Comparison of Sparse Arrays From Viewpoint of Coarray Stability and Robustness

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Outline

1 Introduction

2 Review of Sparse Arrays and Robustness

3 Comparison of Sparse Arrays

4 Concluding Remarks

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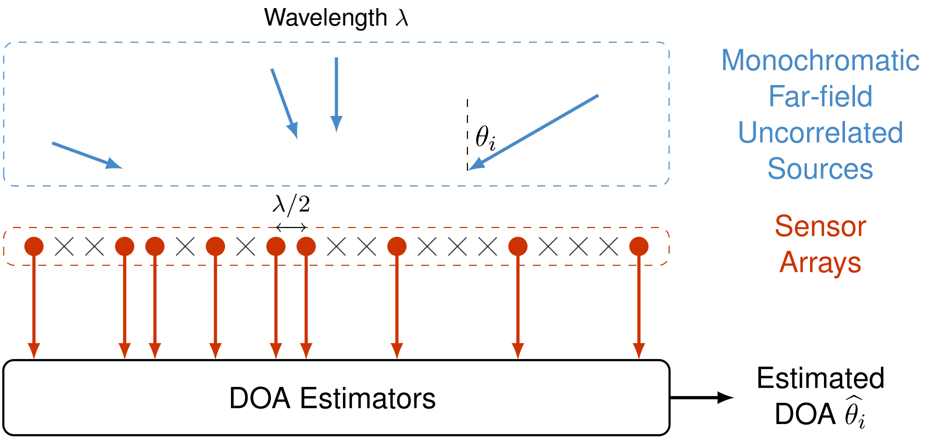
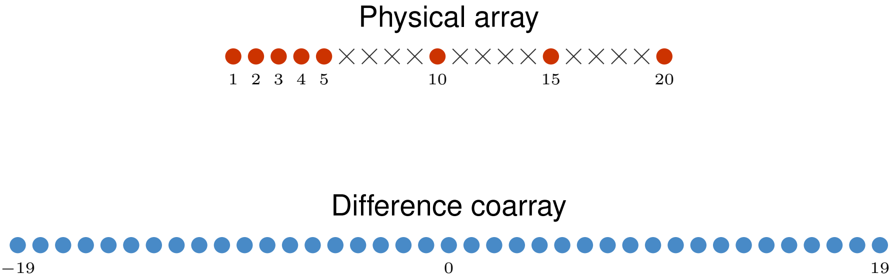
Direction-Of-Arrival (DOA) Estimation



[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

Physical Array and Difference Coarray



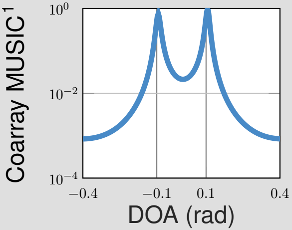
[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

Sensor Failures

Array #1

5 elements

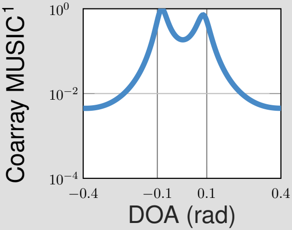
RMSE = 0.00617



Array #2 (2 fails)

4 elements

RMSE = 0.014367



Array #3 (1 fails)

4 elements

Coarray MUSIC is not applicable here!

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[1] Liu and Vaidyanathan, *IEEE Signal Process. Letters*, 2015.

[2] 100 snapshots, 0dB SNR, $D = 2$ sources, $\theta_1 = -0.1$, $\theta_2 = 0.1$, equal-power, uncorrelated sources.

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ULA and Sparse Arrays

ULA (not sparse)

- Identify at most $N_s - 1$ uncorrelated sources.[1]
  (N_s is the number of sensors)
- Can only find fewer sources than sensors.

Linear sparse arrays

1. Minimum redundancy arrays[2]
2. Nested arrays[3]
3. Coprime arrays[4]
4. Super nested arrays[5]

- Identify $\mathcal{O}(N_s^2)$ uncorrelated sources.
- More sources than sensors!

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.
[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968.
[3] Pal and Vaidyanathan, *IEEE Trans. Signal Process.*, 2010.
[4] Vaidyanathan and Pal, *IEEE Trans. Signal Process.*, 2011.
[5] Liu and Vaidyanathan, *IEEE Trans. Signal Process.*, 2016.

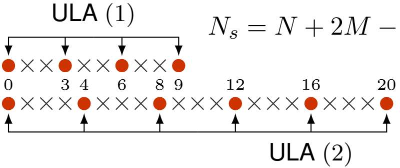
Coprime Arrays

The coprime array with $(M, N) = 1$ is the union of

1. an N-element ULA with spacing M and
2. a $2M$-element ULA with spacing N.

$N_s = N + 2M - 1$

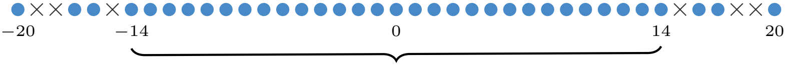
Physical array $\mathbb{S}$ $(M = 3, N = 4)$:



× Holes

Difference coarray

$\mathbb{D} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{S}\}$

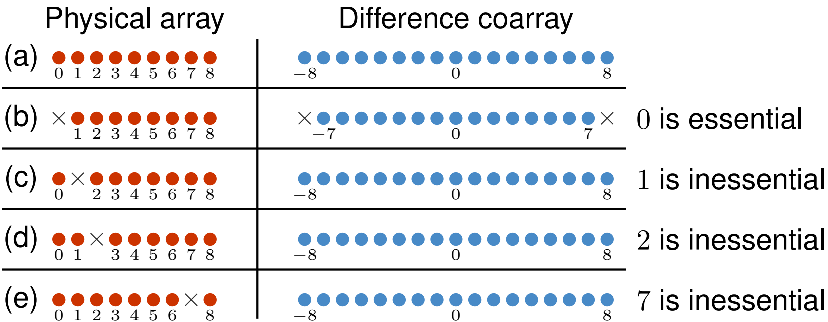


Central ULA segment $\mathbb{U}$

[1] Vaidyanathan and Pal, *IEEE Trans. Signal Process.*, 2011.

The Essentialness Property

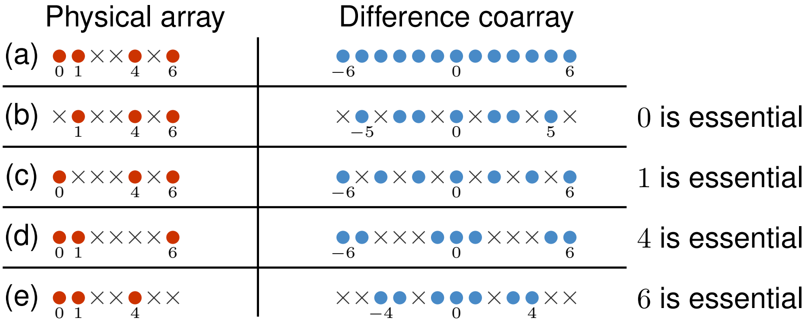
The sensor $n \in \mathbb{S}$ is essential with respect to $\mathbb{S}$ if $\overline{\mathbb{D}} \neq \mathbb{D}$.



[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018; $\mathbb{D}$ is the difference coarray of $\mathbb{S}$ and $\overline{\mathbb{D}}$ is the difference coarray of $\mathbb{S}\backslash\{n\}$.

Maximally Economic Sparse Arrays

An array $\mathbb{S}$ is maximally economic
if all the sensors in $\mathbb{S}$ are essential



Array (a) is maximally economic

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

Maximally Economic Sparse Arrays

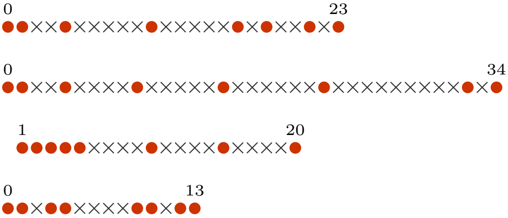
- Array geometries that are maximally economic:

Minimum redundancy array

Minimum hole array

Nested array

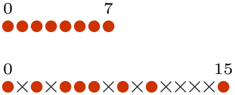
Cantor array



- Array geometries that are not maximally economic:

Uniform linear array

Coprime array



[1] Liu and Vaidyanathan, *IEEE CAMSAP*, 2017; Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

Minimum Redundancy Arrays and Minimum Hole Arrays

Definition of MRA

$$\mathbb{S}_{\mathrm{MRA}} \triangleq \arg\max_{\mathbb{S}} |\mathbb{D}|$$

subject to

$$|\mathbb{S}| = N_s,$$

$$\mathbb{D} = \mathbb{U}.$$

- N_s physical sensors
- Hole-free $\mathbb{D}$

Definition of MHA

$$\mathbb{S}_{\mathrm{MHA}} \triangleq \arg\min_{\mathbb{S}} |\mathbb{H}|$$

subject to

$$|\mathbb{S}| = N_s,$$

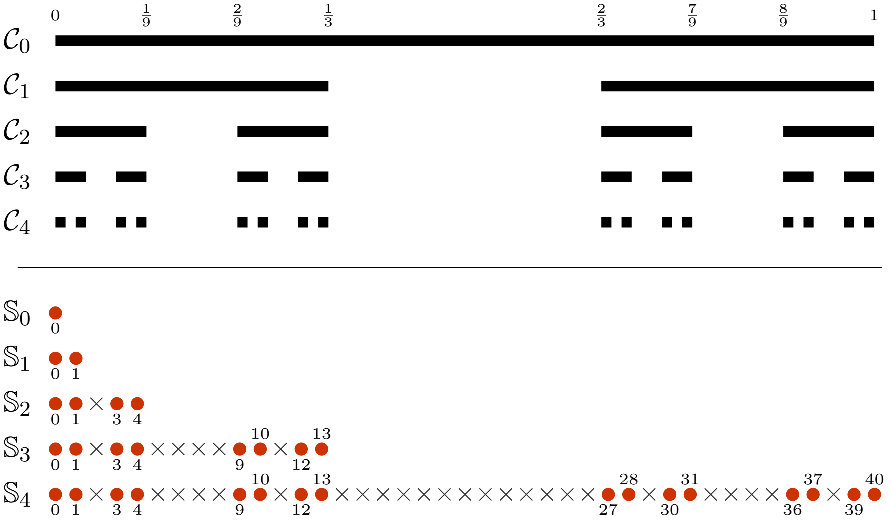
$$w(m) = 1 \text{ for } m \in \mathbb{D}\backslash\{0\}. \quad (1)$$

- N_s physical sensors
- $\mathbb{H}$: the set of holes
- $w(m)$: the number of sensor pairs with separation m
- (1): m is not a hole.

[1] Moffet, *IEEE Trans. Antennas Propag.*, 1968.

[2] Taylor and Golomb, 1985.

Cantor Arrays



[1] Smith, *Proceedings of the London Mathematical Society*, 1874; Cantor, *Mathematische Annalen*, 1883; Puente-Baliarda and Pous, *IEEE Trans. Antennas Propag.*, 1996; Liu and Vaidyanathan, *IEEE CAMSAP*, 2017.

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The Size of the Difference Coarray

$$2N_s - 1 \leq |\mathbb{D}| \leq N_s^2 - N_s + 1$$

- $N_s = |\mathbb{S}|$ is the number of sensors.
- If $\mathbb{S}$ is a ULA, then $|\mathbb{D}| = 2N_s - 1$.
- If $\mathbb{S}$ is a MHA, then $|\mathbb{D}| = N_s^2 - N_s + 1$.

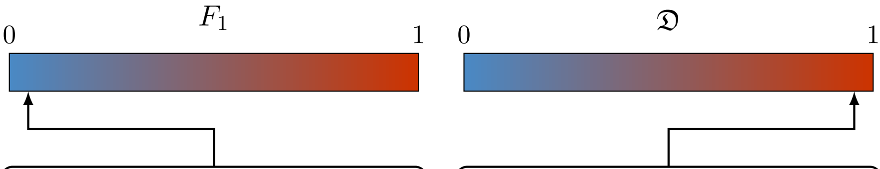
  (MRA does not in general achieve it)

[1] Liu and Vaidyanathan, *IEEE SAM*, 2018.

The Fragility F_1 and the Normalized Size of $\mathbb{D}$

$$F_1 \triangleq \frac{\text{\# of essential sensors}}{\text{\# of all sensors } (N_s)}$$

$$\mathfrak{D} \triangleq \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

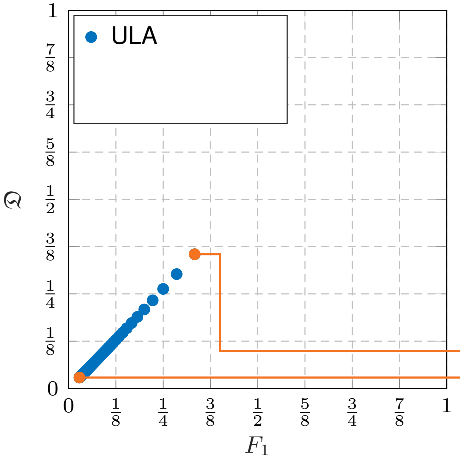


- More robust
- Less fragile

- More sources than sensors
- Higher resolution

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

The F_1-$\mathfrak{D}$ Plane: ULA ($N_s = 6, 7, \ldots, 70$)



$$F_1 = 2/N_s \qquad \text{for } N_s \geq 4,$$
$$|\mathbb{D}| = 2N_s - 1.$$

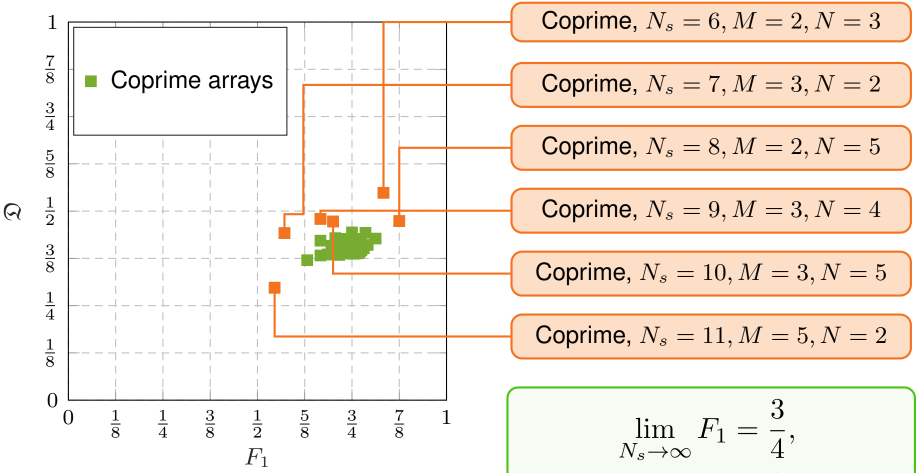
$$\lim_{N_s \to \infty} F_1 = 0,$$
$$\lim_{N_s \to \infty} \mathfrak{D} = 0.$$

ULA, $N_s = 6$

ULA, $N_s = 70$

$$F_1 = \frac{\text{\# of Ess.}}{N_s}, \quad \mathfrak{D} = \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

The F_1-$\mathfrak{D}$ Plane: Coprime Arrays ($N_s = 6, 7, \ldots, 70$)

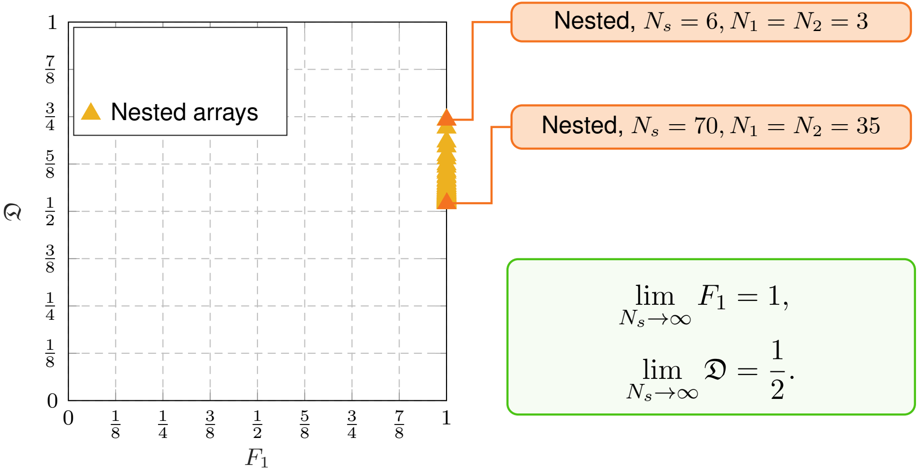


$$F_1 = \frac{\text{\# of Ess.}}{N_s}, \quad \mathfrak{D} = \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

$$\lim_{N_s \to \infty} F_1 = \frac{3}{4},$$

$$\lim_{N_s \to \infty} \mathfrak{D} = \frac{3}{8}.$$

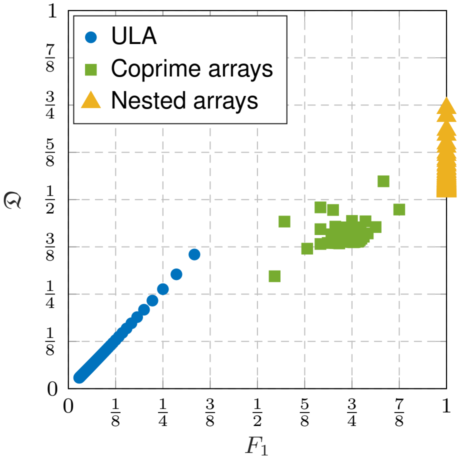
The F_1-$\mathfrak{D}$ Plane: Nested Arrays ($N_s = 6, 7, \ldots, 70$)



$$\lim_{N_s \to \infty} F_1 = 1,$$
$$\lim_{N_s \to \infty} \mathfrak{D} = \frac{1}{2}.$$

$$F_1 = \frac{\text{\# of Ess.}}{N_s}, \quad \mathfrak{D} = \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

The F_1-$\mathfrak{D}$ Plane: ULA/Coprime Arrays/Nested Arrays

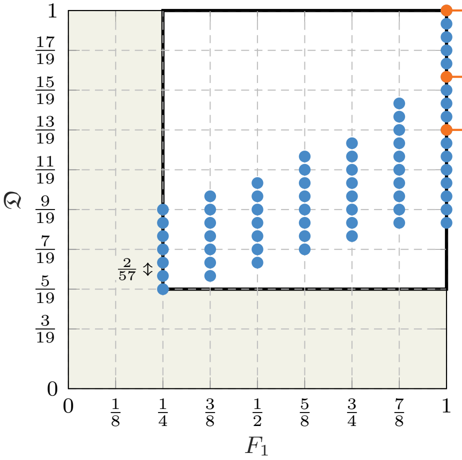


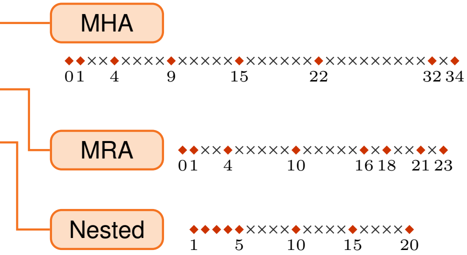
Nested arrays
The largest $\mathbb{D}$
The least robust

ULA
The smallest $\mathbb{D}$
The most robust

$$F_1 = \frac{\text{\# of Ess.}}{N_s}, \quad \mathfrak{D} = \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

The F_1-$\mathfrak{D}$ Plane: $N_s = 8$ Sensors, Aperture $A \leq 34$



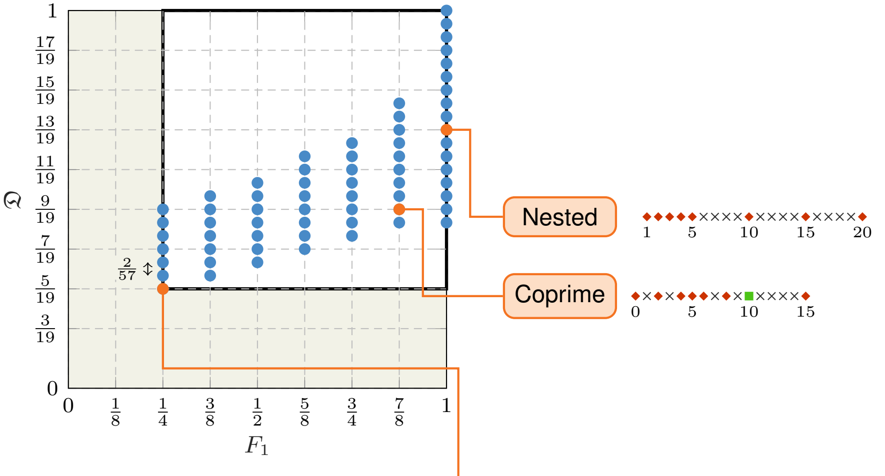


Maximally economic sparse arrays ($F_1 = 1$)

$$F_1 = \frac{\text{\# of Ess.}}{N_s}, \quad \mathfrak{D} = \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

◆ Essential
■ Inessential

The F_1-$\mathfrak{D}$ Plane: $N_s = 8$ Sensors, Aperture $A \leq 34$



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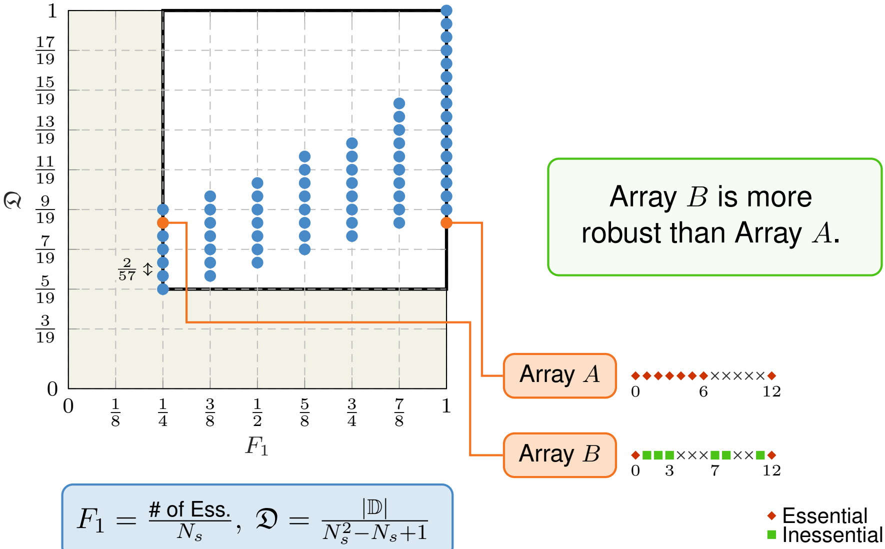
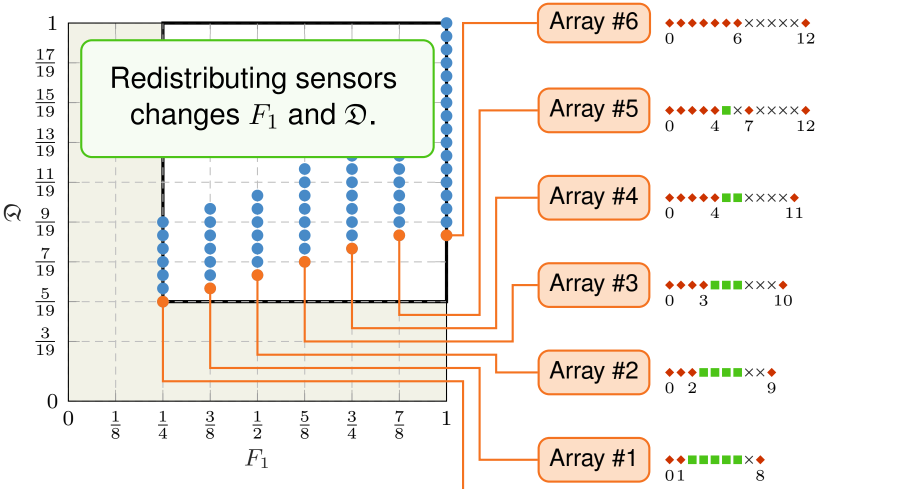


Array B is more robust than Array A.

$$F_1 = \frac{\text{\# of Ess.}}{N_s}, \quad \mathfrak{D} = \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

The F_1-$\mathfrak{D}$ Plane: $N_s = 8$ Sensors, Aperture $A \leq 34$



$$F_1 = \frac{\text{\# of Ess.}}{N_s}, \quad \mathfrak{D} = \frac{|\mathbb{D}|}{N_s^2 - N_s + 1}$$

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Concluding Remarks

- Comparison of sparse arrays
  - Robustness (F_1)
  - Size of the difference coarray ($\mathfrak{D}$)
  - The F_1-$\mathfrak{D}$ plane
- Future work
  - Array geometries with large and robust difference coarrays
  - Analysis of the achievable/unachievable regions on the F_1-$\mathfrak{D}$ plane

Thank you!