A General Framework for the Robustness of Structured Difference Coarrays to Element Failures

Chun-Lin Liu[1]

[1]Dept. of Electrical Engineering
[1]Graduate Institute of Communication Engineering
National Taiwan University, Taipei, Taiwan 10617
chunlinliu@ntu.edu.tw[1]

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Outline

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Physical Array and Difference Coarray

Physical array $\mathbb{S}$

Difference coarray

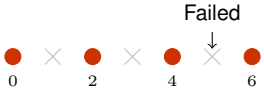
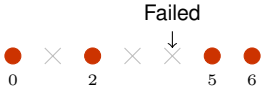
$$\mathbb{D} = \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$$

- Uniform linear array (ULA)[1]
  - $|\mathbb{D}| = \mathcal{O}(|\mathbb{S}|)$
  - Fewer sources than sensors
- Sparse array[2]
  - Minimum redundancy array
  - Nested array
  - Coprime array
  - $|\mathbb{D}| = \mathcal{O}(|\mathbb{S}|^2)$
  - More sources than sensors

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory* , 2002.

[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968; Pal and Vaidyanathan, *IEEE Trans. Signal Process.*, 2010; Vaidyanathan and Pal, *IEEE Trans. Signal Process.*, 2011.

Robustness of Difference Coarrays to Element Failures

Physical array	Difference coarray
$\mathbb{S}_1$ 	$\mathbb{D}_1$ 
$\mathbb{S}_2$ 	$\mathbb{D}_2$ $\neq \mathbb{D}_1$ 
$\mathbb{S}_3$ 	$\mathbb{D}_3$ $= \mathbb{D}_1$ 

Related Work

- Robustness of detection and estimation[1]
- Robustness of the peak sidelobe level in the beampattern to sensor failures[2]
- Robustness of difference coarrays to sensor failures[3-5]
  - The essentialness property and the k-essentialness property[3]
  - The fragility and the k-fragility[3]
  - The essentialness and the fragility for array configurations[4]
  - Numerical algorithms[5]

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[1] A. Alexiou and A. Manikas, "Investigation of array robustness to sensor failure," *Journal of the Franklin Institute*, vol. 342, no. 3, pp. 255 272, May 2005.

[2] M. Carlin, G. Oliveri, and A. Massa, "On the robustness to element failures of linear ADS-thinned arrays," *IEEE Trans. Antennas Propag.*, vol. 59, no. 12, pp. 48494853, Dec. 2011.

[3] C.-L. Liu and P. P. Vaidyanathan, "Robustness of Difference Coarrays of Sparse Arrays to Sensor Failures Part I: A Theory Motivated by Coarray MUSIC," *IEEE Trans. on Signal Processing*, vol. 67, no. 12, pp. 3213-3226, June 2019.

[4] C.-L. Liu and P. P. Vaidyanathan, "Robustness of Difference Coarrays of Sparse Arrays to Sensor Failures Part II: Array Geometries," *IEEE Trans. on Signal Processing*, vol. 67, no. 12, pp. 3227-3242, June 2019.

[5] C.-L. Liu and P. P. Vaidyanathan, "Novel Algorithms for Analyzing the Robustness of Difference Coarrays to Sensor Failures," *Signal Processing*, vol. 171, pp. 107517, 2020.

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The Importance Function: Definition

The Main Idea

The importance function $\mathscr{I}$ quantifies the importance of a subarray $\mathbb{A}$ of $\mathbb{S}$.

Definition

A function $\mathscr{I}$ is said to be an importance function with respect to a non-empty set $\mathbb{S}$ if the four defining properties hold

1. $0 \leq \mathscr{I}(\mathbb{A}) \leq 1$ for all $\mathbb{A} \subseteq \mathbb{S}$. [Range of the importance]
2. $\mathscr{I}(\varnothing) = 0$, where $\varnothing$ is the empty set. [$\varnothing$ is the least important]
3. $\mathscr{I}(\mathbb{S}) = 1$. [$\mathbb{S}$ is the most important]
4. $\mathscr{I}$ is monotone. [If $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{S}$, then $\mathscr{I}(\mathbb{A}) \leq \mathscr{I}(\mathbb{B})$]

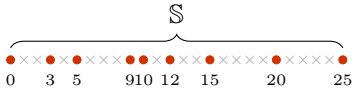
Examples of Importance Functions

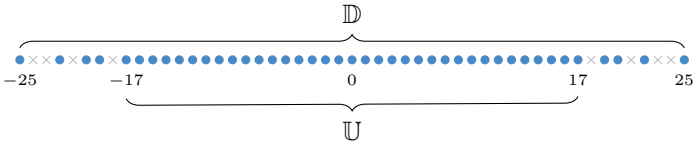
$\mathscr{I}$ related to the k-essentialness

$$\mathscr{I}_{\text{ess}}(\mathbb{A}) \triangleq \begin{cases} 1, & \text{if } \mathbb{A} \text{ is } |\mathbb{A}|\text{-essential,} \\ 0, & \text{otherwise.} \end{cases}$$

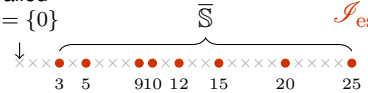
$\mathscr{I}$ related to the size of $\mathbb{U}$

(Proposed) $$\mathscr{I}_{\text{U}}(\mathbb{A}) \triangleq 1 - \frac{|\overline{\mathbb{U}}|}{|\mathbb{U}|}.$$

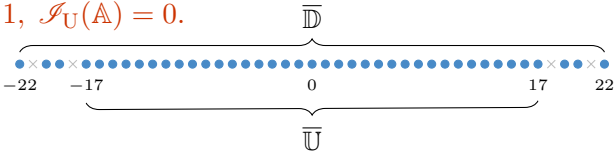




Failed
$\mathbb{A} = \{0\}$
↓



$$\mathscr{I}_{\text{ess}}(\mathbb{A}) = 1, \ \mathscr{I}_{\text{U}}(\mathbb{A}) = 0.$$



Properties of Importance Functions $\mathscr{I}_{\text{ess}}$ and $\mathscr{I}_{\text{U}}$

Properties of the importance functions $\mathscr{I}_{\text{ess}}$ and $\mathscr{I}_{\text{U}}$

1. $\mathscr{I}_{\text{U}}(\mathbb{A}) = 1$ if and only if $\mathbb{A} = \mathbb{S}$.
2. $\mathscr{I}_{\text{U}}(\mathbb{A}) \leq \mathscr{I}_{\text{ess}}(\mathbb{A})$.

Proofs can be found in the paper[1].

[1] C.-L. Liu, "A General Framework for the Robustness of Structured Difference Coarrays to Element Failures," *the Eleventh IEEE Sensor Array and Multichannel Signal Processing Workshop (SAM 2020)*.

Outline

The Generalized k-Fragility

Definition[1]

The generalized k-fragility $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ is defined as

$$\mathcal{F}_k(\mathbb{S}, \mathscr{I}) \triangleq \sum_{\mathbb{A}\subseteq\mathbb{S},\ |\mathbb{A}|=k} \frac{\mathscr{I}(\mathbb{A})}{\binom{|\mathbb{S}|}{k}}, \quad \text{for } k = 0, 1, \ldots, |\mathbb{S}|.$$

- $\mathbb{S}$: The array geometry.
- $\mathscr{I}$: The importance function.
- k: The number of element failures.

Remarks on $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$

- A measure for the array robustness w.r.t. $\mathscr{I}$.
- $\mathscr{I}_{\text{ess}}$ or $\mathscr{I}_{\text{U}}$.
- The k-fragility[2] (F_k).

$$F_k = \mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\text{ess}}).$$

[1] C.-L. Liu, "A General Framework for the Robustness of Structured Difference Coarrays to Element Failures," *the Eleventh IEEE Sensor Array and Multichannel Signal Processing Workshop (SAM 2020)*.

[2] C.-L. Liu and P. P. Vaidyanathan, "Robustness of Difference Coarrays of Sparse Arrays to Sensor Failures Part I: A Theory Motivated by Coarray MUSIC," *IEEE Trans. on Signal Processing*, vol. 67, no. 12, pp. 3213-3226, June 2019.

Properties of The Generalized k-Fragility

$$\mathcal{F}_k(\mathbb{S}, \mathscr{I}) \triangleq \sum_{\mathbb{A}\subseteq\mathbb{S},\ |\mathbb{A}|=k} \frac{\mathscr{I}(\mathbb{A})}{\binom{|\mathbb{S}|}{k}}, \qquad \text{for } k = 0, 1, \ldots, |\mathbb{S}|.$$

For $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$

1. $0 \leq \mathcal{F}_k(\mathbb{S}, \mathscr{I}) \leq 1$.
   [Robust if $\mathcal{F}_k(\mathbb{S}, \mathscr{I}) \to 0$]
2. $\mathcal{F}_0(\mathbb{S}, \mathscr{I}) = 0$ and $\mathcal{F}_{|\mathbb{S}|}(\mathbb{S}, \mathscr{I}) = 1$.
3. $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ is increasing in k.

For $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\text{ess}})$ and $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\text{U}})$

1. $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\text{U}}) = 1$ if and only if $k = |\mathbb{S}|$.
2. $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\text{U}}) \leq \mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\text{ess}})$ for $k = 0, 1, \ldots, |\mathbb{S}|$.

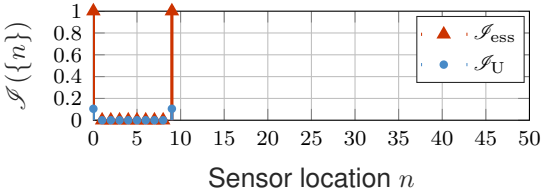
Proofs can be found in the paper[1].

[1] C.-L. Liu, "A General Framework for the Robustness of Structured Difference Coarrays to Element Failures," *the Eleventh IEEE Sensor Array and Multichannel Signal Processing Workshop (SAM 2020).*

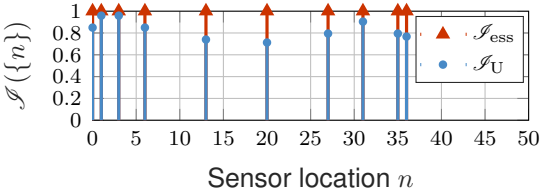
Outline

Importance Function

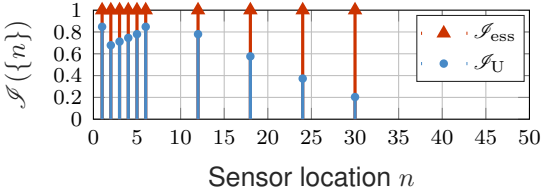
Uniform Linear Array (10 sensors)



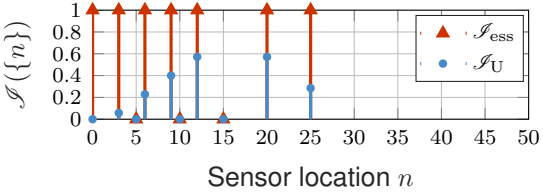
Minimum Redundancy Array (10 sensors)



Nested Array (10 sensors, $N_1 = N_2 = 5$)



Coprime Array (10 sensors, $M = 3, N = 5$)

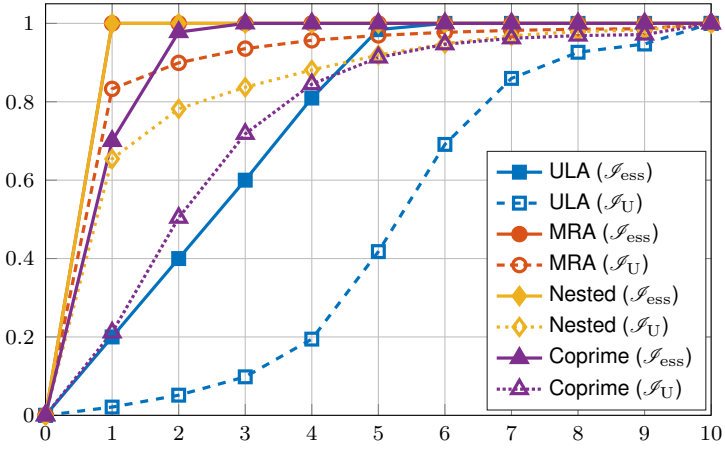


Generalized k-Fragility

Less robust

More robust

$\mathcal{F}_k(\mathbb{S}, \mathscr{I})$



Fewer failures

More failures

Outline

Concluding Remarks

- This presentation
  - The importance function
  - The generalized k-fragility
- Future work
  - Importance function based on performance metrics
  - Sparse array design based on the importance function

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