \leq L_{Q-1}^{(Q)}$, and we have these cases:

1) $L_Q^{(Q)} < L_{Q-1}^{(Q)}$: The corresponding n_1' and n_2' can be expressed into two ways. They are

$$\begin{cases} n_1' = (Q-2)(N_1+1)+2^{Q-2}+2^{Q-1}\ell_1, \\ n_2' = (q-1)(N_1+1)+2^{q-1}+2^q(\ell_2-2^{Q-q-2}), \end{cases} \tag{10}$$

$$\begin{cases} n_1' = (Q-2)(N_1+1)+2^{Q-2}+2^{Q-1}(\ell_1+1), \\ n_2' = (q-1)(N_1+1)+2^{q-1}+2^q(\ell_2+2^{Q-q-2}), \end{cases} \tag{11}$$

The membership of n_1' and n_2' can be derived as follows. Since $L_Q^{(Q)} < L_{Q-1}^{(Q)}$, we have $n_1' \in \mathbb{X}_{Q-1}^{(Q)}$ in (10) and (11). Next, n_2' in (10) belongs to $\mathbb{X}_q^{(Q)}$ if $0 \leq \ell_2 - 2^{Q-q-2} \leq L_q^{(Q)}$. If $0 \leq \ell_2 + 2^{Q-q-2} \leq L_q^{(Q)}$, then n_2' in (11) belongs to $\mathbb{X}_q^{(Q)}$. That is, if

$$2^{Q-q-2} \leq L_q^{(Q)} - 2^{Q-q-2}, \quad 1 \leq q \leq Q-2, \tag{12}$$

then we can find $n_1', n_2' \in \mathbb{S}^{(Q)}$ using either (10) or (11). Solving (12) leads to another sufficient condition $N_1 \geq \frac{7}{4} \cdot 2^Q - 1$.

2) $L_Q^{(Q)} = L_{Q-1}^{(Q)}$ and $0 \leq \ell_1 \leq L_Q^{(Q)} - 1$: The argument is the same as Case 1-1.

3) $L_Q^{(Q)} = L_{Q-1}^{(Q)}$ and $\ell_1 = L_Q^{(Q)}$: Depending on ℓ_2, we obtain two more cases,

   a) $2^{Q-q-2} \leq \ell_2 \leq L_q^{(Q)}$: Under this condition, (10) is still applicable. We obtain $\ell_1 = L_Q^{(Q)} \leq L_{Q-1}^{(Q)}$, implying $n_1' \in \mathbb{X}_{Q-1}^{(Q)}$. In addition, the maximum value of $\ell_2 - 2^{Q-q-2}$ is $L_q^{(Q)} - 2^{Q-q-2}$, which is less than or equal to $L_q^{(Q)}$. This property proves $n_2' \in \mathbb{X}_q^{(Q)}$.

   b) $0 \leq \ell_2 \leq 2^{Q-q-2} - 1$: The associated n_1' and n_2' are

$$\begin{cases} n_1' = (Q-1)(N_1+1)+2^{Q-1}+2^{Q-1}(\ell_1-1), \\ n_2' = q(N_1+1)-2^{q-1}-2^q(2^{Q-q-1}-\ell_2-1). \end{cases} \tag{13}$$

   It can be seen that $n_1' \in \mathbb{X}_Q^{(Q)}$, since $\ell_1 - 1 \leq L_Q^{(Q)}$. We need to show that $n_2' \in \mathbb{Y}_q^{(Q)}$ under some sufficient conditions. Since $0 \leq \ell_2 \leq 2^{Q-q-2} - 1$, we obtain $2^{Q-q-2} \leq 2^{Q-q-1} - \ell_2 - 1 \leq 2^{Q-q-1} - 1$. If $2^{Q-q-1} - 1 \leq L_q^{(Q)}$, it can be inferred that n_2' belongs to $\mathbb{Y}_q^{(Q)}$. Therefore, the associated sufficient condition becomes $N_1 \geq \frac{5}{4} \cdot 2^Q - 1$.

(Case 2, 5, 11, 14) In Case 2, n_1, n_2, n_1', n_2' are

$$\begin{cases} n_1 = (Q-2)(N_1+1) + 2^{Q-1} + 2^{Q-1}\ell_1, \\ n_2 = (Q-2)(N_1+1) + 2^{Q-2} + 2^{Q-1}\ell_2, \end{cases} \tag{14}$$

$$\begin{cases} n_1' = 1 + 2(2^{Q-2} + 2^{Q-2}\ell_1 - 1), \\ n_2' = 1 + 2(2^{Q-3} + 2^{Q-2}\ell_2 - 1), \end{cases} \tag{15}$$

where $0 \le \ell_1 \le L_Q^{(Q)}$ and $0 \le \ell_2 \le L_{Q-1}^{(Q)}$. It can be concluded that $n_1', n_2' \in \mathbb{X}_1^{(Q)}$ since

$$\begin{aligned}
&2^{Q-2} + 2^{Q-2}\ell_1 - 1 \\
&\le \left\lfloor 2^{Q-2} + 2^{Q-2}\left(\frac{N_1+1}{2^Q} - 1\right) - 1 \right\rfloor \le L_1^{(Q)}, \\
&2^{Q-3} + 2^{Q-2}\ell_2 - 1 \\
&\le \left\lfloor 2^{Q-3} + \frac{2^{Q-2}}{2}\left(\frac{N_1+1}{2^{Q-1}} - 1\right) - 1 \right\rfloor \le L_1^{(Q)}.
\end{aligned}$$

Here we apply some properties of the floor function: $\lfloor 2x \rfloor \ge 2\lfloor x \rfloor$ and $\lfloor x + n \rfloor = \lfloor x \rfloor + n$ for integer n. Note that in Case 5, 11, 14, we can relate n_1' and n_2' with either $\mathbb{X}_1^{(Q)}$ or $\mathbb{Y}_1^{(Q)}$.

(Case 3, 6, 8, 12, 15, 17, 21, 24, 26) For any n_1 and n_2 in this combination, the corresponding $n_1' = n_1 + (N_1+1)$ and $n_2' = n_2 + (N_1 + 1)$. We have

$$n_1', n_2' \in \mathbb{X}_Q^{(Q)} \cup \mathbb{Y}_Q^{(Q)} \cup \{Q(N_1+1)\} \subset \mathbb{S}^{(Q)}.$$

(Case 4, 10) Let us consider Case 4 first. The associated n_1 and n_2 are given by

$$\begin{cases} n_1 = (Q-2)(N_1+1) + 2^{Q-1} + 2^{Q-1}\ell_1, \\ n_2 = q(N_1+1) - 2^{q-1} - 2^q\ell_2, \end{cases}$$

where $0 \le \ell_1 \le L_Q^{(Q)}$ and $0 \le \ell_2 \le L_q^{(Q)}$. According to ℓ_1, we have

1) $L_Q^{(Q)} < L_{Q-1}^{(Q)}$: n_1' and n_2' can be written as

$$\begin{cases} n_1' = (Q-2)(N_1+1) + 2^{Q-2} + 2^{Q-1}\ell_1, \\ n_2' = q(N_1+1) - 2^{q-1} - 2^q(\ell_2 + 2^{Q-q-2}), \end{cases} \tag{16}$$

$$\begin{cases} n_1' = (Q-2)(N_1+1) + 2^{Q-2} + 2^{Q-1}(\ell_1+1), \\ n_2' = q(N_1+1) - 2^{q-1} - 2^q(\ell_2 - 2^{Q-q-2}), \end{cases} \tag{17}$$

Following Case 1-1, if $N_1 \ge \frac{7}{4} \cdot 2^Q - 1$, then $n_1' \in \mathbb{X}_{Q-1}^{(Q)}$ and $n_2' \in \mathbb{Y}_q^{(Q)}$ in either (16) or (17).

2) $L_Q^{(Q)} = L_{Q-1}^{(Q)}$ and $0 \le \ell_1 \le L_Q^{(Q)} - 1$: Case 4-1 applies.
3) $L_Q^{(Q)} = L_{Q-1}^{(Q)}$ and $\ell_1 = L_Q^{(Q)}$:
   a) $0 \le \ell_2 \le L_q^{(Q)} - 2^{Q-q-2}$: (16) can be applied to this case. It can be shown that $n_1' \in \mathbb{X}_{Q-1}^{(Q)}$ and $n_2' \in \mathbb{Y}_q^{(Q)}$.
   b) $L_q^{(Q)} - 2^{Q-q-2} + 1 \le \ell_2 \le L_q^{(Q)}$: To identify n_1' and n_2' in this case, we first introduce the *remainder* $R = (Q-q-1)(N_1+1) - n_1 + n_2$, which is rewritten as

$$R = (N_1+1) - 2^{Q-1} - 2^{Q-1}L_Q^{(Q)} - 2^{q-1} - 2^q\ell_2. \tag{18}$$

Corollary 3. $1 \le R < 2^{Q-1}$.

Proof: If x is a real number, we obtain $x-1<\lfloor x\rfloor \leq x$. This property implies

$$\frac{N_1+1}{2}-2^{Q-1}<2^{Q-1}+2^{Q-1}L_Q^{(Q)}\leq\frac{N_1+1}{2}, \tag{19}$$

$$\frac{N_1+1}{2}-2^{Q-2}<2^{Q-1}+2^{Q-1}L_{Q-1}^{(Q)}\leq\frac{N_1+1}{2}+2^{Q-2}, \tag{20}$$

$$\frac{N_1+1}{2}-2^{Q-2}<2^{q-1}+2^{q}\ell_2\leq\frac{N_1+1}{2}. \tag{21}$$

Combining (19) to (21) and $L_Q^{(Q)}=L_{Q-1}^{(Q)}$ gives $0\leq R<2^{Q-1}$. However, if $R=0$, both (19) and (21) achieve their upper bound. The condition that (19) being equal is N_1+1 is a multiple of 2^Q, which contradicts with the condition that (21) being the equal. Hence, $1\leq R<2^{Q-1}$. ■

Next, according to R, we can identify n_1' and n_2'. Let us consider the *binary expansion* of R, which is

$$R=\sum_{r=0}^{Q-2}a_r2^r, \quad a_r\in\{0,1\}. \tag{22}$$

Then we define P satisfying

$$a_0=a_1=\cdots=a_{P-1}=0,\ a_P=1. \tag{23}$$

It can be deduced that (a) P is unique for a given R, and (b) $0\leq P\leq Q-2$. Here we have three more cases, where q is consistent with Case 4 in Table I:

i) $q=1$: In this case, the proof technique in (16) is applicable. $n_1'\in\mathbb{X}_{Q-1}^{(Q)}$ and n_2' is an odd number less than N_1+1 so $n_2'\in\mathbb{X}_1^{(Q)}\cup\mathbb{Y}_1^{(Q)}$.

ii) $P\leq q, q\geq 2$: Since $q\geq 2$, R is an even number and $P\geq 1$. n_1' and n_2' become

$$\begin{cases} n_1'=(Q+P-q-1)(N_1+1)+2^{Q+P-q-1}, \\ n_2'=P(N_1+1)+R+2^{Q+P-q-1}. \end{cases} \tag{24}$$

It will be shown that $n_1'\in\mathbb{X}_{Q-P-q}^{(Q)}$ and $n_2'\in\mathbb{X}_{P+1}^{(Q)}$ under some sufficient conditions. It is obvious that n_1' is the minimum element in $\mathbb{X}_{Q-P-q}^{(Q)}$. $n_2'\in\mathbb{X}_{P+1}^{(Q)}$ is equivalent to

$$R+2^{Q+P-q-1}=2^P+2^{P+1}\ell_3, \tag{25}$$

for some integer ℓ_3 satisfying $0\leq\ell_3\leq L_{P+1}^{(Q)}$. According to the definition of P, in (23), the left-hand side of (25) is a multiple of 2^P and ℓ_3 is an integer.

Next, we need to show $2^P<R+2^{Q+P-q-1}\leq 2^P+2^{P+1}L_{P+1}^{(Q)}$ under some sufficient conditions. According to (23) and the range of P and q, we have $R\geq 2^P$ and $2^{Q+P-q-1}\geq 1$, yielding the lower bound. A sufficient condition for the upper bound is given by

$$2^{Q-1}+2^{Q+P-q-1}\leq 2^P+2^{P+1}L_{P+1}^{(Q)}. \tag{26}$$

Using $x-1<\lfloor x\rfloor$, $P\leq q$, and $1\leq q\leq Q-2$, (26) becomes $N_1\geq 3\cdot 2^Q-1$.

iii) $P \geq q+1, q \geq 2$: n_1' and n_2' in this case are

$$\begin{cases} n_1' = (Q+P-q-1)(N_1+1), \\ n_2' = P(N_1+1)+R. \end{cases} \tag{27}$$

A sufficient condition for n_1' belonging to $\mathbb{Z}_1^{(Q)}$ is $Q \leq Q+P-q-1 \leq N_2$, implying $N_2 \geq 2Q-5$. On the other hand, n_2' lives in $\mathbb{X}_{P+1}^{(Q)}$ when there exists some ℓ_3 satisfying $0 \leq \ell_3 \leq L_{P+1}^{(Q)}$ and $R = 2^P + 2^{P+1}\ell_3$. It suffices to solve $2^{Q-1} \leq 2^P + 2^{P+1}L_{P+1}^{(Q)}$, which gives another sufficient condition $N_1 \geq 2 \cdot 2^Q - 1$.

The proof for Case 10 is similar to Case 4.

(Case 7, 16) First we consider Case 7, where n_1 and n_2 are given by

$$\begin{cases} n_1 = (Q-2)(N_1+1) + 2^{Q-1} + 2^{Q-1}\ell_1, \\ n_2 = \ell_2(N_1+1). \end{cases} \tag{28}$$

Here $0 \leq \ell_1 \leq L_Q^{(Q)}$ and $Q \leq \ell_2 \leq N_2$. According to ℓ_2, we obtain

1) $Q \leq \ell_2 \leq N_2 - 1$: n_1' and n_2' can be written as

$$\begin{cases} n_1' = (Q-1)(N_1+1) + 2^{Q-1} + 2^{Q-1}\ell_1, \\ n_2' = (\ell_2+1)(N_1+1). \end{cases} \tag{29}$$

It is trivial that $n_1' \in \mathbb{X}_Q^{(Q)}$ and $n_2' \in \mathbb{Z}_1^{(Q)}$.

2) $\ell_2 = N_2$: We obtain n_1' and n_2' to be

$$\begin{cases} n_1' = 1 + 2^{Q-1}\ell_1, \\ n_2' = (N_2+2-Q)(N_1+1) - 2^{Q-1} + 1. \end{cases} \tag{30}$$

It can be seen from (30) that $n_1' \in \mathbb{X}_1^{(Q)}$ and $n_2' \in \mathbb{Z}_2^{(Q)}$.

The proof for Case 16 follows the same argument for Case 7.

(Case 9, 18) For Case 9, n_1 and n_2 are given by

$$\begin{cases} n_1 = (Q-2)(N_1+1) + 2^{Q-1} + 2^{Q-1}\ell_1, \\ n_2 = (N_2+1-q)(N_1+1) - 2^q + 1. \end{cases} \tag{31}$$

where $0 \leq \ell_1 \leq L_Q^{(Q)}$ and $1 \leq q \leq Q-2$. Rewriting (31) gives n_1' and n_2'

$$\begin{cases} n_1' = 2^q - 1 + 2^{Q-1} + 2^{Q-1}\ell_1, \\ n_2' = (N_2+3-Q-q)(N_1+1). \end{cases} \tag{32}$$

We need to show that $n_1' \in \mathbb{X}_1^{(Q)} \cup \mathbb{Y}_1^{(Q)}$ and $n_2' \in \mathbb{Z}_1^{(Q)}$. If $Q \leq N_2+3-Q-q \leq N_2$, then $n_2' \in \mathbb{Z}_1^{(Q)}$, which leads to a sufficient condition $N_2 \geq 3Q-5$. The membership of n_1' can be verified as follows. It is trivial that n_1' is an odd number. In addition, $n_1' \leq 2^{Q-2} - 1 + 2^{Q-1} + 2^{Q-1}L_Q^{(Q)} \leq N_1+1$. We obtain $n_1' \in \mathbb{X}_1^{(Q)} \cup \mathbb{Y}_1^{(Q)}$. Case 18 has the same proof as Case 9.

(Case 13) In this case, n_1, n_2 are given by

$$\begin{cases} n_1 = (Q-1)(N_1+1) - 2^{Q-1} - 2^{Q-1}\ell_1, \\ n_2 = q(N_1+1) - 2^{q-1} - 2^q\ell_2, \end{cases} \tag{33}$$

where $0 \le \ell_1 \le L_Q^{(Q)}$ and $0 \le \ell_2 \le L_q^{(Q)}$. According to ℓ_1, we have two sub-cases

1) $L_Q^{(Q)} < L_{Q-1}^{(Q)}$: The pair n'_1 and n'_2 can be written as

$$\begin{cases} n'_1 = (Q-1)(N_1+1) - 2^{Q-2} - 2^{Q-1}\ell_1, \\ n'_2 = q(N_1+1) - 2^{q-1} - 2^q(\ell_2 - 2^{Q-q-2}), \end{cases} \tag{34}$$

$$\begin{cases} n'_1 = (Q-1)(N_1+1) - 2^{Q-2} - 2^{Q-1}(\ell_1+1), \\ n'_2 = q(N_1+1) - 2^{q-1} - 2^q(\ell_2 + 2^{Q-q-2}), \end{cases} \tag{35}$$

Following the same discussion as Case 1-1, we obtain a sufficient condition $N_1 \ge \frac{7}{4} \cdot 2^Q - 1$.

2) $L_Q^{(Q)} = L_{Q-1}^{(Q)}$ and $0 \le \ell_1 < L_Q^{(Q)} - 1$: This case is the same as Case 13-1.

3) $L_Q^{(Q)} = L_{Q-1}^{(Q)}$ and $\ell_1 = L_Q^{(Q)}$:

   a) $2^{Q-q-2} \le \ell_2 \le L_q^{(Q)}$: It can be shown that $n'_1 \in \mathbb{Y}_{Q-1}^{(Q)}$ and $n'_2 \in \mathbb{Y}_q^{(Q)}$ due to the same reason in Case 1-3a.

   b) $0 \le \ell_2 \le 2^{Q-q-2} - 1$, $q = Q-2$, and $Q = 3$: In this case n'_1 and n'_2 are given by

$$\begin{cases} n'_1 = N_1 + 3, \\ n'_2 = 1 + 4(L_3^{(3)} + 1), \end{cases} \tag{36}$$

   We know that $n'_1 \in \mathbb{X}_2^{(3)}$, which is trivial, and $n'_2 \in \mathbb{X}_1^{(3)} \cup \mathbb{Y}_1^{(3)}$, since n'_2 is an odd number less than N_1+1.

   c) $0 \le \ell_2 \le 2^{Q-q-2} - 1$, $q = Q-2$, and $Q \ge 4$: n'_1 and n'_2 can be written as

$$\begin{cases} n'_1 = (N_1+2) - 2^{Q-1} \\ \qquad\qquad -2^{Q-1}L_Q^{(Q)} + 2^{Q-3} + 2^{Q-2}\ell_2, \\ n'_2 = 1. \end{cases} \tag{37}$$

   It can be inferred that n'_1 is an odd number less than N_1+1. Therefore, $n'_1, n'_2 \in \mathbb{X}_1^{(Q)} \cup \mathbb{Y}_1^{(Q)}$.

   d) $0 \le \ell_2 \le 2^{Q-q-2} - 1$ and $1 \le q \le Q-3$: We found that n'_1 and n'_2 can be expressed as

$$\begin{cases} n'_1 = (Q-2)(N_1+1) - 2^{Q-3} - 2^{Q-1}L_Q^{(Q)}, \\ n'_2 = (q-1)(N_1+1) + 2^{q-1} \\ \qquad\qquad +2^q(2^{Q-q-2} + 2^{Q-q-3} - \ell_2 - 1), \end{cases} \tag{38}$$

   which satisfies $n'_1 \in \mathbb{Y}_{Q-2}^{(Q)}$ and $n'_2 \in \mathbb{X}_q^{(Q)}$ under the sufficient conditions $2L_Q^{(Q)} \le L_{Q-2}^{(Q)}$ and $0 \le 2^{Q-q-2} + 2^{Q-q-3} - \ell_2 - 1 \le L_q^{(Q)}$. We obtain a sufficient condition $N_1 \ge \frac{7}{8} \cdot 2^Q - 1$.

(Case 19, 22) In Case 19, n_1, n_2, n'_1, and n'_2 can be written as

$$\begin{cases} n_1 = (Q-1)(N_1+1), \\ n_2 = (q-1)(N_1+1) + 2^{q-1} + 2^q \ell_2, \end{cases} \tag{39}$$

$$\begin{cases} n'_1 = (Q-1)(N_1+1)+2^{Q-1}, \\ n'_2 = (q-1)(N_1+1)+2^{q-1}+2^q(\ell_2+2^{Q-q-1}), \end{cases} \tag{40}$$

$$\begin{cases} n'_1 = (Q-1)(N_1+1)-2^{Q-2}, \\ n'_2 = (q-1)(N_1+1)+2^{q-1}+2^q(\ell_2-2^{Q-q-2}), \end{cases} \tag{41}$$

where $0 \le \ell_2 \le L_q^{(Q)}$. The next argument is similar to that in Case 1-1. A sufficient condition for $n'_1 \in \mathbb{X}_Q^{(Q)}, n'_2 \in \mathbb{X}_q^{(Q)}$ or $n'_1 \in \mathbb{Y}_{Q-1}^{(Q)}, n'_2 \in \mathbb{X}_q^{(Q)}$ is that

$$2^{Q-q-2} \le L_q^{(Q)} - 2^{Q-q-1},$$

which leads to another sufficient condition $N_1 \ge \frac{9}{4} \cdot 2^Q - 1$. Case 22 is the same as Case 19.

(Case 20, 23) In Case 20, n_1, n_2, n'_1, and n'_2 become

$$\begin{cases} n_1 = (Q-1)(N_1+1), \\ n_2 = (Q-2)(N_1+1) + 2^{Q-2} + 2^{Q-1}\ell_2, \end{cases} \tag{42}$$

$$\begin{cases} n'_1 = (N_1+1) - 1, \\ n'_2 = 1 + 2(-1 + 2^{Q-3} + 2^{Q-2}\ell_2), \end{cases} \tag{43}$$

where $0 \le \ell_2 \le L_{Q-1}^{(Q)}$. It is trivial that $n'_1 \in \mathbb{Y}_1^{(Q)}$. Besides, $n'_2 \in \mathbb{X}_1^{(Q)}$ since

$$-1+2^{Q-3}+2^{Q-2}\ell_1 \le -1+2^{Q-3}+2^{Q-2}L_{Q-1}^{(Q)} \le L_1^{(Q)}.$$

(Case 25) We can write n_1 and n_2 as

$$\begin{cases} n_1 = (Q-1)(N_1+1), \\ n_2 = q(N_1+1), \end{cases} \tag{44}$$

where $Q \le q \le N_2$. Based on q, n'_1 and n'_2 are given by the following.

1) $Q \le q \le N_2 - 1$: In this case $n'_1 = Q(N_1+1)$ and $n'_2 = (q+1)(N_1+1)$. It is evident that $n'_1, n'_2 \in \mathbb{Z}_1^{(Q)}$.
2) $q = N_2$: We obtain

$$\begin{cases} n'_1 = (N_1+1) - 1 - 2(2^{Q-2} - 1), \\ n'_2 = (N_2 - Q + 2)(N_1+1) - 2^{Q-1} + 1. \end{cases} \tag{45}$$

It can be seen that n'_2 is contained in $\mathbb{Z}_2^{(Q)}$. The sufficient condition for n'_1 in $\mathbb{Y}_1^{(Q)}$ is $2^{Q-2} - 1 \le L_1^{(Q)}$. We obtain $N_1 \ge 2^Q + 1$.

(Case 27) In this case, we have

$$\begin{cases} n_1 = (Q-1)(N_1+1), \\ n_2 = (N_2+1-q)(N_1+1) - 2^q + 1, \end{cases} \tag{46}$$

$$\begin{cases} n_1' = 2^q - 1, \\ n_2' = (N_2+2-Q-q)(N_1+1), \end{cases} \tag{47}$$

where $1 \le q \le Q-2$. If $2^q - 1 \le N_1$ and $Q \le N_2 + 2 - Q - q \le N_2$, n_1' and n_2' belong to $\mathbb{X}_1^{(Q)} \cup \mathbb{Y}_1^{(Q)}$ and $\mathbb{Z}_1^{(Q)}$, respectively. Solving these inequalities leads to the following sufficient conditions for Qth-order super nested arrays: $N_1 \ge \frac{1}{4} \cdot 2^Q - 1$ and $N_2 \ge 3Q - 4$.

The last step in the proof is to take the intersection of all these sufficient conditions. We obtain $N_1 \ge 3 \cdot 2^Q - 1$ and $N_2 \ge 3Q - 4$. Then all the (n_1', n_2') pairs in those 27 cases exist simultaneously, implying this array configuration is a restricted array. ■

VI. Proof of Theorem 3

This proof follows the same strategy as that of Theorem 1, where induction on Q is applied. First of all, it is essential to characterize some properties of $\mathbb{X}_q^{(Q)}$ as well as $\mathbb{Y}_q^{(Q)}$ before the induction step.

Lemma 7 (Properties on $\mathbb{X}_q^{(Q)}$ when N_1 is even)**.** *Suppose that $2 \le q \le Q$, N_1 is an even number and $\mathbb{S}^{(Q)}$ is defined as Definition 2. Then $\mathbb{X}_q^{(Q)}$ possess the following properties:*

1) *$\mathbb{X}_Q^{(Q)}$ is a ULA with inter-element spacing 2^{Q-1}. The first element is $(Q-1)(N_1+1) + 2^{Q-1}$.*
2) *For $2 \le q \le Q-1$, $\mathbb{X}_q^{(Q)}$ has a ULA portion with inter-element spacing 2^q. The minimum (or leftmost) element of ULA in $\mathbb{X}_q^{(Q)}$ is $(q-1)(N_1+1) + 2^{q-1}$.*
3) *If there is an extra term in $\mathbb{X}_q^{(Q)}$, it is the maximum (rightmost) element of $\mathbb{X}_q^{(Q)}$ and it is 2^{q-1} larger than the maximum element in the ULA section of $\mathbb{X}_q^{(Q)}$.*
4) *If $n \in \mathbb{X}_Q^{(Q)}$, then $n - (N_1+1) \pm 2^{Q-2} \in \mathbb{X}_{Q-1}^{(Q)}$.*

Proof:

1) We can prove this property by induction. When $Q = 2$, the closed-form expression is given by Definition 7 in [11], which satisfies Lemma 7-1. Suppose $\mathbb{X}_{Q-1}^{(Q-1)}$ is an ULA with sensor separation 2^{Q-2} and first element $(Q-2)(N_1+1) + 2^{Q-2}$. According to Rule 2 of Definition 2, $\mathbb{X}_Q^{(Q)}$ is derived from odd terms in $\mathbb{X}_{Q-1}^{(Q-1)}$. Therefore, the first element of $\mathbb{X}_Q^{(Q)}$ is $(Q-1)(N_1+1) + 2^{Q-1}$ and the inter-element spacing is 2^{Q-1}.
2) According to Rule 1 of Definition 2, we have $\mathbb{X}_q^{(Q)} = \mathbb{X}_q^{(q+1)}$. Following Rule 2, $\mathbb{X}_q^{(q+1)}$ has at least all the the even terms of $\mathbb{X}_q^{(q)}$, which constitute the ULA portion. Based on Lemma 7-1, $\mathbb{X}_q^{(q)}$ is a ULA of separation 2^{q-1} and its minimum element is $(q-1)(N_1+1) + 2^{q-1}$. Therefore, the ULA part in $\mathbb{X}_q^{(q+1)}$ owns sensor separation 2^q. The minimum element in $\mathbb{X}_q^{(Q)}$ is then given by $(q-1)(N_1+1) + 2^{q-1}$.
3) In this case, Rule 2b of Definition 2 indicates that the extra term is the largest one of $\mathbb{X}_q^{(q)}$ while the last term of the ULA section of $\mathbb{X}_q^{(Q)}$ is the second largest one in $\mathbb{X}_q^{(q)}$. Based on Lemma 7-1, their difference is 2^{q-1}.

4) If $n \in \mathbb{X}_Q^{(Q)}$, then $n-(N_1+1)$ is an odd term of $\mathbb{X}_{Q-1}^{(Q-1)}$. Based on Rule 2 of Definition 2 and Lemma 7-1, $n-(N_1+1) \pm 2^{Q-2}$ are even terms of $\mathbb{X}_{Q-1}^{(Q-1)}$, which is contained in $\mathbb{X}_{Q-1}^{(Q)}$.

This completes the proof. ■

Next, assuming $\mathbb{S}^{(Q-1)}$ is a restricted array, we need to show that $\mathbb{S}^{(Q)}$ is also a restricted array. Similarly, there are 27 cases, as listed in Table I.

(Case 1, 13) Given n_1 and n_2 in this case, we have

1) n_2 belongs to the ULA portion of $\mathbb{X}_q^{(Q)}$: This case is the same as Case 1-1 in the proof of Theorem 1. n_1' and n_2' can be written in two ways:

$$n_1' = n_1 - 2^{Q-2}, \quad n_2' = n_2 - 2^{Q-2}, \tag{48}$$

$$n_1' = n_1 + 2^{Q-2}, \quad n_2' = n_2 + 2^{Q-2}, \tag{49}$$

where (48) and (49) resemble (10) and (11), respectively. According to Lemma 7-4, n_1' lives in $\mathbb{X}_{Q-1}^{(Q)}$. In addition, at least one of the n_2' in (48) or (49) belongs to $\mathbb{X}_q^{(Q)}$. If neither n_2+2^{Q-2} nor n_2-2^{Q-2} belongs to $\mathbb{X}_q^{(Q)}$, then the ULA part of $\mathbb{X}_q^{(Q)}$ has aperture less than 2^{Q-1}. On the other hand, n_1' in (48) and (49) implies $\mathbb{X}_{Q-1}^{(Q)}$ has aperture at least 2^{Q-1}. This is a contradiction since $\mathbb{X}_q^{(Q)}$ must have larger aperture than $\mathbb{X}_{Q-1}^{(Q)}$.

2) n_2 is an extra term in $\mathbb{X}_q^{(Q)}$: In this case, we only need to consider $2 \leq q \leq Q-2$ because when $q=1$, there is no extra term, by definition. Based on (49) and Lemma 7-4, we know that n_1+2^{Q-2} belongs to $\mathbb{X}_{Q-1}^{(Q)}$ and $n_1+2^{Q-2}-(N_1+1)+2^{Q-3}$ is contained in $\mathbb{X}_{Q-2}^{(Q)}$. Applying these rules multiple times yields,

$$\begin{cases} n_1' = n_1 - (q-1)(N_1+1) + \sum_{p=Q-q-1}^{Q-2} 2^p, \\ n_2' = n_2 - (q-1)(N_1+1) + \sum_{p=Q-q-1}^{Q-2} 2^p. \end{cases} \tag{50}$$

It ensures that n_1' lives in $\mathbb{X}_{Q-q}^{(Q)}$. We need to show that $n_2' \in \mathbb{Y}_1^{(Q)}$. According to Rule 2 in Definition 2, n_2' is an even number. Its minimum value is attained when all the sets $\mathbb{X}_1^{(Q)}, \mathbb{X}_2^{(Q)}, \ldots, \mathbb{X}_{q-1}^{(Q)}$ own extra terms, implying

$$n_2 - (q-1)(N_1+1) \geq (1+2A_1) - \left(1 + \sum_{p=2}^{q} 2^p\right),$$

where A_1 is given by Definition 7 in [11]. Therefore, n_2' is lower-bounded by

$$(1+2A_1) - \left(1 + \sum_{p=2}^{q} 2^p\right) + \sum_{p=Q-q-1}^{Q-2} 2^p > 1 + 2A_1.$$

Thus, n_2' belongs to $\mathbb{Y}_1^{(Q)}$.

(Case 4, 10) Let us consider Case 4. According to n_2, we obtain two cases:

1) If n_2 belongs to the ULA part of $\mathbb{Y}_q^{(Q)}$, it is the same as Case 1-1.
2) If n_2 is an extra term in $\mathbb{Y}_q^{(Q)}$, following the idea of Case 1-2, we can write

$$\begin{cases} n_1' = n_1 - (q-1)(N_1+1) - \sum_{p=Q-q-1}^{Q-2} 2^p, \\ n_2' = n_2 - (q-1)(N_1+1) - \sum_{p=Q-q-1}^{Q-2} 2^p. \end{cases} \tag{51}$$

Here n_1' belongs to $\mathbb{X}_{Q-q}^{(Q)}$ and n_2' is an odd number. Next we will show that n_2' belongs to $\mathbb{X}_1^{(Q)}$. Similar to Case 1-2, $n_2-(q-1)(N_1+1)$ is upper-bounded by

$$\begin{aligned} & n_2-(q-1)(N_1+1) \\ & \leq (N_1+1)-(1+2B_1)+\left(1+\sum_{p=2}^{q} 2^p\right). \end{aligned}$$

Therefore, n_2' has an upper bound

$$\begin{aligned} & (N_1+1)-(1+2B_1)+\left(1+\sum_{p=2}^{q} 2^p\right)-\sum_{p=Q-q-1}^{Q-2} 2^p \\ & < (N_1+1)-(1+2B_1), \end{aligned}$$

which proves that n_2' belongs to $\mathbb{X}_1^{(Q)}$. The proof for Case 10 is similar to Case 4.

(Case 19, 22) Let $n_1=(Q-1)(N_1+1)$ and $n_2 \in \mathbb{X}_q^{(Q)}$. Based on n_2, we have two cases:

1) n_2 belongs to the ULA portion of $\mathbb{X}_q^{(Q)}$: Following the steps of Case 19, 22 in Section V, we obtain

$$\begin{cases} n_1' = (Q-1)(N_1+1)+2^{Q-1}, \\ n_2' = n_2+2^{Q-1}, \end{cases} \tag{52}$$

$$\begin{cases} n_1' = (Q-1)(N_1+1)-2^{Q-2}, \\ n_2' = n_2-2^{Q-2}. \end{cases} \tag{53}$$

Then, n_1' can be either in $\mathbb{X}_Q^{(Q)}$ or $\mathbb{Y}_{Q-1}^{(Q)}$, according to (52) or (53). It is trivial that n_2' belongs to $\mathbb{X}_q^{(Q)}$.

2) n_2 is an extra term in $\mathbb{X}_q^{(Q)}$: n_1' and n_2' are given by

$$\begin{cases} n_1' = (Q-q)(N_1+1)+2^{Q-q}+2^{Q-1}, \\ n_2' = n_2-(q-1)(N_1+1)+2^{Q-q}+2^{Q-1}. \end{cases} \tag{54}$$

It can be seen from (54) that n_1' belongs to $\mathbb{X}_{Q-q+1}^{(Q)}$ and n_2' is contained in $\mathbb{Y}_1^{(Q)}$. It can be proved by checking the lower bound of n_2', which is

$$\begin{aligned} & n_2-(q-1)(N_1+1)+2^{Q-q}+2^{Q-1} \\ & \geq (1+2A_1)-\left(1+\sum_{p=2}^{q} 2^p\right)+2^{Q-q}+2^{Q-1} \\ & = (1+2A_1)+(2^{Q-1}-2^{q+1})+(2^{Q-q}+3) \\ & > 1+2A_1. \end{aligned}$$

for $1 \leq q \leq Q-2$.

(Other Cases) Proofs are the same as those in Section V.

Next we discuss the sufficient conditions of $\mathbb{S}^{(Q)}$ being a restricted array. According to Definition 2, $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ are not empty. Suppose there is only one element in $\mathbb{X}_Q^{(Q)}$, Lemma 7-4 implies there are at least 2 elements in

TABLE II

ARRAY PROFILES FOR THE EXAMPLE IN SECTION VII

Array	ULA	MRA	Nested array, $N_1 = N_2 = 17$	Coprime array, $M = 9, N = 17$	$\mathbb{S}^{(2)}$, $N_1 = N_2 = 17$	$\mathbb{S}^{(3)}$, $N_1 = N_2 = 17$	$\mathbb{S}^{(3)}$, $N_1 = 16, N_2 = 18$
Aperture	33	329	305	289	305	305	305
DOF	67	659	611	451	611	611	611
Uniform DOF	67	659	611	323	611	611	611
Restricted arrays	Yes	Yes	Yes	No	Yes	Yes	Yes
Max. sources	33	329	305	161	305	305	305
$w(1)$	33	1	17	2	1	1	2
$w(2)$	32	12	16	2	16	9	8
$w(3)$	31	1	15	2	1	2	5

$\mathbb{X}_{Q-1}^{(Q)}$. Applying this argument many times yields that $\mathbb{X}_q^{(Q)}$ has at least 2^{Q-q} elements. The same property holds for $\mathbb{Y}_q^{(Q)}$. In addition, if $N_1 = 4r + 2$, the number of elements between $\mathbb{X}_1^{(Q)}$ and $\mathbb{Y}_1^{(Q)}$ differs by 2. Hence, to guarantee this proof is valid, we need

$$N_1 \geq 2 \sum_{q=1}^{Q} 2^{Q-q} + 4 = 2 \cdot 2^Q + 2.$$

Besides, the sufficient condition for N_2 is $N_2 \geq 3Q - 4$, following the same argument in the proof of Theorem 1.

■

VII. NUMERICAL EXAMPLES

In this section, we make a comparison among ULA, MRA, nested arrays, coprime arrays, and super nested arrays when the mutual coupling effect is present. The total number of sensors is 34 for each array configuration. The sensor locations for MRA cannot be found in the literature, so instead we select the approximate MRA with $n = 18$ and $p = 13$ (Reference L, Table 6 of [14])[2]. The nested array has parameter $N_1 = N_2 = 17$. We choose $M = 9, N = 17$ in coprime arrays. For super nested arrays, there are three different cases: 1) the super nested array with $Q = 2, N_1 = N_2 = 17$, 2) the super nested array with $Q = 3, N_1 = N_2 = 17$, and 3) the super nested array with $Q = 3, N_1 = 16, N_2 = 18$. The sensor locations for these arrays are given by (4) for the nested array, (7) in [11] for the coprime array, Definition 7 in [11] for the super nested array with $Q = 2$, Definition 1 for the super nested

[2]The sensor locations for the approximate MRA are 0, 1, 14, 30, 46, 62, 78, 94, 110, 126, 142, 158, 174, 190, 206, 222, 238, 254, 270, 286, 302, 304, 306, 308, 310, 312, 314, 317, 319, 321, 323, 325, 327, and 329.

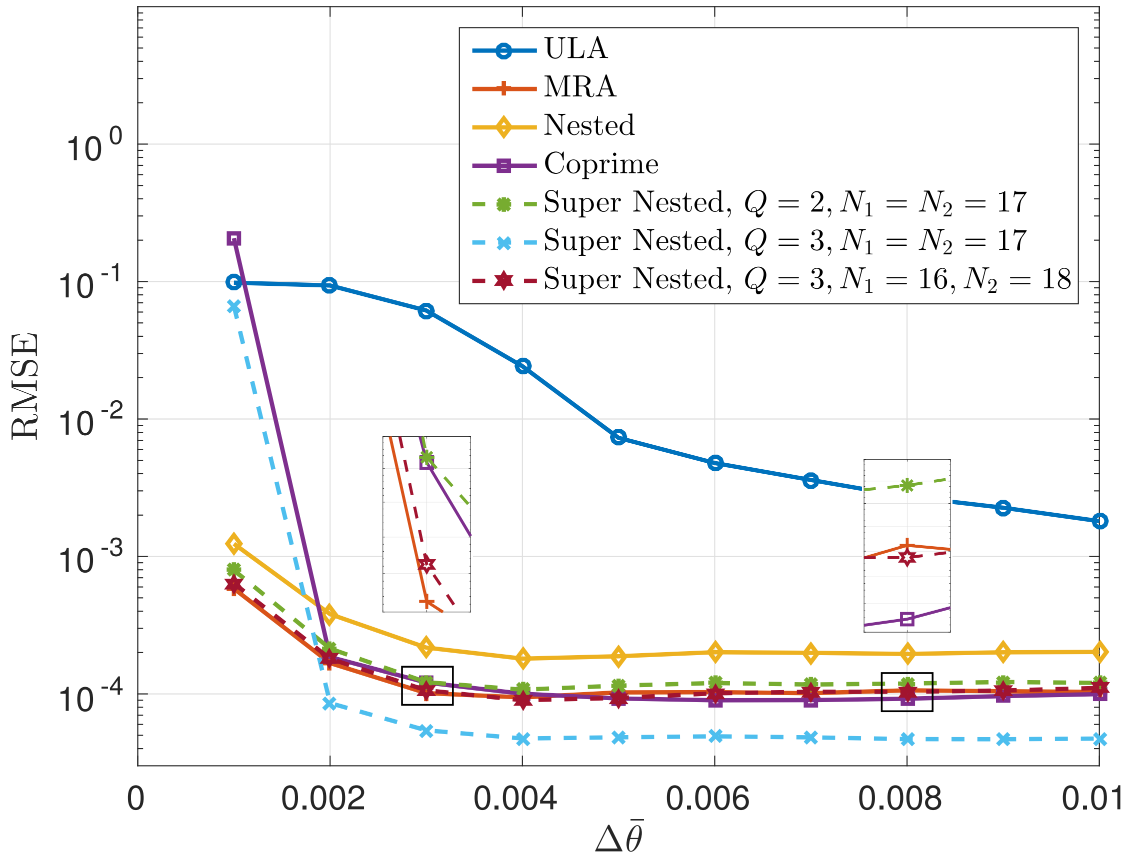


Fig. 6. Estimation error as a function of source spacing $\Delta\bar{\theta}$ between two sources. The parameters are $\mathrm{SNR} = 0\mathrm{dB}, K = 500$. The sources have equal power and their normalized DOA are $\bar{\theta}_1 = \bar{\theta}_0 + \Delta\bar{\theta}/2$ and $\bar{\theta}_2 = \bar{\theta}_0 - \Delta\bar{\theta}/2$, where $\bar{\theta}_0 = 0.2$. Each point is an average over 1000 runs.

array with $Q = 3, N_1 = N_2 = 17$, and Definition 2 for the super nested array with $Q = 3, N_1 = 16, N_2 = 18$. More details on these arrays are listed in Table II.

The experiments in this section are conducted as in the companion paper [11]. Sensor measurements are generated from the model with mutual coupling, as in (8) of [11]. Then, for ULA, the MUSIC algorithm [15] is applied while for sparse arrays, the spatially smoothed MUSIC algorithm [8], [16], [17] is utilized to estimate the source directions. Note that *no decoupling algorithms* are involved. The parameters to be estimated are the normalized DOA: $\bar{\theta}_i = (d/\lambda)\sin\theta_i$, where $d = \lambda/2$ is the minimum sensor separation, λ is the wavelength, and $-\pi/2 \leq \theta_i \leq \pi/2$ is the DOA for the ith source. To compare the result quantitatively, the root-mean-squared error (RMSE) is defined as $E = (\sum_{i=1}^{D}(\hat{\bar{\theta}}_i - \bar{\theta}_i)^2/D)^{1/2}$, where $\hat{\bar{\theta}}_i$ is the estimated normalized DOA of the ith source, calculated from the root MUSIC algorithm, and $\bar{\theta}_i$ is the true normalized DOA.

A. Two Closely-Spaced Sources

In this example, two closely-spaced sources with equal power are presented. The parameters are 0 dB SNR, and $K = 500$ snapshots. The mutual coupling matrix is based on linear dipole antennas, as in (9) of [11]. We choose the carrier frequency $f = 2.4$GHz so $\lambda = 0.1249$m. The dipole length $l = \lambda/2$. The impedance $Z_A = Z_L = 50$ ohms. Two sources are located at $\bar{\theta}_1 = \bar{\theta}_0 + \Delta\bar{\theta}/2$ and $\bar{\theta}_2 = \bar{\theta}_0 - \Delta\bar{\theta}/2$, where $\bar{\theta}_0 = 0.2$. This experiment is repeated for 1000 runs, yielding 1000 instances of RMSE. In Fig. 6(a), the relationship between the source separation $\Delta\bar{\theta}$ and its RMSE, which is the sample mean of 1000 RMSE instances, is plotted. Some observations can be made from Fig. 6. First, all sparse arrays show a significant error reduction in almost all $\Delta\bar{\theta}$, compared to ULA. It can also be deduced from Fig. 6(a) that, as $\Delta\bar{\theta}$ increases, the coprime array becomes slightly better than the second-order super nested array and the third-order super nested array with even N_1. The third-order super nested arrays with odd N_1 shows the best performance over $0.002 \leq \Delta\bar{\theta} \leq 0.01$ in Fig. 6, among all these array configurations.

B. Performance Evaluation under Various Parameters

The next simulation considers the performance over various SNR, number of snapshots, number of sources, and the mutual coupling matrices. The default parameter setting is 0 dB SNR, $K = 500$ snapshots, and $D = 20$ sources with equal power. The sources are located at $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$ for $1 \leq i \leq D$. It will be observed from the simulations that the coprime array outperforms the other array configurations if the number of sources is small and the mutual coupling is small. The super nested array with $Q = 3$ and odd N_1 exhibits the best performance when there are many sources and mutual coupling is severe.

In Fig. 7(a), the RMSE is plotted as a function of SNR. We see that the super nested arrays with $Q = 3$ are the best and ULA is the worst. Fig. 7(b) shows the RMSE versus the number of snapshots K, where the super nested arrays with $Q = 3$ demonstrate a significant reduction on RMSE. The coprime array becomes more accurate as the number of snapshots increases, and it works better than the second-order super nested array when K is above 200.

The relationship between the RMSE and the number of sources D is plotted in Fig. 7(c). The coprime array works the best if the number of sources is small. As D increases, the super nested arrays with $Q = 3$ own the minimum RMSE. The reason is, coprime arrays might own the least mutual coupling effect while third-order super nested arrays possess larger uniform DOF. If D is small, mutual coupling might be more important than uniform DOF. On the other hand, as the number of sources gets closer to the theoretical limit, as shown in Table II, the performance worsens for any array. This phenomenon happens sooner in the coprime array (around $D = 30$) than in the third-order super nested arrays (around $D = 50$), since the coprime array detects at most 161 sources while the third-order super nested arrays can resolve up to 305 sources.

Fig. 8 examines the RMSE of various arrays under various amount of mutual coupling effect and different number of sources $D = 10$, 20, and 40. Note that the total number of sensors is 34, so $D = 40$ exceeds the resolution limit of ULA, as listed in Table II. Here Eq. (10) of [11] is selected to be our mutual coupling model with $B = 3$. Notice that the larger the magnitudes of the mutual coupling coefficients $c_1, c_2, \ldots c_B$ are, the more severe mutual coupling is. In this simulation, we first parametrize $|c_1|$ and then $|c_2|, \ldots, |c_B|$ are obtained from the assumption

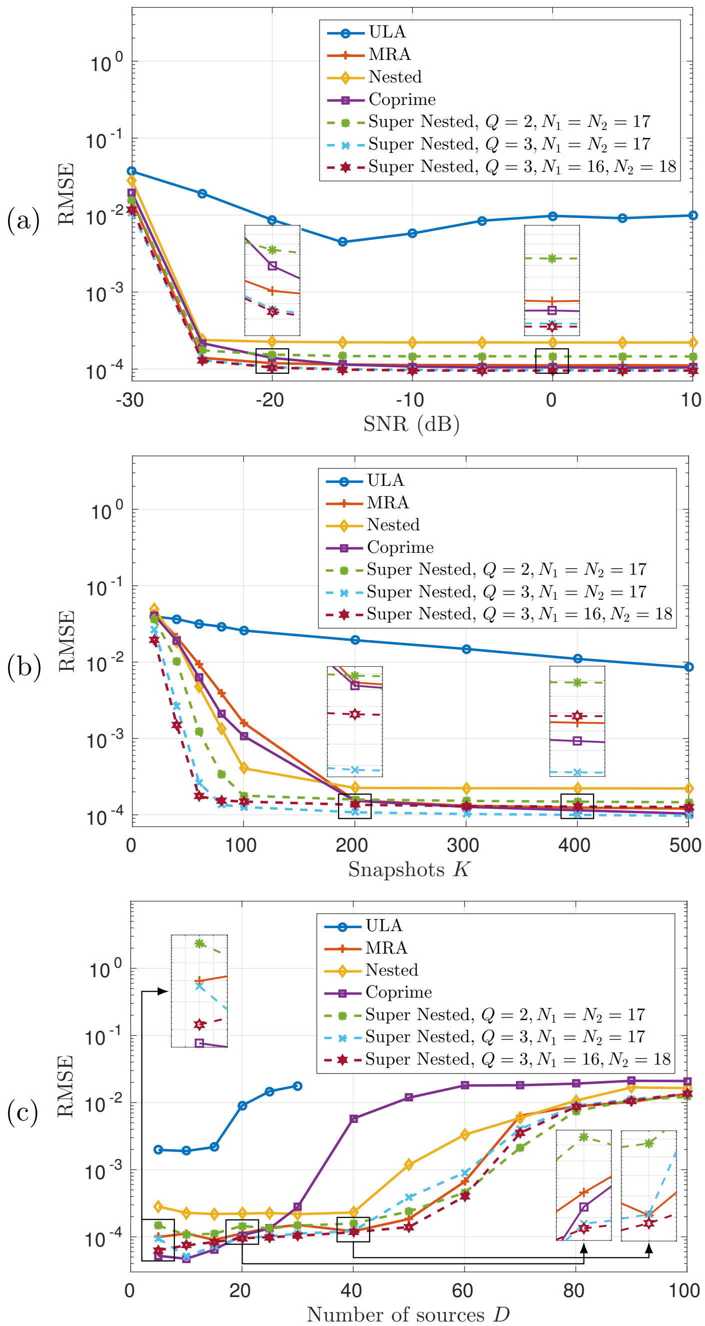


Fig. 7. Estimation error as a function of (a) SNR, (b) the number of snapshots K, and (c) the number of sources D. The parameters are (a) $K = 500, D = 20$, (b) $\text{SNR} = 0\text{dB}, D = 20$, and (c) $\text{SNR} = 0\text{dB}, K = 500$. The sources have equal power and normalized DOA $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$ for $1 \leq i \leq D$. Each point is an average over 1000 runs.

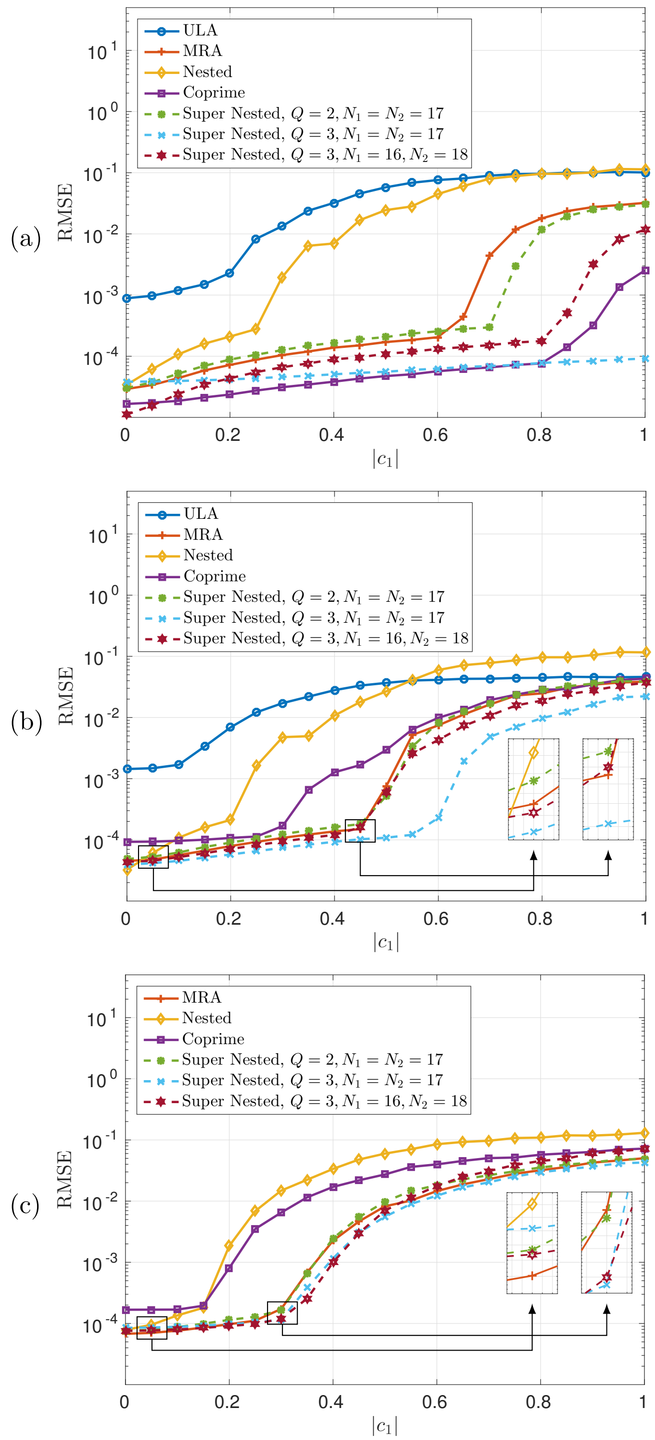


Fig. 8. Estimation error as a function of mutual coupling coefficient c_1 (see Eq. (10) of [11]). The parameters are $\mathrm{SNR} = 0\mathrm{dB}$, $K = 500$, and the number of sources (a) $D = 10$, (b) $D = 20$ (c) $D = 40$. The sources have equal power and are located at $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$ for $1 \leq i \leq D$. The mutual coupling coefficients satisfy $|c_\ell / c_k| = k/\ell$ while the phases are randomly chosen from their domain. Each point is an average over 1000 runs.

that the magnitude of mutual coupling coefficients is inversely proportional to the sensor separation. In each run, the phases of $c_1, c_2, \ldots, c_B$ are randomly drawn from $[-\pi, \pi)$, the root MUSIC algorithm is used to estimate the DOA of D sources, located at $\bar{\theta}_i = -0.45 + 0.9(i-1)/(D-1)$ for $1 \leq i \leq D$. Finally the RMSE is evaluated. Each data point in Fig. 8 is the sample mean of 1000 runs.

Some observations can be drawn from Fig. 8. First, for any array geometry, as $|c_1|$ increases, the associated RMSE increases. This is reasonable since larger $|c_1|$ introduces more severe mutual coupling effect. Second, array configurations seem to have a direct impact on the robustness under mutual coupling. Most curves have turning points, or thresholds, such that the performance starts to become much worse. Hence larger thresholds imply the associated arrays are more tolerant to severe mutual coupling. Note that this threshold depends on the number of sources D. For instance, in coprime arrays, the thresholds in $|c_1|$ are 0.8, 0.3, and 0.15 for $D = 10$, $D = 20$, and $D = 40$, respectively. In the super nested array with $Q = 2$, the threshold moves from 0.7, to 0.45, to 0.35, as D goes from 10, to 20, to 40. An interesting observation is that, the super nested array with $Q = 3$ and odd N_1 are quite robust in the case of severe mutual coupling and many sources.

Another way to interpret Fig. 8 is to consider a fixed D and a fixed $|c_1|$. In most cases, the super nested array with $Q = 3$ and odd N_1 give the minimum RMSE. The exception occurs in Fig. 8(a) when $D = 10$ and $0.1 < |c_1| < 0.8$, where coprime arrays become the best. This result is consistent with that in Fig. 7(c), where coprime arrays work slightly better if the number of sources is small.

VIII. Concluding Remarks

In this paper, we presented an extension of super nested arrays, called the Qth-order super nested arrays. These arrays preserve all the properties of nested arrays, while significantly reducing the effects of mutual coupling between sensors, by decreasing the number of sensor pairs with small separation. In the future, it will be of interest to apply to these arrays the decoupling algorithms developed in earlier literature for mitigating mutual coupling effects [1]–[7]. This will further improve the detection and estimation performance of these arrays. Another future direction of interest would be the extension of linear super nested arrays to the case of planar arrays. These extensions are currently under investigation.

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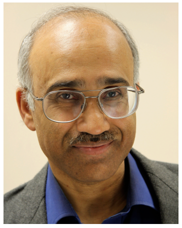

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