High Order Super Nested Arrays

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Outline

1. Introduction (DOA, Sensor Arrays, ...)
2. Review of Super Nested Arrays
3. High Order Super Nested Arrays
4. Numerical Examples
5. Concluding Remarks
DOA estimation in the presence of mutual coupling

We will develop new sparse arrays with less mutual coupling.

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## ULA and sparse arrays

### ULA (not sparse)
- Identify at most $N - 1$ uncorrelated sources, given $N$ sensors.\(^1\)
- Can only find fewer sources than sensors.

### Sparse arrays
1. Minimum redundancy arrays\(^2\)
2. Nested arrays\(^3\)
3. Coprime arrays\(^4\)
4. Super nested arrays\(^5\)
   - Identify $O(N^2)$ uncorrelated sources with $O(N)$ physical sensors.
   - More sources than sensors!

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The nested array

\[ N_1 = 4, \quad N_2 = 4. \]

Difference coarray

\[ \mathcal{D} = \{ n_1 - n_2 \mid n_1, n_2 \in \mathcal{S} \} \]

\[ |\mathcal{S}| = N_1 + N_2 = 8 \]

- **Dense ULA**
  - \( N_1 \) sensors
  - spacing 1

- **Sparse ULA**
  - \( N_2 \) sensors
  - spacing \( N_1 + 1 \)

\[ |\mathcal{D}| = O(N_1 N_2) \]

For sufficient number of snapshots,

\[ (|\mathcal{U}| - 1)/2 = O(N_1 N_2) \]

uncorrelated sources can be identified.

\( \mathcal{U} = \) Central ULA part of \( \mathcal{D} \)

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Super nested arrays

1. Super nested arrays have the same number of sensors as nested arrays.
2. Super nested arrays have the same difference coarrays as nested arrays. In particular, no holes.
3. Super nested arrays are more sparse than nested arrays, i.e., super nested arrays have less mutual coupling.

Nested array $N_1 = 13, N_2 = 5$.

Super nested array $N_1 = 13, N_2 = 5$.

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How to characterize mutual coupling in arrays?¹

The weight function $w(m)$

The number of sensor pairs with separation $m$.

Nested array, $N_1 = N_2 = 7$

Super nested array, $N_1 = N_2 = 7$

More sparse
Less mutual coupling

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Goal: Desired properties of super nested arrays

They should have **the same number of sensors** as nested arrays,

\[ |S_{\text{High order super nested}}| = |S_{\text{Super nested}}| = |S_{\text{Nested}}|. \]

They should have **the same difference coarray** as nested arrays,

\[ D_{\text{High order super nested}} = D_{\text{Super nested}} = D_{\text{Nested}}. \]

(In particular, no holes)

They should be **more sparse** than nested arrays,

\[
\begin{align*}
    w_{\text{High order super nested}}(1) & \leq w_{\text{Super nested}}(1) \leq w_{\text{Nested}}(1), \\
    w_{\text{High order super nested}}(2) & \leq w_{\text{Super nested}}(2) \leq w_{\text{Nested}}(2),
\end{align*}
\]
2D representations for 1D nested arrays

The nested array with $N_1 = N_2 = 5$

Dense ULA

Sparse ULA

Layer 1
Layer 2
Layer 3
Layer 4
Layer 5

High order super nested arrays

Second-order super nested array

\[ N_1 = 15, \]
\[ N_2 = 7, \]
\[ Q = 2 \]

High-order super nested array

\[ N_1 = 15, \]
\[ N_2 = 7, \]
\[ Q = 3. \]
The hierarchy of $Q$th-order super nested arrays

$$S(Q) = \left( \bigcup_{q=1}^{Q} X_q^{(Q)} \cup Y_q^{(Q)} \right) \cup Z_1^{(Q)} \cup Z_2^{(Q)},$$

1 MATLAB routines are available at [http://systems.caltech.edu/dsp/students/clliu/SuperNested.html](http://systems.caltech.edu/dsp/students/clliu/SuperNested.html)
Main properties of super nested arrays:

1) Difference coarray

\[ D_{\text{Nested}} \]

\[ D^{(Q)}_{\text{Super nested}} \]

- \( D^{(Q)}_{\text{Super nested}} = D_{\text{Nested}} \) if
  - \( Q \geq 3 \),
  - \( N_1 \) and \( N_2 \) are sufficiently large.\(^1\)

- Properties of \( D^{(Q)}_{\text{Super nested}} \):
  - Contiguous integers.
  - Hole-free.

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\(^1\) The lower bounds are given in the papers.
Main properties of super nested arrays:

2) Weight functions

<table>
<thead>
<tr>
<th></th>
<th>$w(1)$</th>
<th>$w(2)$</th>
<th>$w(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nested</td>
<td>$N_1$</td>
<td>$N_1 - 1$</td>
<td>$N_1 - 2$</td>
</tr>
<tr>
<td>Super nested</td>
<td>$\begin{cases} 2, &amp; \text{if } N_1 \text{ is even} \ 1, &amp; \text{if } N_1 \text{ is odd} \end{cases}$</td>
<td>$\begin{cases} N_1 - 3, &amp; \text{if } N_1 \text{ is even} \ N_1 - 1, &amp; \text{if } N_1 \text{ is odd} \end{cases}$</td>
<td>$\begin{cases} 3, &amp; \text{if } N_1 = 4, 6, \ 4, &amp; \text{if } N_1 \text{ is even} \ 1, &amp; \text{if } N_1 \text{ is odd} \end{cases}$</td>
</tr>
<tr>
<td>High order</td>
<td>$\begin{cases} 2, &amp; \text{if } N_1 \text{ is even} \ 1, &amp; \text{if } N_1 \text{ is odd} \end{cases}$</td>
<td>$\begin{cases} 2 \left\lfloor \frac{N_1}{4} \right\rfloor + 1, &amp; \text{if } N_1 \text{ is odd} \ N_1/2 + 1, &amp; \text{if } N_1 = 8k - 2 \ N_1/2 - 1, &amp; \text{if } N_1 = 8k + 2 \ N_1/2, &amp; \text{otherwise} \end{cases}$</td>
<td>$\begin{cases} 5, &amp; \text{if } N_1 \text{ is even} \ 2, &amp; \text{if } N_1 \text{ is odd} \end{cases}$</td>
</tr>
<tr>
<td>super nested</td>
<td>$Q \geq 3$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Numerical Examples

Simulation procedure

\[ \bar{\theta}_1 = 0.25 \quad \bar{\theta}_D = -0.25 \]

Spatial Smoothing MUSIC (SS MUSIC) \(^1\)

Uncorrelated Sources

34 Sensors

Estimated normalized DOA \( \hat{\theta}_i \)

0 dB SNR, 200 snapshots, RMSE

\[ E = \left( \frac{1}{D} \sum_{i=1}^{D} (\hat{\theta}_i - \bar{\theta}_i)^2 \right)^{1/2} \]

---

MUSIC spectra (34 sensors, 30 sources)

- **Nested array** ($E = 0.10209$)
- **Coprime array** ($E = 0.019742$)
- **Super nested array**
  \[ Q = 2, E = 0.013414 \]
- **High order super nested array**
  \[ Q = 3, E = 0.00015819 \]
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Concluding remarks

- High order super nested arrays
  - They have the same number of sensors as (super) nested arrays.
  - They have the same difference coarray as (super) nested arrays if $N_1$ and $N_2$ are sufficiently large.
  - They have reduced mutual coupling than (super) nested arrays.
  - They can be constructed recursively from (super) nested arrays.

- In the future, decoupling algorithms will improve the performance.\(^1\)

- For more information, please go to our project website: [http://systems.caltech.edu/dsp/students/clliu/SuperNested.html](http://systems.caltech.edu/dsp/students/clliu/SuperNested.html)

Thank you!

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The data model (ideal)

\[ x_S = \sum_{i=1}^{D} A_i v_S(\bar{\theta}_i) + n_S, \]

- \( S \): An integer set for the sensor locations, in units of \( \lambda/2 \).
- \( \bar{\theta}_i = (d/\lambda) \sin \theta_i \): the normalized DOA \((-1/2 \leq \bar{\theta}_i < 1/2)\).
- \( A_i \): The complex amplitude for the \( i \)th source.
- \( v_S(\bar{\theta}_i) = [e^{j2\pi\bar{\theta}_in}]_{n \in S} \): steering vectors.

**Statistical Assumptions**

- \( A_i \): zero mean, variance \( \sigma_i^2 \).
- \( n_S \): zero mean, covariance \( \sigma^2 I \).
- Sources are uncorrelated: \( \mathbb{E}[A_i A_j^*] = \sigma_i^2 \delta_{i,j} \).
- Sources are uncorrelated to the noise: \( \mathbb{E}[A_i n_S^H] = 0 \).
- \( \bar{\theta}_i \) is considered to be fixed but unknown.
The data model in the presence of mutual coupling

\[ x_S = \sum_{i=1}^{D} A_i C v_S(\bar{\theta}_i) + n_S, \]

- **C**: mutual coupling matrix satisfying
  \[ \langle C \rangle_{n_1,n_2} = \begin{cases} 
  c_{|n_1-n_2|}, & \text{if } |n_1 - n_2| \leq B, \\
  0, & \text{otherwise,}
  \end{cases} \]

- \( n_1 \) and \( n_2 \) are sensor locations.
- \( 1 = c_0 > |c_1| > |c_2| > \cdots > |c_B| \).
- In this paper, we assume that \( |c_k/c_\ell| = \ell/k. \)
- Mutual coupling is a function of sensor separations.

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The mutual coupling models in simulations

\[ B = 100, \quad c_1 = 0.6e^{j\frac{\pi}{3}}, \quad c_\ell = \frac{c_1}{\ell} e^{-j\frac{\pi}{8} (\ell - 1)}, \quad \text{for} \ \ell = 2, 3, \ldots, B \]

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(c_3)</th>
<th>(c_4)</th>
<th>(c_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real</td>
<td>0.3000</td>
<td>0.2380</td>
<td>0.1932</td>
<td>0.1487</td>
<td>0.1039</td>
</tr>
<tr>
<td>Imaginary</td>
<td>0.5196</td>
<td>0.1826</td>
<td>0.0518</td>
<td>−0.0196</td>
<td>−0.0600</td>
</tr>
</tbody>
</table>

Magnitudes of mutual coupling matrices, \(|[C]_{i,j}|\)

- Nested array
- Coprime array
- Second-order super nested array
- Third-order super nested array

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Another mutual coupling model: King’s formula

If the sensor array is a linear dipole array, \( C \) can be written as

\[
C = (Z_A + Z_L)(Z + Z_L I)^{-1},
\]

where \( Z_A \) and \( Z_L \) are the element/load impedance, respectively. \( \langle Z \rangle_{n_1,n_2} \) is given by

\[
\left\{
\begin{array}{l}
\frac{\eta_0}{4\pi} (0.5772 + \ln(2\beta l) - \text{Ci}(2\beta l) + j\text{Si}(2\beta l)), \quad \text{if } n_1 = n_2, \\
\frac{\eta_0}{4\pi} \left( \langle R \rangle_{n_1,n_2} + j \langle X \rangle_{n_1,n_2} \right), \quad \text{if } n_1 \neq n_2.
\end{array}
\right.
\]

Here \( \eta_0 = \sqrt{\mu_0 / \epsilon_0} \approx 120\pi \) is the intrinsic impedance. \( \beta = 2\pi / \lambda \) is the wavenumber, where \( \lambda \) is the wavelength. \( l \) is the length of dipole antennas. \( R \) and \( X \) are

\[
\langle R \rangle_{n_1,n_2} = \sin(\beta l) (-\text{Si}(u_0) + \text{Si}(v_0) + 2\text{Si}(u_1) - 2\text{Si}(v_1))
\]
\[
+ \cos(\beta l) (\text{Ci}(u_0) + \text{Ci}(v_0) - 2\text{Ci}(u_1) - 2\text{Ci}(v_1) + 2\text{Ci}(\beta d_{n_1,n_2})) - \left( 2\text{Ci}(u_1) + 2\text{Ci}(v_1) - 4\text{Ci}(\beta d_{n_1,n_2}) \right),
\]
\[
\langle X \rangle_{n_1,n_2} = \sin(\beta l) (-\text{Ci}(u_0) + \text{Ci}(v_0) + 2\text{Ci}(u_1) - 2\text{Ci}(v_1))
\]
\[
+ \cos(\beta l) (-\text{Si}(u_0) - \text{Si}(v_0) + 2\text{Si}(u_1) + 2\text{Si}(v_1) - 2\text{Si}(\beta d_{n_1,n_2})) + \left( 2\text{Si}(u_1) + 2\text{Si}(v_1) - 4\text{Si}(\beta d_{n_1,n_2}) \right).
\]

where \( d_{n_1,n_2} = |n_1 - n_2| \lambda / 2 \) is the distance between sensors. The parameters \( u_0, v_0, u_1, \) and \( v_1 \) are

\[
u_0 = \beta \left( \sqrt{d_{n_1,n_2}^2 + l^2} - l \right), \quad v_0 = \beta \left( \sqrt{d_{n_1,n_2}^2 + l^2} + l \right),
\]
\[
u_1 = \beta \left( \sqrt{d_{n_1,n_2}^2 + 0.25l^2} - 0.5l \right), \quad v_1 = \beta \left( \sqrt{d_{n_1,n_2}^2 + 0.25l^2} + 0.5l \right).
\]

Here \( \text{Si}(u) = \int_0^u \frac{\sin t}{t} \, dt \) and \( \text{Ci}(u) = \int_\infty^u \frac{\cos t}{t} \, dt \) are sine/cosine integrals.

\footnote{King, *IRE Trans. Antennas Propag.*, 1957.}
Properties of the weight functions $w(m)$

The weight function $w(m)$

The number of sensor pairs with separation $m$.

For any linear array with $N$ sensors, weight functions satisfy

1. $w(0)$ equals the total number of sensors, i.e.,

$$w(0) = N.$$ 

2. The sum of the weight functions is purely dependent on $N$.

$$\sum_{m \in \mathbb{D}} w(m) = N^2.$$ 

3. Weight functions are symmetric.

$$w(m) = w(-m), \quad \text{for } m \in \mathbb{D}.$$ 

---

Performance over SNR

34 sensors, 20 equal-power sources, 500 snapshots, dipole model, $Z_A = Z_L = 50$, $l = \lambda/2$, 
$\bar{\theta}_i = -0.45 + 0.9(i - 1)/(D - 1)$, 1000 runs.

Performance over Snapshots

(b) RMSE

34 sensors, 20 equal-power sources, 0dB SNR, dipole model, $Z_A = Z_L = 50, l = \lambda/2,$

$\bar{\theta}_i = -0.45 + 0.9(i - 1)/(D - 1), 1000$ runs.

Performance over Number of sources

(c)

ULA
MRA
Nested
Coprime
Super Nested, \( Q = 2, N_1 = N_2 = 17 \)
Super Nested, \( Q = 3, N_1 = N_2 = 17 \)
Super Nested, \( Q = 3, N_1 = 16, N_2 = 18 \)

34 sensors, equal-power sources, 0dB SNR, 500 snapshots, dipole model, \( Z_A = Z_L = 50 \), \( l = \lambda/2 \),
\( \hat{\theta}_i = -0.45 + 0.9(i - 1)/(D - 1) \), 1000 runs.

Performance over two closely spaced sources

**Figure Description**

- **Graph Title:** Performance over two closely spaced sources

- **Graph Axes:**
  - **Y-axis:** RMSE
  - **X-axis:** $\Delta \bar{\theta}$

- **Lines and Data Points:**
  - ULA
  - MRA
  - Nested
  - Coprime
  - Super Nested, $Q = 2$, $N_1 = N_2 = 17$
  - Super Nested, $Q = 3$, $N_1 = N_2 = 17$
  - Super Nested, $Q = 3$, $N_1 = 16$, $N_2 = 18$

- **Data:**
  - 34 sensors, two equal-power sources at $\bar{\theta}_1 = 0.2 + \Delta \bar{\theta}/2$, $\bar{\theta}_1 = 0.2 - \Delta \bar{\theta}/2$, 0dB SNR, 500 snapshots, dipole model, $Z_A = Z_L = 50$, $l = \lambda/2$, 1000 runs.

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**Notes:**

Performance over mutual coupling models

(a) RMSE vs. $|c_1|$ for different arrays:
- ULA
- MRA
- Nested
- Coprime
- Super Nested, $Q = 2, N_1 = N_2 = 17$
- Super Nested, $Q = 3, N_1 = N_2 = 17$
- Super Nested, $Q = 3, N_1 = 16, N_2 = 18$

10 sources, 34 sensors

0dB SNR, 500 snapshots, Toeplitz model, phases of $c_\ell$ are random. $\bar{\theta}_i = -0.45 + 0.9(i - 1)/(D - 1)$, 1000 runs.

Performance over mutual coupling models

20 sources, 34 sensors

0 dB SNR, 500 snapshots, Toeplitz model, phases of $c_\ell$ are random. $\bar{\theta}_i = -0.45 + 0.9(i - 1)/(D - 1)$, 1000 runs.

Performance over mutual coupling models

40 sources, 34 sensors

0dB SNR, 500 snapshots, Toeplitz model, phases of $c_\ell$ are random. $\bar{\theta}_i = -0.45 + 0.9(i - 1)/(D - 1)$, 1000 runs.