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A General Framework for the Robustness of Structured Difference Coarrays to Element Failures

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***Abstract*—Sparse arrays have received attention in array signal processing since they can resolve $\mathcal{O}(N^2)$ uncorrelated sources using N physical sensors. The reason is that the difference coarray, which consists of the differences between sensor locations, has a central uniform linear array (ULA) segment of size $\mathcal{O}(N^2)$. From the theory of the k-essentialness property and the k-fragility, the difference coarrays of some sparse arrays are not robust to sensor failures, possibly affecting the applicability of coarray-based direction-of-arrival (DOA) estimators. However, the k-essentialness property might not fully reflect the conditions under which these estimators fail. This paper proposes a framework for the robustness of array geometries based on the importance function and the generalized k-fragility. The importance function characterizes the importance of the subarrays in an array subject to some defining properties. The importance function is also compatible with the k-essentialness property and the size of the central ULA segment in the difference coarray. The latter is closely related to the performance of some coarray-based DOA estimators. Based on the importance function, the generalized k-fragility is proposed to quantify the robustness of an array. Properties of the importance function and the generalized k-fragility are also studied and demonstrated through numerical examples.**

***Index Terms*—Sparse arrays, difference coarrays, robustness, the importance function, the generalized k-fragility.**

I. Introduction

Sparse arrays have drawn attention in many fields of science and engineering [1]–[6], since they can resolve $\mathcal{O}(N^2)$ source directions using N physical sensors under mild assumptions [7]–[10]. Therefore, it is possible to identify *more source directions than sensors* using sparse arrays such as the minimum redundancy array (MRA) [7], the nested array [9], and the coprime array [10]. This property arises from the *difference coarray* of the sparse arrays having $\mathcal{O}(N^2)$ distinct elements, where the difference coarray is defined as the set of differences between the sensor locations. On the contrary, the ULA can only identify $\mathcal{O}(N)$ source directions with N sensors [6].

However, in practice, the robustness of array geometries to *sensor failures* plays an important role in the overall system performance [11]–[13]. It can be assessed from detection and estimation [11], the peak sidelobe level in the beampattern [12], and the difference coarray [13]. It is assumed that the sensors in the array deviate from their operational state, causing missing or erroneous data, and then the changes in the performance metrics are analyzed.

Among these, the robustness of difference coarrays to sensor failures is recently addressed by the k-essentialness property [13], which characterizes the patterns of k failed sensors that change the difference coarray. Based on this concept, the k-fragility, defined as the ratio of the number of k-essential subarrays to the number of all subarrays with size k, admits to compare the robustness of array geometries to sensor failures [13]. Therefore, the robustness of arrays is analyzed and compared [14] and robust array configurations are designed [15]. However, the k-essentialness property might not fully reflect the conditions under which the coarray-based DOA estimators fail. Even if the difference coarray changes, information on the impaired difference coarray might still be processed using *the central ULA segment in the difference coarray* [16], [17], positive-definite Toeplitz completion [18], or coarray interpolation [19], [20].

This paper aims to extend the theory of k-essentialness and the k-fragility to a broader family of array robustness. To begin with, the *importance function* of the subarrays of an array characterizes the importance of subarrays, from basic principles of array robustness. The importance function not only provides insights into the implementation cost of the array but also offers a more general framework for array robustness than the k-essentialness property. Based on these concepts, the *generalized k-fragility* is defined to measure the robustness of sensor arrays subject to the importance function. In particular, the generalized k-fragility for any importance function is an increasing function bounded between 0 and 1, which shares similar properties with the k-fragility in [13]. Furthermore, the proposed framework can be readily extended to the robustness of the central ULA segment of the difference coarray.

Note that the structural importance [11] was also proposed to evaluate the array robustness. The structural importance is defined as the *sensitivity* of the performance criterion with respect to the operational state of *each sensor*. In this paper, the importance function is defined for *multiple sensor failures with basic properties*, as elaborated in Section III.

The outline of this paper is as follows. Section II reviews the difference coarray, the k-essentialness property, and the k-fragility. Section III presents the importance function and its properties. Section IV studies the generalized k-fragility and its attributes. Section V demonstrates numerical examples

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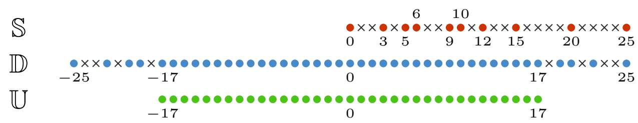

Fig. 1. An illustration of the array geometry $\mathbb{S}$, the difference coarray $\mathbb{D}$, and the central ULA segment $\mathbb{U}$, where the dots denote elements and crosses stand for empty space. The array geometry is the coprime array with $M = 3$ and $N = 5$.

while Section VI concludes this paper.

II. Preliminaries

Consider a linear array whose sensors are located at $n\lambda/2$. Here λ is the wavelength of the far-field, narrowband, and uncorrelated sources. The index n belongs to an integer-valued set $\mathbb{S}$. Next the difference coarray of $\mathbb{S}$ is defined as [7]–[10]:

Definition 1. The difference coarray $\mathbb{D}$ of an array $\mathbb{S}$ is defined as the set of differences between the sensor locations. Namely, $\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$.

Based on these assumptions, it is known [8]–[10] that the measurements on $\mathbb{S}$ can be transformed into the measurements on $\mathbb{D}$. This property has led to the development of coarray-based DOA estimators [8], [9], [16]–[18], [21]. Among these, DOA estimators based on the MUSIC algorithm on the difference coarray [8], [9], [16], [17] operate on the central ULA segment in the difference coarray, which is defined as follows:

Definition 2. The central ULA segment in the difference coarray is defined as $\mathbb{U} \triangleq \{m : \{0, \pm 1, \pm 2, \ldots, \pm|m|\} \subseteq \mathbb{D}\}$.

For example, Fig. 1 depicts the sets $\mathbb{S}$, $\mathbb{D}$, and $\mathbb{U}$ for the coprime array [10] with parameters $M = 3$ and $N = 5$. The coprime array has $\mathbb{S} = \{0, M, 2M, \ldots, (N-1)M, N, 2N, \ldots, (2M-1)N\}$, as shown in red dots. The difference coarray $\mathbb{D}$ is illustrated in blue dots while the central ULA segment $\mathbb{U}$ is also depicted. It can be observed that the sizes of these sets are $|\mathbb{S}| = 10$, $|\mathbb{D}| = 43$, and $|\mathbb{U}| = 35$. The notation $|\cdot|$ represents the cardinality of a set.

In the literature, the sensor locations can be designed properly such that the set $\mathbb{U}$ possesses size up to $\mathcal{O}(|\mathbb{S}|^2)$. This attribute admits to resolve *more sources than sensors* using coarray-based DOA estimators on $\mathbb{U}$ [8], [9], [16], [17]. Note that sparse arrays such as the MRA [7], the nested array [9], and the coprime array [10], satisfy the property that $|\mathbb{U}| = \mathcal{O}(|\mathbb{S}|^2)$.

Next the k-essentialness property is reviewed for the robustness of sparse arrays [13], [14]:

Definition 3. Let $\mathbb{A}$ be a subset of $\mathbb{S}$ with cardinality k. Then $\mathbb{A}$ is said to be *k-essential* with respect to $\mathbb{S}$ if the removal of $\mathbb{A}$ from $\mathbb{S}$ changes the difference coarray. That is, $\mathbb{A}$ is k-essential if $\mathbb{D} \neq \overline{\mathbb{D}}$, where $\mathbb{D}$ and $\overline{\mathbb{D}}$ are the difference coarrays of $\mathbb{S}$ and $\overline{\mathbb{S}} \triangleq \mathbb{S} \backslash \mathbb{A}$, respectively.

Based on Definition 3, the *k-fragility* of $\mathbb{S}$ is defined as [13]

$$F_k(\mathbb{S}) \triangleq \frac{|\{\mathbb{A} : \mathbb{A} \text{ is } k\text{-essential with respect to } \mathbb{S}\}|}{\binom{|\mathbb{S}|}{k}}, \quad (1)$$

for $k = 1, 2, \ldots, |\mathbb{S}|$. It was shown in [13] that $0 \le F_k(\mathbb{S}) \le 1$ and $F_k(\mathbb{S})$ is an increasing function in k. The larger the k-fragility is, the more likely the difference coarray is to change under k sensor failures. The array is said to be more robust if $F_k(\mathbb{S})$ is close to 0. Interested readers are referred to [13], [14] for the more details of these attributes.

The k-essentialness property and the k-fragility correspond to changes in the difference coarray. However, changes in the difference coarray are necessary, but not sufficient for the inapplicability of some coarray-based DOA estimators. For instance, changes in $\mathbb{D}$ are necessary but not sufficient for changes in $\mathbb{U}$, and the coarray-based MUSIC algorithms [16], [17] are based on the autocorrelation estimates on $\mathbb{U}$. Furthermore, techniques such as coarray-interpolation [19], [20] and sparsity-based methods [22]–[25] can be utilized to estimate the DOAs when the difference coarray changes.

III. The importance Function

In this section, the *importance function* will be presented for the framework of the robustness of sparse arrays. The importance function not only generalizes the k-essentialness property but also leads to new insights into the robustness of sparse arrays with respect to the central ULA segment $\mathbb{U}$.

The definition of the importance function is as follows.

Definition 4. A function $\mathscr{I}$ is said to be an importance function with respect to a non-empty set $\mathbb{S}$ if the defining properties hold

1) $0 \le \mathscr{I}(\mathbb{A}) \le 1$ for all $\mathbb{A} \subseteq \mathbb{S}$.
2) $\mathscr{I}(\varnothing) = 0$, where $\varnothing$ is the empty set.
3) $\mathscr{I}(\mathbb{S}) = 1$.
4) $\mathscr{I}$ is monotone. That is, if $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{S}$, then $\mathscr{I}(\mathbb{A}) \le \mathscr{I}(\mathbb{B})$.

The importance of a given subset of $\mathbb{S}$ satisfies the properties in Definition 4. We say that a subset $\mathbb{A}$ is the least important for $\mathbb{S}$ if the importance function is 0, while $\mathbb{A}$ is the most important for $\mathbb{S}$ if $\mathscr{I}(\mathbb{A}) = 1$. Therefore, the empty set $\varnothing$ is the least important and $\mathbb{S}$ is the most important, (Properties 2 and 3 in Definition 4). Furthermore, Property 4 in Definition 4 indicates that the importance function increases as extra sensors are added to the subarray.

The importance function is related to the implementation cost of an array. The subarrays with larger importance functions are implemented with higher cost, since the removal of these subarrays might cause significant performance degradation. Based on this concept, array can be designed to strike a balance between robustness and the implementation cost.

Depending on the criteria for the importance, *the importance function can be defined in different forms*, as long as the properties in Definition 4 hold true. These forms may or may not be related to the difference coarray. As an example, the ratio $|\mathbb{A}|/|\mathbb{S}|$ is an importance function, since the properties in Definition 4 can be shown to be true for $\mathscr{I}(\mathbb{A}) = |\mathbb{A}|/|\mathbb{S}|$.

However, in this paper, the importance function related to the difference coarray is of primary interest, due to its significance in DOA estimation. We will focus on 1) the k-essentialness property and 2) the central ULA segment $\mathbb{U}$.

To begin with, the importance function corresponding to the k-essentialness property in Definition 3 can be defined as

$$\mathscr{I}_{\text{ess}}(\mathbb{A}) \triangleq \begin{cases} 1, & \text{if } \mathbb{A} \text{ is } |\mathbb{A}|\text{-essential}, \\ 0, & \text{otherwise}, \end{cases} \quad (2)$$

where the set $\mathbb{A} \subseteq \mathbb{S}$.

The function $\mathscr{I}_{\text{ess}}$ is an importance function due to the following. The first three properties in Definition 4 are trivially true for $\mathscr{I}_{\text{ess}}(\mathbb{A})$ in (2). Property 4 in Definition 4 for $\mathscr{I}_{\text{ess}}(\mathbb{A})$ is also satisfied due to [13, Lemma 3].

As reviewed in Section II, coarray-based DOA estimators might still work when $\mathbb{D}$ changes. It is alternative to consider the central ULA segment $\mathbb{U}$ as the performance metric. As the size of $\mathbb{U}$ decreases, it was observed that the number of resolvable sources drops [26]–[28] and the DOA estimation performance tends to degrade [20]. Based on these, the function $\mathscr{I}_{\text{U}}$ is proposed to incorporate the size of the central ULA segment $\mathbb{U}$ with the importance:

Definition 5. Assume that the array is denoted by the set $\mathbb{S}$. Let the set $\mathbb{U}$ be defined in Definition 2. The importance function associated with the size of $\mathbb{U}$ is defined as

$$\mathscr{I}_{\text{U}}(\mathbb{A}) \triangleq 1 - \frac{|\overline{\mathbb{U}}|}{|\mathbb{U}|}, \tag{3}$$

where $\mathbb{A}$ is a subset of $\mathbb{S}$. The set $\overline{\mathbb{U}}$ denotes the central ULA segment in the difference coarray of $\overline{\mathbb{S}} = \mathbb{S} \backslash \mathbb{A}$.

It will be shown next that the function $\mathscr{I}_{\text{U}}(\mathbb{A})$ is an importance function. First, by definition, the set $\overline{\mathbb{U}}$ satisfies $0 \leq |\overline{\mathbb{U}}| \leq |\mathbb{U}|$, implying that $0 \leq \mathscr{I}_{\text{U}}(\mathbb{A}) \leq 1$. Second, if $\mathbb{A} = \varnothing$, then we have $\overline{\mathbb{S}} = \mathbb{S}$, so the difference coarray and its central ULA segment remain unchanged. Therefore $\mathscr{I}_{\text{U}}(\varnothing) = 0$. On the other hand, if $\mathbb{A} = \mathbb{S}$, then we obtain $\overline{\mathbb{S}} = \varnothing$, $\overline{\mathbb{D}} = \varnothing$, and $\overline{\mathbb{U}} = \varnothing$, so that $\mathscr{I}_{\text{U}}(\mathbb{S}) = 1$. Property 4 in Definition 4 can be verified through the following chain of arguments. Suppose that $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{S}$. We define the sets $\mathbb{S}_1 \triangleq \mathbb{S} \backslash \mathbb{A}$ and $\mathbb{S}_2 \triangleq \mathbb{S} \backslash \mathbb{B}$. Due to [13, Proposition 1], we have $\mathbb{D}_2 \subseteq \mathbb{D}_1$, where $\mathbb{D}_1$ and $\mathbb{D}_2$ are the difference coarrays of $\mathbb{S}_1$ and $\mathbb{S}_2$, respectively. As a result, the central ULA segments in the difference coarrays of $\mathbb{S}_1$ and $\mathbb{S}_2$, denoted by $\mathbb{U}_1$ and $\mathbb{U}_2$, satisfy $\mathbb{U}_2 \subseteq \mathbb{U}_1$. Therefore, we have $\mathscr{I}_{\text{U}}(\mathbb{A}) \leq \mathscr{I}_{\text{U}}(\mathbb{B})$.

Apart from the properties in Definition 4, the importance function $\mathscr{I}_{\text{U}}$ has these additional properties:

Proposition 1. The importance function $\mathscr{I}_{\text{U}}$ in (3) satisfies the following attributes.

1) $\mathscr{I}_{\text{U}}(\mathbb{A}) = 1$ if and only if $\mathbb{A} = \mathbb{S}$.
2) $\mathscr{I}_{\text{U}}(\mathbb{A}) \leq \mathscr{I}_{\text{ess}}(\mathbb{A})$.

Proof: The first property is due to the following. According to (3), $\mathscr{I}_{\text{U}}(\mathbb{A}) = 1$ if and only if $\overline{\mathbb{U}} = \varnothing$. This condition is equivalent to the condition that $\overline{\mathbb{S}} \triangleq \mathbb{S} \backslash \mathbb{A} = \varnothing$. As a result, $\mathscr{I}_{\text{U}}(\mathbb{A}) = 1$ if and only if $\mathbb{A} = \mathbb{S}$.

The second property can be proved as follows. Based on (2), the importance function $\mathscr{I}_{\text{ess}}(\mathbb{A})$ is either 1 or 0. If $\mathscr{I}_{\text{ess}}(\mathbb{A}) = 1$, then the inequality holds according to Property 1 in Definition 4. On the other hand, if $\mathscr{I}_{\text{ess}}(\mathbb{A}) = 0$, then we have $\overline{\mathbb{D}} = \mathbb{D}$, owing to Definition 3. Therefore, $\overline{\mathbb{U}} = \mathbb{U}$ and $\mathscr{I}_{\text{U}}(\mathbb{A}) = 0$. ∎

The importance functions $\mathscr{I}_{\text{ess}}$ and $\mathscr{I}_{\text{U}}$ correspond to different performance metrics. The former focuses on changes in $\mathbb{D}$ while the latter considers the size of $\mathbb{U}$. Furthermore, $\mathscr{I}_{\text{ess}}$ is binary-valued but $\mathscr{I}_{\text{U}}$ may have more than two values.

IV. The Generalized k-Fragility

The importance function not only provides insights into the importance of subarrays of a given array configuration, but also makes it possible to quantify the array robustness. In what follows, the *generalized k-fragility* is defined:

Definition 6. The generalized k-fragility $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ related to the array $\mathbb{S}$ and the importance function $\mathscr{I}$ is defined as

$$\mathcal{F}_k(\mathbb{S}, \mathscr{I}) \triangleq \sum_{\mathbb{A} \subseteq \mathbb{S},\ |\mathbb{A}| = k} \frac{\mathscr{I}(\mathbb{A})}{\binom{|\mathbb{S}|}{k}}, \tag{4}$$

for $k = 0, 1, \ldots, |\mathbb{S}|$.

The generalized k-fragility can be viewed as the extensions of the k-fragility associated with the importance function $\mathscr{I}$. In particular, the generalized k-fragility $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\text{ess}})$ is equivalent to the k-fragility $F_k(\mathbb{S})$ in (1), for $k = 1, 2, \ldots |\mathbb{S}|$.

Next some properties about $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ are presented.

Lemma 1. For the importance function $\mathscr{I}$ in Definition 4, the generalized k-fragility $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ satisfies these properties:

1) $0 \leq \mathcal{F}_k(\mathbb{S}, \mathscr{I}) \leq 1$ for $k = 0, 1, \ldots, |\mathbb{S}|$.
2) $\mathcal{F}_0(\mathbb{S}, \mathscr{I}) = 0$ and $\mathcal{F}_{|\mathbb{S}|}(\mathbb{S}, \mathscr{I}) = 1$.
3) $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ is an increasing function in k.

The generalized k-fragility can be interpreted as a measure for the array robustness, subject to the performance metric defined in the importance function $\mathscr{I}$. An array is said to be more robust (or less fragile) if $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ is close to 0, and less robust (or more fragile) if $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ is close to 1. Furthermore, $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ is an increasing function in k, showing that as more elements are removed, the array becomes less robust. These interpretations are consistent with the k-fragility in (1).

Proof of Lemma 1: The first property is a direct consequence of Property 1 in Definition 4 while the second property follows from Properties 2 and 3 in Definition 4.

The third property can be proved as follows. It is assumed that the set $\mathbb{A} \subseteq \mathbb{S}$ and $|\mathbb{A}| = k$. Let the element n be in the set $\mathbb{S}$, but not in $\mathbb{A}$. Therefore, $\mathbb{A} \subseteq \mathbb{A} \cup \{n\}$, and according to Property 4 in Definition 4, the importance function $\mathscr{I}$ satisfies

$$\mathscr{I}(\mathbb{A}) \leq \mathscr{I}(\mathbb{A} \cup \{n\}). \tag{5}$$

Summing up all possible $\mathbb{A}$ and n in (5) leads to

$$\sum_{\mathbb{A} \subseteq \mathbb{S},\, |\mathbb{A}| = k,\, n \in \mathbb{S} \backslash \mathbb{A}} \mathscr{I}(\mathbb{A}) \leq \sum_{\mathbb{A} \subseteq \mathbb{S},\, |\mathbb{A}| = k,\, n \in \mathbb{S} \backslash \mathbb{A}} \mathscr{I}(\mathbb{A} \cup \{n\}). \tag{6}$$

Next the duplicated terms in the summations of (6) are analyzed. The left-hand side of (6) can be simplified as $(|\mathbb{S}| - k) \sum_{\mathbb{A} \subseteq \mathbb{S}, |\mathbb{A}| = k} \mathscr{I}(\mathbb{A})$. The right-hand side of (6), on the other hand, can be expressed in terms of another set $\mathbb{B} \triangleq \mathbb{A} \cup \{n\}$. Each $\mathbb{B}$ can be constructed from $\mathbb{A} \cup \{n\}$ in $k+1$ ways. Therefore, the right-hand side of (6) can be rewritten as $(k+1) \sum_{\mathbb{B} \subseteq \mathbb{S}, |\mathbb{B}| = k+1} \mathscr{I}(\mathbb{B})$. Based on these expressions and Definition 6, we obtain

$$(|\mathbb{S}| - k) \binom{|\mathbb{S}|}{k} \mathcal{F}_k(\mathbb{S}, \mathscr{I}) \leq (k+1) \binom{|\mathbb{S}|}{k+1} \mathcal{F}_{k+1}(\mathbb{S}, \mathscr{I}),$$

which simplifies to $\mathcal{F}_k(\mathbb{S}, \mathscr{I}) \leq \mathcal{F}_{k+1}(\mathbb{S}, \mathscr{I})$. ∎

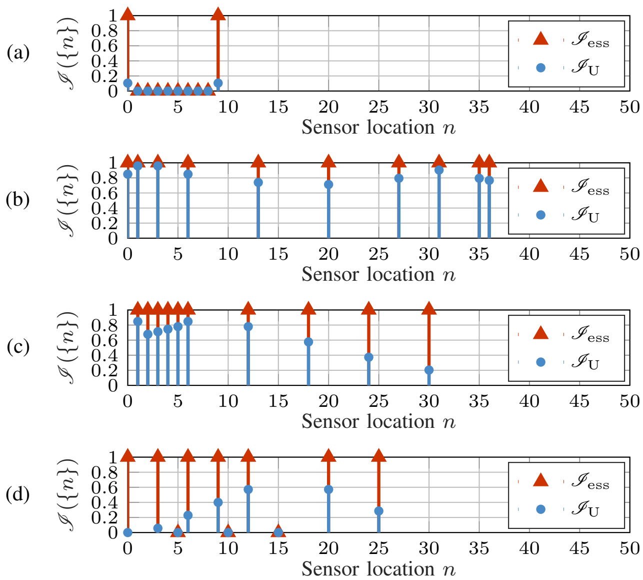

Fig. 2. The importance functions for (a) the ULA with 10 sensors, (b) the MRA with 10 sensors, (c) the nested array with $N_1 = N_2 = 5$, and (d) the coprime array with $M = 3, N = 5$.

Next the properties of the generalized k-fragility associated with the importance function $\mathscr{I}_{\mathrm{U}}$ are presented, whose the proof is a direct consequence of Proposition 1 and Lemma 1:

Proposition 2. Let $\mathscr{I}_{\mathrm{U}}$ be defined in Definition 5. The generalized k-fragility associated with $\mathscr{I}_{\mathrm{U}}$ has these properties

1) $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{U}}) = 1$ if and only if $k = |\mathbb{S}|$.
2) $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{U}}) \leq \mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{ess}})$ for $k = 0, 1, \ldots, |\mathbb{S}|$, where $\mathscr{I}_{\mathrm{ess}}$ is defined in (2).

Summarizing, the generalized k-fragility associated with the importance function $\mathscr{I}$ quantifies the robustness of arrays in the range $0 \leq \mathcal{F}_k(\mathbb{S}, \mathscr{I}) \leq 1$. For any importance function $\mathscr{I}$, Lemma 1 is satisfied. Additional properties for $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{ess}})$ and $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{U}})$ are can be found in Proposition 2.

V. Numerical Examples

In this section, the importance function and the generalized k-fragility will be demonstrated through numerical examples. We consider the following arrays. The ULA [6] with 10 sensors has $\mathbb{S} = \{0, 1, \ldots, 9\}$. The MRA [7] with 10 sensors owns $\mathbb{S} = \{0, 1, 3, 6, 13, 20, 27, 31, 35, 36\}$. The nested array [9] with $N_1 = 5$ and $N_2 = 5$ possesses $\mathbb{S} = \{1, 2, 3, 4, 5, 6, 12, 18, 24, 30\}$. The coprime array [10] with $M = 3$ and $N = 5$ has $\mathbb{S} = \{0, 3, 5, 6, 9, 10, 12, 15, 20, 25\}$. All these arrays have 10 sensors.

The importance functions for (a) the ULA, (b) the MRA, (c) the nested array, and (d) the coprime array, are depicted in Fig. 2, where the sensor locations are denoted by $n \in \mathbb{S}$. The importance function $\mathscr{I}_{\mathrm{ess}}(\{n\})$ is marked with red triangles while the importance function $\mathscr{I}_{\mathrm{U}}(\{n\})$ is shown in blue dots.

The properties in Definition 4 and Proposition 1 are consistent with the results in Fig. 2. In these examples, the importance functions all satisfy $0 \leq \mathscr{I}_{\mathrm{U}}(\{n\}) \leq \mathscr{I}_{\mathrm{ess}}(\{n\}) \leq 1$ for all $n \in \mathbb{S}$. Furthermore, $\mathscr{I}_{\mathrm{ess}}(\{n\})$ is either 0 or 1.

It is also confirmed that the change in $\mathbb{D}$ is not sufficient for the change in $\mathbb{U}$. For instance, it is observed in Fig. 2(d) that

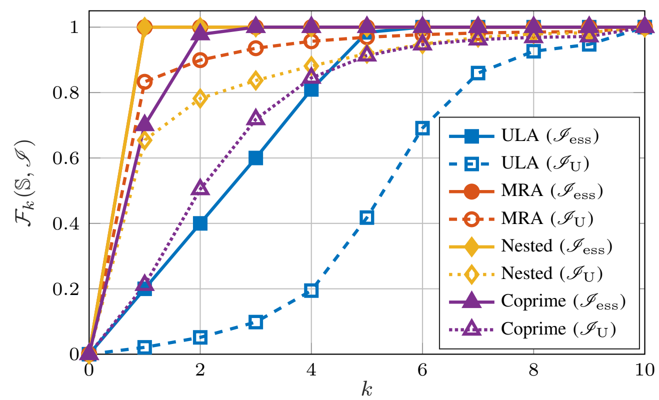

Fig. 3. The generalized k-fragility $\mathcal{F}_k(\mathbb{S}, \mathscr{I})$ for the ULA, the MRA, the nested array, and the coprime array. The array geometries are identical to those in Fig. 2.

$\mathscr{I}_{\mathrm{ess}}(\{0\}) = 1$ and $\mathscr{I}_{\mathrm{U}}(\{0\}) = 0$. Therefore, if the element 0 is deleted from the coprime array, then the difference coarray changes ($\mathscr{I}_{\mathrm{ess}}(\{0\}) = 1$) but the central ULA segment in the difference coarray remains the same, since $\mathscr{I}_{\mathrm{U}}(\{0\}) = 0$ implies $\overline{\mathbb{U}} = \mathbb{U}$ in Definition 5.

Fig. 3 depicts the generalized k-fragility $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{ess}})$ and $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{U}})$ for the ULA, the MRA, the nested array, and the coprime array. These curves are consistent with the properties in Lemma 1 and Proposition 2. Second, based on $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{ess}})$, the ULA is the most robust array, followed by the coprime array, and finally the nested array and the MRA. In particular, the MRA and the nested array share the same level of robustness, in the sense of the k-essentialness property and $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{ess}})$. However, if the size of $\mathbb{U}$ is considered for the importance function, the nested array is more robust than the MRA, due to the results of $\mathcal{F}_k(\mathbb{S}, \mathscr{I}_{\mathrm{U}})$ in Fig. 3. Note that this relation is in accordance with the numerical results for the estimation performance of coarray MUSIC in [14].

VI. Concluding Remarks

This paper proposed a generalized framework for analyzing the robustness of arrays to sensor failures, based on the importance function and the generalized k-fragility. The importance function is compatible with the k-essentialness property and the size of the central ULA segment of the difference coarray. The latter was known to be closely related to the performance of some coarray-based DOA estimators. Based on these, the generalized k-fragility was presented to quantify the robustness among arrays. Numerical examples demonstrated the properties of the importance function and the generalized k-fragility.

The proposed framework is applicable to any importance functions. Therefore, it is of interest to investigate other realistic criteria for the robustness of both array geometries and DOA estimators that can be utilized in the importance function and the generalized k-fragility.

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