A General Framework for the Robustness of Structured Difference Coarrays to Element Failures

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Outline

1. Introduction

2. The Proposed Framework
   - The Importance Function
   - The Generalized \( k \)-Fragility

3. Numerical Examples

4. Concluding Remarks
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1 Introduction

2 The Proposed Framework
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3 Numerical Examples

4 Concluding Remarks
Physical Array and Difference Coarray

- Uniform linear array (ULA)\(^1\)
  - \(|\mathcal{D}| = O(|\mathcal{S}|)\)
  - Fewer sources than sensors

- Sparse array\(^2\)
  - Minimum redundancy array
  - Nested array
  - Coprime array
  - \(|\mathcal{D}| = O(|\mathcal{S}|^2)\)
  - More sources than sensors

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Robustness of Difference Coarrays to Element Failures

<table>
<thead>
<tr>
<th>Physical array</th>
<th>Difference coarray</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>$D_1$</td>
</tr>
<tr>
<td>0 2 4 5 6</td>
<td>−6 0 6</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_2$</td>
<td>$D_2$ ≠ $D_1$</td>
</tr>
<tr>
<td>0 2 4 6 ×</td>
<td>−6 −4 −2 0 2 4 6</td>
</tr>
<tr>
<td>Failed</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$S_3$</td>
<td>$D_3$ = $D_1$</td>
</tr>
<tr>
<td>0 2 5 6 ×</td>
<td>−6 0 6</td>
</tr>
<tr>
<td>Failed</td>
<td></td>
</tr>
</tbody>
</table>
Related Work

- Robustness of detection and estimation\(^1\)
- Robustness of the peak sidelobe level in the beampattern to sensor failures\(^2\)
- **Robustness of difference coarrays to sensor failures\(^3\-5\)**
  - The essentialness property and the \(k\)-essentialness property\(^3\)
  - The fragility and the \(k\)-fragility\(^3\)
  - The essentialness and the fragility for array configurations\(^4\)
  - Numerical algorithms\(^5\)

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The Importance Function: Definition

The Main Idea

The importance function $\mathcal{I}$ quantifies the importance of a subarray $A$ of $S$.

Definition

A function $\mathcal{I}$ is said to be an importance function with respect to a non-empty set $S$ if the four defining properties hold:

1. $0 \leq \mathcal{I}(A) \leq 1$ for all $A \subseteq S$. [Range of the importance]
2. $\mathcal{I}(\emptyset) = 0$, where $\emptyset$ is the empty set. [$\emptyset$ is the least important]
3. $\mathcal{I}(S) = 1$. [$S$ is the most important]
4. $\mathcal{I}$ is monotone. [If $A \subseteq B \subseteq S$, then $\mathcal{I}(A) \leq \mathcal{I}(B)$]
Examples of Importance Functions

\( \mathcal{I} \) related to the \( k \)-essentialness

\[ \mathcal{I}_{\text{ess}}(A) \triangleq \begin{cases} 1, & \text{if } A \text{ is } |A|\text{-essential}, \\ 0, & \text{otherwise}. \end{cases} \]

\( \mathcal{I} \) related to the size of \( U \)

(Proposed)

\[ \mathcal{I}_U(A) \triangleq 1 - \frac{|U|}{|A|}. \]

Failed

\( A = \{0\} \)

\( \mathcal{I}_{\text{ess}}(A) = 1, \quad \mathcal{I}_U(A) = 0. \)
Properties of Importance Functions $I_{\text{ess}}$ and $I_{U}$

Properties of the importance functions $I_{\text{ess}}$ and $I_{U}$

1. $I_{U}(A) = 1$ if and only if $A = S$.
2. $I_{U}(A) \leq I_{\text{ess}}(A)$.

Proofs can be found in the paper\(^1\).

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The Generalized $k$-Fragility

**Definition**

The generalized $k$-fragility $\mathcal{F}_k(S, \mathcal{I})$ is defined as

$$\mathcal{F}_k(S, \mathcal{I}) \triangleq \sum_{A \subseteq S, |A|=k} \frac{\mathcal{I}(A)}{\binom{|S|}{k}}, \quad \text{for } k = 0, 1, \ldots, |S|.$$ 

- $S$: The array geometry.
- $\mathcal{I}$: The importance function.
- $k$: The number of element failures.

**Remarks on $\mathcal{F}_k(S, \mathcal{I})$**

- A measure for the array robustness w.r.t. $\mathcal{I}$.
- $\mathcal{I}_{\text{ess}}$ or $\mathcal{I}_{\text{U}}$.
- The $k$-fragility (2) $F_k = \mathcal{F}_k(S, \mathcal{I}_{\text{ess}})$.

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The Proposed Framework

The Generalized $k$-Fragility

Properties of The Generalized $k$-Fragility

$$
F_k(S, \mathcal{I}) \triangleq \sum_{A \subseteq S, |A|=k} \frac{\mathcal{I}(A)}{\binom{|S|}{k}},
$$
for $k = 0, 1, \ldots, |S|$.

For $F_k(S, \mathcal{I})$

1. $0 \leq F_k(S, \mathcal{I}) \leq 1$. [Robust if $F_k(S, \mathcal{I}) \to 0$]
2. $F_0(S, \mathcal{I}) = 0$ and $F_{|S|}(S, \mathcal{I}) = 1$.
3. $F_k(S, \mathcal{I})$ is increasing in $k$.

For $F_k(S, \mathcal{I}_{\text{ess}})$ and $F_k(S, \mathcal{I}_{U})$

1. $F_k(S, \mathcal{I}_{U}) = 1$ if and only if $k = |S|$.
2. $F_k(S, \mathcal{I}_{U}) \leq F_k(S, \mathcal{I}_{\text{ess}})$ for $k = 0, 1, \ldots, |S|$.

Proofs can be found in the paper\(^1\).

Importance Function

Uniform Linear Array (10 sensors)

Sensor location $n$

Minimum Redundancy Array (10 sensors)

Sensor location $n$

Nested Array (10 sensors, $N_1 = N_2 = 5$)

Sensor location $n$

Coprime Array (10 sensors, $M = 3, N = 5$)

Sensor location $n$
Generalized $k$-Fragility

$\mathcal{F}_k(\mathbb{S}, \mathcal{I})$

Less robust

More robust

Fewer failures

More failures
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Concluding Remarks

- This presentation
  - The importance function
  - The generalized $k$-fragility
- Future work
  - Importance function based on performance metrics
  - Sparse array design based on the importance function

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