High Order Super Nested Arrays

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***Abstract*—Mutual coupling between sensors has a negative impact on the estimation of directions of arrival (DOAs). Sparse arrays such as nested arrays, coprime arrays, and minimum redundancy arrays (MRAs) have less mutual coupling than uniform linear arrays (ULAs). These sparse arrays also have a difference coarray of size $O(N^2)$, where N is the number of sensors, and can therefore resolve $O(N^2)$ uncorrelated source directions. The various sparse arrays proposed in the literature have their pros and cons. The nested array is practical and easy to use but has a dense ULA part which suffers from mutual coupling effects like the traditional ULA. The recently introduced super nested arrays reduce this mutual coupling problem, while maintaining the desirable hole-free $O(N^2)$ difference coarray of the nested array. In this paper, a generalization of super nested arrays is introduced, called the Qth-order super nested array. This has all the properties of the second-order super nested array with the additional advantage that mutual coupling effects are further reduced for $Q > 2$. A numerical example is included to demonstrate the superior performance of these arrays.**[1]

***Index Terms*—Sparse arrays, nested arrays, coprime arrays, super nested arrays, mutual coupling, DOA estimation.**

I. Introduction

In array processing, mutual coupling between sensors has an adverse effect on the estimation of parameters (e.g., DOA) [1]–[7]. Sparse arrays such as nested arrays [8], coprime arrays [9], and minimum redundancy arrays (MRA) [10] have reduced mutual coupling compared to uniform linear arrays (ULAs). Sparse arrays also have a difference coarray with $O(N^2)$ virtual elements, where N is the number of physical sensors, and can therefore resolve $O(N^2)$ uncorrelated source directions. But these sparse arrays have shortcomings: MRAs do not have simple closed-form expressions for the array geometry [10]; coprime arrays have holes in the coarray [9]; and nested arrays contain a dense ULA in the physical array [8], resulting in significantly higher mutual coupling than coprime arrays and MRAs.

The (second-order) super nested array was introduced in [11], [12], which has many of the advantages of these sparse arrays, while removing some of the disadvantages. Namely, the sensor locations are well-defined and readily computed for any N (unlike MRAs), and the difference coarray is exactly that of a nested array, and therefore hole-free (unlike coprime arrays). At the same time, the mutual coupling is reduced compared to nested arrays. Super nested arrays were designed by rearranging the dense ULA part of a nested array in such a way that the coarray remains unchanged, but mutual coupling is reduced by reducing the number of elements with small inter-element spacings. Quantitatively, this is described in terms of the weight function $w(m)$, which will be given in Definition 2 later. It was proved in [12] that the first three weight functions of second-order super nested arrays are

$$w(1) = \begin{cases} 2, & \text{if } N_1 \text{ is even,} \\ 1, & \text{if } N_1 \text{ is odd,} \end{cases} \tag{1}$$

$$w(2) = \begin{cases} N_1 - 3, & \text{if } N_1 \text{ is even,} \\ N_1 - 1, & \text{if } N_1 \text{ is odd,} \end{cases} \tag{2}$$

$$w(3) = \begin{cases} 3, & \text{if } N_1 = 4, 6, \\ 4, & \text{if } N_1 \text{ is even}, N_1 \geq 8, \\ 1, & \text{if } N_1 \text{ is odd,} \end{cases} \tag{3}$$

Contrast this with the nested array which has $w(1) = N_1, w(2) = N_1 - 1$ and $w(3) = N_1 - 2$. While $w(1)$ and $w(3)$ are significantly better in (1) and (3), there is plenty of room for improving $w(2)$, and possibly $w(m), m > 3$.

In this paper, a generalization of super nested arrays is introduced, called the *Qth-order super nested array*. It has all the good properties of the second-order super nested array with the additional advantage that mutual coupling effects are further reduced for $Q > 2$. For a given number of physical array elements N, Qth-order super nested arrays have the following properties: (a) the sensor locations can be defined using a simple algorithm, (b) the physical array has the same aperture as the nested array, (c) the difference coarray is exactly identical to that of the nested array (hence hole free), and (d) the weight functions are further improved, compared even to second-order super nested arrays (Theorem 2).

II. Preliminaries

Assume that D monochromatic far-field sources illuminate the sensor array, where the sensor locations are nd. Here n belongs to some integer set $\mathbb{S}$, $d = \lambda/2$ denotes the minimum distance between sensors, and λ is the wavelength. For the ith source, its complex amplitude is written as A_i and its direction-of-arrival (DOA) is denoted by $\theta_i \in [-\pi/2, \pi/2]$. The measurement vector $\mathbf{x}_\mathbb{S}$ on the sensor array $\mathbb{S}$ can be modeled as follows:

$$\mathbf{x}_\mathbb{S} = \sum_{i=1}^{D} A_i \mathbf{v}_\mathbb{S}\left(\bar{\theta}_i\right) + \mathbf{n}_\mathbb{S}, \tag{4}$$

where $\mathbf{v}_\mathbb{S}(\bar{\theta}_i) = [e^{j2\pi\bar{\theta}_i n}]_{n\in\mathbb{S}}$ are steering vectors and $\mathbf{n}_\mathbb{S}$ is the additive noise term. $\bar{\theta}_i = (d/\lambda)\sin\theta_i$ is the normalized DOA. We obtain $-1/2 \leq \bar{\theta} \leq 1/2$. The parameters A_i and $\mathbf{n}_\mathbb{S}$ are assumed to be zero-mean, uncorrelated random variables

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with $\mathbb{E}[A_i A_j^*] = \sigma_i^2 \delta_{i,j}$ and $\mathbb{E}[\mathbf{n}_{\mathbb{S}}\mathbf{n}_{\mathbb{S}}^H] = \sigma^2 \mathbf{I}$. Here σ_i^2 is the power of the ith source, σ^2 is the noise power, and $\delta_{p,q}$ is the Kronecker delta. $\bar{\theta}_i$ is considered to be fixed but unknown.

If there is mutual coupling, the data model becomes

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{C} \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathbb{S}}, \tag{5}$$

where $\mathbf{C}$ is a mutual coupling matrix. The entries of $\mathbf{C}$ are

$$\langle \mathbf{C} \rangle_{n_1,n_2} = \begin{cases} c_{|n_1-n_2|}, & \text{if } |n_1 - n_2| \leq B, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $\langle \cdot \rangle$ is the triangular bracket notation [13], $n_1, n_2 \in \mathbb{S}$, and the coupling coefficients $c_0, c_1, \ldots, c_B$ satisfy $1 = c_0 > |c_1| > |c_2| > \cdots > |c_B|$. It is assumed that the magnitudes of coupling coefficients are inversely proportional to their sensor separations [1], i.e. $|c_k/c_\ell| = \ell/k$.

For sparse arrays, the following definitions will be useful:

Definition 1 (Difference coarray)**.** *For an array specified by an integer set $\mathbb{S}$, its difference coarray $\mathbb{D} = \{n_1 - n_2 | n_1, n_2 \in \mathbb{S}\}$.*

Definition 2 (Weight functions)**.** *The weight function $w(m)$ of an array $\mathbb{S}$ is defined as the number of sensor pairs that lead to coarray index m, i.e., $|\{(n_1, n_2) \in \mathbb{S}^2 \mid n_1 - n_2 = m\}|$.*

A *restricted array* is an array whose difference coarray $\mathbb{D}$ is a ULA with adjacent elements separated by $\lambda/2$, or an array with hole-free difference coarray. *The uniform degree of freedom (uniform DOF)* is the cardinality of the central ULA part of $\mathbb{D}$, which is related to the limit of identifiable sources. It was shown in [8], [14] that, if the uniform DOF is $\mathcal{F}$, the maximum number of uncorrelated sources that coarray MUSIC can identify is $(\mathcal{F} - 1)/2$.

Next, we will review some well-known sparse array configurations, like minimum redundancy arrays (MRAs) [10], nested arrays [8], coprime arrays [9], and super nested arrays [12]. All these arrays can resolve $O(N^2)$ sources provided with $O(N)$ sensors and increase the spatial resolution [8]–[10].

MRAs [10] maximize their uniform DOF subject to a given total number of sensors. However, it is not possible to obtain explicit expressions of the sensor locations for arbitrary number of sensors [10], [15]. Nested arrays [8] and coprime arrays [9], on the other hand, characterize their sensor locations in a closed-form, simple, and scalable fashion. For nested arrays, the sensor locations are given by

$$\begin{aligned} \mathbb{S}_{\text{nested}} = \{&1, 2, \ldots, N_1, \\ &(N_1+1), 2(N_1+1), \ldots N_2(N_1+1)\}, \end{aligned} \tag{7}$$

where N_1 and N_2 are positive integers. The sensor locations for coprime arrays are

$$\begin{aligned} \mathbb{S}_{\text{coprime}} = \{&0, M, 2M, \ldots, (N-1)M, \\ &N, 2N, \ldots, (2M-1)N\}, \end{aligned} \tag{8}$$

where M and N are a coprime pair of positive integers. These simple expressions facilitate the design process with arbitrary number of sensors. However, the nested array has severe mutual coupling, due to the dense ULA part [8],

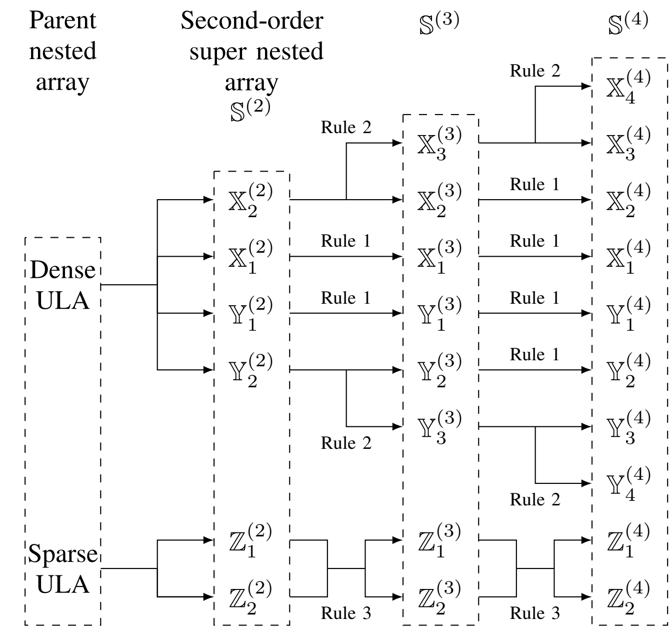


Fig. 1. Hierarchy of nested arrays, second-order super nested arrays $\mathbb{S}^{(2)}$, and Qth-order super nested arrays $\mathbb{S}^{(Q)}$. Arrows indicate the origin of the given sets. For instance, $\mathbb{X}_2^{(4)}$ originates from $\mathbb{X}_2^{(3)}$ while $\mathbb{Y}_3^{(3)}$ is split into $\mathbb{Y}_3^{(4)}$ and $\mathbb{Y}_4^{(4)}$. It can be observed that the sets $\mathbb{X}_q^{(Q)}$ and $\mathbb{Y}_q^{(Q)}$ result from the dense ULA part of nested arrays. The sparse ULA portion of nested arrays is rearranged into the sets $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$.

[12]. The coprime array owns holes in the difference coarray, which prevents us from using the full coarray in the MUSIC algorithm [14]. The (second-order) super nested array aims to overcome all the above issues, as mentioned in Section I. The sensor locations for (second-order) super nested arrays are

$$\mathbb{S}^{(2)} = \mathbb{X}_1^{(2)} \cup \mathbb{Y}_1^{(2)} \cup \mathbb{X}_2^{(2)} \cup \mathbb{Y}_2^{(2)} \cup \mathbb{Z}_1^{(2)} \cup \mathbb{Z}_2^{(2)}, \tag{9}$$

where the sets $\mathbb{X}_1^{(2)}$, $\mathbb{Y}_1^{(2)}$, $\mathbb{X}_2^{(2)}$, $\mathbb{Y}_2^{(2)}$, and $\mathbb{Z}_1^{(2)}$ are ULAs. The formal definition of $\mathbb{S}^{(2)}$ can be found in [11], [12].

III. High-Order Super Nested Arrays

Fig. 1 summarizes the hierarchy among nested arrays, second-order super nested arrays, and Qth-order super nested arrays. It has been mentioned in [12] that the sets $\mathbb{X}_1^{(2)}$, $\mathbb{Y}_1^{(2)}$, $\mathbb{X}_2^{(2)}$, and $\mathbb{Y}_2^{(2)}$ are obtained by rearranging the dense ULA part of parent nested arrays. The sparse ULA part of parent nested arrays is reorganized into $\mathbb{Z}_1^{(2)}$ and $\mathbb{Z}_2^{(2)}$ of second-order super nested arrays [12]. High-order super nested arrays can be obtained from the second-order ones, using some recursive rules, as depicted in Fig. 1.

The formal definition of Qth-order nested arrays will be given in Definition 3 later. To develop some feeling for it, first consider $Q = 3$. Third-order super nested arrays, as specified by the integer set $\mathbb{S}^{(3)}$, consist of eight sets as follows: $\mathbb{X}_1^{(3)}$, $\mathbb{Y}_1^{(3)}$, $\mathbb{X}_2^{(3)}$, $\mathbb{Y}_2^{(3)}$, $\mathbb{X}_3^{(3)}$, $\mathbb{Y}_3^{(3)}$, $\mathbb{Z}_1^{(3)}$, and $\mathbb{Z}_2^{(3)}$, which can be recursively generated from the sets $\mathbb{X}_1^{(2)}$, $\mathbb{Y}_1^{(2)}$, $\mathbb{X}_2^{(2)}$, $\mathbb{Y}_2^{(2)}$, $\mathbb{Z}_1^{(2)}$, $\mathbb{Z}_2^{(2)}$ in second-order super nested arrays. For instance, $\mathbb{X}_1^{(3)}$ is identical to $\mathbb{X}_1^{(2)}$ (Rule 1 in Definition 3). $\mathbb{X}_2^{(2)}$ is split into two sets $\mathbb{X}_2^{(3)}$ and $\mathbb{X}_3^{(3)}$ (Rule 2 in Definition 3). The same connections also apply to $\mathbb{Y}_1^{(2)}$, $\mathbb{Y}_2^{(2)}$, $\mathbb{Y}_1^{(3)}$, $\mathbb{Y}_2^{(3)}$, and

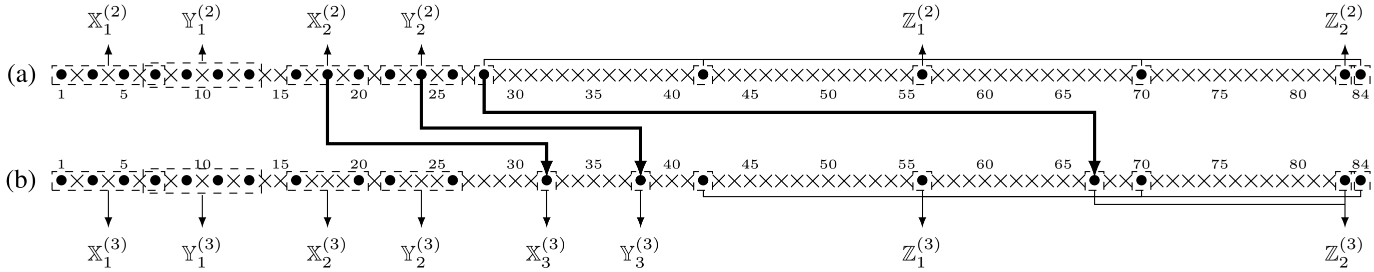


Fig. 2. Array configurations of (a) second-order super nested arrays, $\mathbb{S}^{(2)}$, and (b) third-order super nested arrays, $\mathbb{S}^{(3)}$, where $N_1 = 13$ and $N_2 = 6$. Bullets denote sensor locations while crosses indicate empty locations.

$\mathbb{Y}_3^{(3)}$. Finally, the elements in $\mathbb{Z}_1^{(2)}$ and $\mathbb{Z}_2^{(2)}$ are rearranged into $\mathbb{Z}_1^{(3)}$ and $\mathbb{Z}_2^{(3)}$ (Rule 3 in Definition 3). Hence, it can be interpreted that the sets $\mathbb{X}_q^{(3)}$ and $\mathbb{Y}_q^{(3)}$ for $q = 1, 2, 3$ originate from the dense ULA of parent nested arrays while $\mathbb{Z}_1^{(3)}$ and $\mathbb{Z}_2^{(3)}$ emanate from the sparse ULA of parent nested arrays.

Fourth-order super nested arrays (or super nested arrays with $Q = 4$) generalize third-order super nested arrays further. It can be deduced from Fig. 1 that $\mathbb{X}_3^{(3)}$ and $\mathbb{Y}_3^{(3)}$ are divided into $\mathbb{X}_3^{(4)}$, $\mathbb{X}_4^{(4)}$ and $\mathbb{Y}_3^{(4)}$, $\mathbb{Y}_4^{(4)}$, respectively (Rule 2). $\mathbb{Z}_1^{(3)}$ and $\mathbb{Z}_2^{(3)}$ are rearranged into $\mathbb{Z}_1^{(4)}$ and $\mathbb{Z}_2^{(4)}$ (Rule 3). The remaining sets in fourth-order super nested arrays are the same as their correspondences in third-order super nested arrays (Rule 1).

Next, we give a concrete example of how Qth-order super nested arrays are obtained from $(Q-1)$th-order super nested arrays. Fig. 2 depicts the array configurations of the second-order super nested array (in part (a)) and the third-order one (in part (b)), respectively. In this example, it is obvious that $\mathbb{X}_1^{(2)} = \mathbb{X}_1^{(3)}$ and $\mathbb{Y}_1^{(2)} = \mathbb{Y}_1^{(3)}$, which satisfy Rule 1. To explain Rule 2, we consider the following sets in Fig. 2:

$$\mathbb{X}_2^{(2)} = \{16, 18, 20\},\ \mathbb{X}_2^{(3)} = \{16, 20\},\ \mathbb{X}_3^{(3)} = \{32\}. \quad (10)$$

The middle element of $\mathbb{X}_2^{(2)}$, which is the element 18 in this case, is selected and relocated to the element 32 in $\mathbb{X}_3^{(3)}$. The remaining elements in $\mathbb{X}_2^{(2)}$, which correspond to sensor locations 16 and 20, constitute $\mathbb{X}_2^{(3)}$. Finally, Rule 3 can also be clarified using Fig. 2. In the second-order super nested array, we consider the sensor located at $2(N_1 + 1) = 28$, which is the leftmost element of $\mathbb{Z}_1^{(2)}$. However, this sensor is removed from $\mathbb{S}^{(2)}$ and inserted to $\mathbb{S}^{(3)}$ at location 67, as indicated by a thick arrow in Fig. 2. This new sensor location is included in $\mathbb{Z}_2^{(3)} = \{67, 83\}$, which explains Rule 3. Furthermore, after all these operations, the sensors between location 1 and 14 do not change while only some elements (18, 24, and 28 in Fig. 2) between 15 and 28 are rearranged to somewhere else.

Summarizing, Qth-order super nested arrays can be recursively generated from $(Q-1)$th-order super nested arrays, as elaborated in Fig. 1 and 2. Next, we will give a formal definition for super nested arrays, by characterizing Rule 1, 2, and 3. This definition also enables us to determine the sensor locations explicitly.

Definition 3 (Qth-order super nested arrays)**.** *Let N_1 be a positive integer, $N_2 \geq N_{2,\min}$, and $Q \geq 3$. Let $\mathbb{S}^{(2)}$ be a second-order super nested array, defined in* (9)*. A Qth-order super nested array is specified by the integer set $\mathbb{S}^{(Q)}$,*

$$\mathbb{S}^{(Q)} = \left(\bigcup_{q=1}^{Q} \mathbb{X}_q^{(Q)} \cup \mathbb{Y}_q^{(Q)} \right) \cup \mathbb{Z}_1^{(Q)} \cup \mathbb{Z}_2^{(Q)},$$

where $N_{2,\min}$ is $2Q-1$ for odd N_1 and $2Q$ otherwise. These nonempty subsets $\mathbb{X}_q^{(Q)}$, $\mathbb{Y}_q^{(Q)}$, $\mathbb{Z}_1^{(Q)}$, and $\mathbb{Z}_1^{(Q)}$ satisfy

1) ***(Rule 1)*** *For $1 \leq q \leq Q-2$, $\mathbb{X}_q^{(Q)} = \mathbb{X}_q^{(Q-1)}$.*
2) ***(Rule 2)*** *$\mathbb{X}_{Q-1}^{(Q)}$ and $\mathbb{X}_Q^{(Q)}$ can be obtained from $\mathbb{X}_{Q-1}^{(Q-1)}$ by*
   a) *If N_1 is odd, or the cardinality of $\mathbb{X}_{Q-1}^{(Q-1)}$ is odd, then*
   $$\mathbb{X}_{Q-1}^{(Q)} = \{\textit{Even terms of } \mathbb{X}_{Q-1}^{(Q-1)}\},$$
   $$\mathbb{X}_{Q}^{(Q)} = \{(\textit{Odd terms of } \mathbb{X}_{Q-1}^{(Q-1)}) + (N_1+1)\}.$$
   b) *Otherwise, we call the last element in $\mathbb{X}_{Q-1}^{(Q-1)}$ as the* ***extra term****. Then*
   $$\mathbb{X}_{Q-1}^{(Q)} = \{\textit{Even terms of } \mathbb{X}_{Q-1}^{(Q-1)}\} \cup \{\textit{the extra term}\},$$
   $$\mathbb{X}_{Q}^{(Q)} = \{(\textit{Odd terms of } \mathbb{X}_{Q-1}^{(Q-1)}, \textit{ except the extra term}) + (N_1+1)\}.$$

   $\mathbb{Y}_q^{(Q)}$ share similar properties as $\mathbb{X}_q^{(Q)}$ in Rule 1 and 2.
3) ***(Rule 3)*** *The sets $\mathbb{Z}_1^{(Q)}$ and $\mathbb{Z}_2^{(Q)}$ are given by*
   $$\mathbb{Z}_1^{(Q)} = \mathbb{Z}_1^{(Q-1)} \backslash \{(Q-1)(N_1+1)\},$$
   $$\mathbb{Z}_2^{(Q)} = \mathbb{Z}_2^{(Q-1)} \cup \{(N_2+1-(Q-1))(N_1+1) - 2^{Q-1} + 1\},$$
   where $\mathbb{A}\backslash\mathbb{B}$ denotes the relative complement of $\mathbb{B}$ in $\mathbb{A}$.

A MATLAB code to find the sensor locations of Qth-order super nested arrays is given in [16].

Next, we will clarify Rule 2 in Definition 3. Let $N_1 = 16$ and $N_2 = 5$ in super nested arrays. According to [12], the second-order super nested array has $\mathbb{X}_2^{(2)} = \{19, 21, 23, 25\}$. Since N_1 is even and the cardinality of $\mathbb{X}_2^{(2)}$ is 4, Rule 2b is applicable. For $\mathbb{X}_2^{(2)}$, the extra term is 25, the even terms (the first, third smallest ones and so on) are 19 and 23, and the odd terms (the second, fourth smallest ones and so on) are 21 and 25. Using the expressions in Rule 2b of Definition 3, we obtain $\mathbb{X}_2^{(3)} = \{19, 23, 25\}$ and $\mathbb{X}_3^{(3)} = \{38\}$. On the other hand, if we consider $\mathbb{Y}_2^{(2)} = \{28, 30, 32\}$, then the cardinality of $\mathbb{Y}_2^{(2)}$ becomes 3, implying Rule 2a is applicable. Note that for $\mathbb{Y}_2^{(2)}$, the largest term is an even term, the second largest term is an odd term, and so on. Hence, the even terms and odd terms of $\mathbb{Y}_2^{(2)}$ are $28, 32$ and 30, respectively. As a result, $\mathbb{Y}_2^{(3)} = \{28, 32\}$ and $\mathbb{Y}_3^{(3)} = \{47\}$.

IV. Coarray Properties

One of the most striking properties of the Qth-order super nested array is that the coarray is exactly identical to that of the parent nested array [17]:

Theorem 1. *If $N_1 \geq N_{1,\min}$, $N_2 \geq 3Q-4$, and $Q \geq 3$, then Qth-order super nested arrays are restricted arrays, i.e., the difference coarray is hole-free, where $N_{1,\min}$ is given by*

$$N_{1,\min} = \begin{cases} 2 \cdot 2^Q + 2, & \text{if } N_1 \text{ is even,} \\ 3 \cdot 2^Q - 1, & \text{if } N_1 \text{ is odd.} \end{cases} \tag{11}$$

Corollary 1. *If $N_1 \geq N_{1,\min}$, $N_2 \geq 3Q-4$, and $Q \geq 3$, then Qth-order super nested arrays have the same coarray as their parent nested array, where $N_{1,\min}$ is defined in* (11).

Recall that the weight function $w(2)$ of the second-order super nested array was as in Eq. (2). The next theorem shows that the super nested array for $Q > 2$ has significantly improved weight function $w(2)$, which is crucial to reducing the mutual coupling effects [17]:

Theorem 2. *Assume that $N_1 \geq N_{1,\min}$, $N_2 \geq 3Q-4$, and $Q \geq 3$, where $N_{1,\min}$ is defined in* (11). *The weight function $w(m)$ of Qth-order super nested arrays satisfies*

$$w(1) = \begin{cases} 2, & \text{if } N_1 \text{ is even,} \\ 1, & \text{if } N_1 \text{ is odd,} \end{cases} \quad w(3) = \begin{cases} 5, & \text{if } N_1 \text{ is even,} \\ 2, & \text{if } N_1 \text{ is odd,} \end{cases}$$

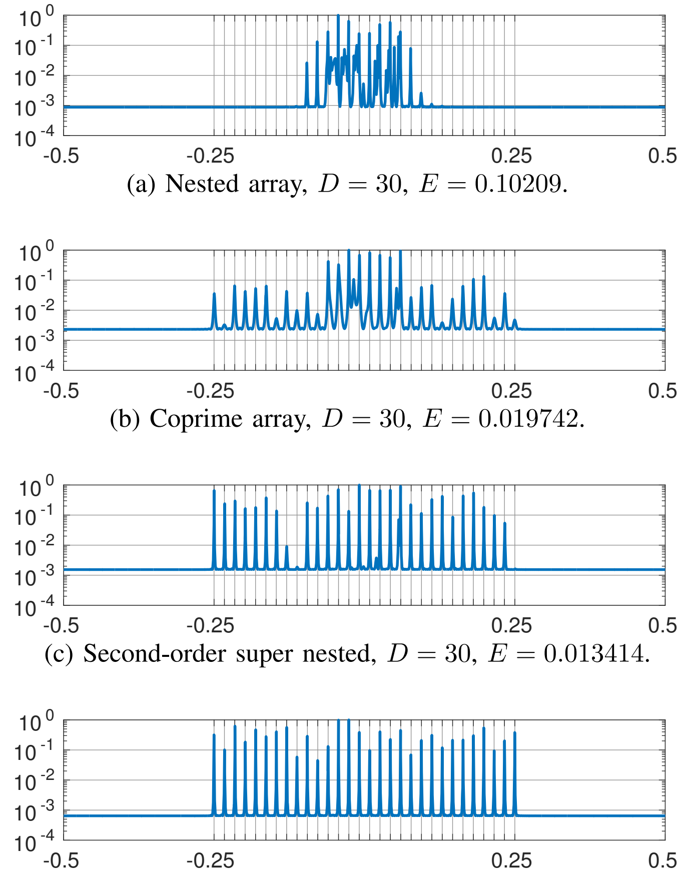
$$w(2) = \begin{cases} 2\lfloor N_1/4 \rfloor + 1, & \text{if } N_1 \text{ is odd,} \\ N_1/2 + 1, & \text{if } N_1 = 8k-2, \\ N_1/2 - 1, & \text{if } N_1 = 8k+2, \\ N_1/2, & \text{otherwise,} \end{cases}$$

where k is an integer.

As we can see, the Qth-order super nested array has the same $w(1)$ as the second-order one, while $w(2)$ of the Qth-order super nested array is approximately half of that of the second-order one.

V. Numerical Examples

In this section, we make a comparison among nested arrays, coprime arrays, and super nested arrays when mutual coupling is present. The total number of sensors is 34 for each array. The nested array and the super nested arrays have parameters $N_1 = N_2 = 17$. We choose $M = 9, N = 17$ in the coprime array. For super nested arrays, there are two different cases: the second-order one and the third-order one ($Q = 3$). The sensor locations for these arrays are given by (7) for the nested array, (8) for the coprime array, Definition 7 in [12] for the second-order super nested array, and Definition 3 for the super nested array with $Q = 3$. The remaining parameters are listed in the caption of Fig. 3. The spectra in Fig. 3 are evaluated as follows: The measurements are influenced by the mutual coupling matrix, as in (5), and the spatial smoothing MUSIC algorithm [8], [13], [14] is utilized directly from the data *without using any decoupling algorithms*. To compare the result quantitatively, the root-mean-squared error (RMSE) is defined as $E = (\sum_{i=1}^{D} (\hat{\bar{\theta}}_i - \bar{\theta}_i)^2/D)^{1/2}$, where $\hat{\bar{\theta}}_i$ denotes the estimated normalized DOA of the ith source, calculated from the root MUSIC algorithm, and $\bar{\theta}_i$ is the true normalized DOA. The best estimation performance is enjoyed by the super nested array with $Q = 3$, followed by the second-order super nested array, then the coprime array, and finally the nested array. Furthermore, only the super nested array with $Q = 3$ displays 30 peaks, without any missing targets or spurious peaks. This example shows that, in the presence of mutual coupling, Qth-order super nested arrays can be superior to nested arrays, coprime arrays, and second-order super nested arrays, even when no decoupling algorithms are employed.



(a) Nested array, $D = 30$, $E = 0.10209$.

(b) Coprime array, $D = 30$, $E = 0.019742$.

(c) Second-order super nested, $D = 30$, $E = 0.013414$.

(d) Third-order super nested, $D = 30$, $E = 0.00015819$.

Fig. 3. The MUSIC spectra $P(\bar{\theta})$ for (a) nested arrays, (b) coprime arrays, (c) second-order super nested arrays, and (d) third-order super nested arrays when $D = 30$ sources are located at $\bar{\theta}_i = -0.25 + 0.5(i-1)/19$, $i = 1, 2, \ldots, 30$, as depicted by ticks and vertical lines. The horizontal axis is the normalized DOA $\bar{\theta}$. The SNR is 0 dB while the number of snapshots is $K = 200$. The mutual coupling is based on (6) with $c_1 = 0.6e^{j\pi/3}$, $B = 100$, and $c_\ell = c_1 e^{-j(\ell-1)\pi/8}/\ell$ for $2 \leq \ell \leq B$.

VI. Concluding Remarks

In this paper, we presented an extension of super nested arrays, called the Qth-order super nested arrays. These arrays preserve all the properties of nested arrays, while significantly reducing the effects of mutual coupling between sensors, by decreasing the number of sensor pairs with small separation. In the future, it will be of interest to extend linear super nested arrays to the case of planar arrays. These extensions are currently under investigation.

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