Robustness of Difference Coarrays of Sparse Arrays to Sensor Failures – Part I: A Theory Motivated by Coarray MUSIC

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***Abstract*—In array processing, sparse arrays are capable of resolving $\mathcal{O}(N^2)$ uncorrelated sources with N sensors. Sparse arrays have this property because they possess uniform linear array (ULA) segments of size $\mathcal{O}(N^2)$ in the difference coarray, defined as the differences between sensor locations. However, the coarray structure of sparse arrays is susceptible to sensor failures, and the reliability of sparse arrays remains a significant but challenging topic for investigation. Broadly speaking, ULAs whose difference coarrays only have $\mathcal{O}(N)$ elements, are more robust than sparse arrays with $\mathcal{O}(N^2)$ coarray sizes. This paper advances a theory for quantifying such robustness by introducing the k-essentialness of sensors and the k-essential family of arrays. The proposed theory is motivated by the coarray MUSIC algorithm, which estimates source directions based on difference coarrays. Furthermore, the concept of essentialness not only characterizes the patterns of k faulty sensors that shrink the difference coarray, but also leads to the notion of k-fragility, which assesses the robustness of array geometries quantitatively. However, the large size of the k-essential family usually complicates the theory. It will be shown that the k-essential family can be compactly represented by the so-called k-essential Sperner family. Finally the proposed theory is used to provide insights into the probability of change of the difference coarray, as a function of the sensor failure probability and array geometry. In a companion paper, the k-essential Sperner family for several commonly used array geometries will be derived in closed-form, resulting in a quantitative comparison of the robustness of these arrays.**

***Index Terms*—Sparse arrays, difference coarrays, the k-essentialness property, the k-fragility, the k-essential Sperner family.**

I. Introduction

Sparse arrays, which have nonuniform sensor spacing, have recently attracted considerable attention in array signal processing [1]–[5]. Unlike uniform linear arrays (ULA), which resolve at most $N-1$ uncorrelated sources with N sensors, some sparse arrays are capable of identifying $\mathcal{O}(N^2)$ uncorrelated sources using N physical sensors. These arrays include minimum redundancy arrays (MRA) [2], nested arrays [4], coprime arrays [5], and their generalizations [6]. This $\mathcal{O}(N^2)$ property is because the difference coarray, defined as the set of differences between the sensor locations, possesses an $\mathcal{O}(N^2)$-long central ULA segment. By analyzing the samples on the difference coarray, quite a few direction-of-arrival (DOA) estimators have been shown to resolve more uncorrelated sources than sensors [4], [5], [7]–[12].

In practice, *sensor failure* could occur randomly and may lead to the breakdown of the overall system [13], [14]. It can be *empirically* observed that, for some sparse arrays, such as MRA, faulty sensors could shrink the size of the $\mathcal{O}(N^2)$-long ULA segment in the difference coarray significantly. Furthermore, small ULA segments in the difference coarray typically lead to degraded performance [4], [7], [8], [15]. Due to these observations, in the past, sparse arrays were considered not to be robust to sensor failure. However, the impact of damaged sensors on sparse arrays remains to be analyzed, since these observations assume specific array configurations.

The issue of sensor failure was addressed in the literature in two respects, including 1) developing new algorithms that are functional in the presence of sensor failure and 2) analyzing the robustness of array geometries. In the first case, various approaches have been developed, including DOA estimators based on minimal resource allocation network [16], impaired array beamforming and DOA estimation [17], array diagnosis based on Bayesian compressive sensing [18], and so on [19], [20]. However, the interplay between the array configuration and the exact condition under which these algorithms are applicable, remains to be investigated. The second aspect assesses the robustness of array configurations with faulty sensors [21], [22]. For instance, Alexiou and Manikas [21] proposed various measures to quantify the robustness of arrays while Carlin *et al.* [22] performed a statistical study on the beampattern with a given sensor failure probability. Even so, the impact of damaged sensors on the difference coarray has not yet been analyzed in a deterministic fashion, which is crucial for sparse arrays.

In this paper, we aim to investigate the influence of faulty sensors on the difference coarray. The main focus of this paper is not to develop new algorithms, but to analyze the robustness of arrays. Note that the proposed theory is *motivated by the coarray MUSIC algorithm*, which relies on the data on the difference coarray to estimate the DOAs. Therefore, changes in the difference coarray may hinder the applicability of coarray MUSIC [4], [7], [8].

A sensor is said to be *essential* if its deletion changes the difference coarray. Note that the essentialness property, which was originally introduced to study the economy of sensors [23], depends purely on the array geometry, rather than the source parameters and the estimation algorithms. One of the main contributions of this paper is to show that the

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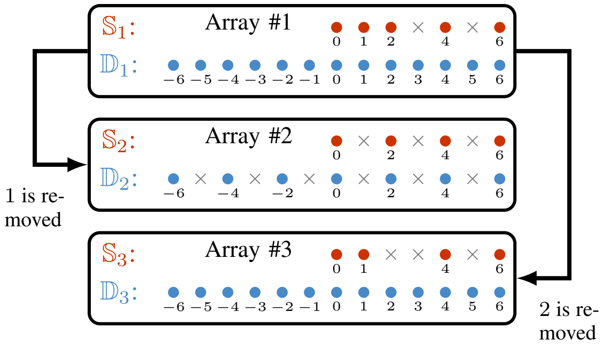

Fig. 1. An illustration of the essentialness property. $\mathbb{S}_i$ and $\mathbb{D}_i$ represent the physical array and the difference coarray of the ith array, respectively. Elements are marked by dots while empty space is depicted by crosses. It can be observed that removing the sensor at 1 from Array #1 changes the difference coarray ($\mathbb{D}_2 \neq \mathbb{D}_1$). However, in Array #3, which is obtained by removing the sensor at 2 from Array #1, the difference coarray remains the same ($\mathbb{D}_3 = \mathbb{D}_1$). We say that the sensor at 1 is essential with respect to Array #1 while the sensor at 2 is inessential with respect to Array #1.

essentialness property can be used to assess the robustness of the array geometry, in the sense of preserving the difference coarray. A generalization of this, called k-essentialness, is then developed in order to study the effect of multiple sensor failures on the difference coarray. The coarray robustness is quantified using the notion of k-fragility which is introduced later in the paper. This quantity ranges from 0 to 1; an array is more robust if the fragility is closer to 0.

For an array with N sensors, the size of the k-essential family can be as large as $\binom{N}{k}$, which makes it challenging to analyze and to store the complete information. To address this issue, we introduced the k-essential Sperner family, which encodes the information in the k-essential family with great economy.

These proposed quantities find applications in quantifying the susceptibility of the difference coarray with respect to random sensor failures. Note that this topic is of considerable interest in reliability engineering [13], [14]. Our study offers several insights into the interplay between the overall reliability, the essentialness property, and the fragility. For instance, under mild assumptions, the system reliability decreases as the number of essential sensors increases.

As an example, Fig. 1 demonstrates the main idea of the essentialness property. Let us consider Array #1 and its difference coarray, as depicted on the top of Fig. 1. The sensor at 1 is essential since its removal from Array #1 alters the difference coarray. However, the sensor at 2 is inessential, since $\mathbb{D}_3 = \mathbb{D}_1$. This example shows that according to the sensor locations, some sensors are more important than others, as far as preserving the difference coarray is concerned. The essentialness property and its connection to the robustness of the array geometry will be developed in depth later.

Paper outline: Section II reviews the theory of sparse arrays. Section III proposes the k-essential family while Section IV introduces the k-fragility. Section V presents the k-essential Sperner family. Section VI offers a number of insights into the system reliability for the difference coarray while Section VII concludes this paper. Parts of the results were presented in a conference paper [24], including (a) the definitions of the k-essentialness property and the k-fragility, (b) Theorems 1 and 2, and (c) sketches of the proofs of these theorems.

A. Remarks on Redundancy and the Proposed Theory

The proposed theory shares concepts similar to *redundancy* discussed in [2], [25]. For instance, it was stated in [25] that "the redundancy of an array may be described as the degree to which it contains elements that can be eliminated without changing its coarray." This statement is closely related to the definition of the essentialness property (Definition 3). Even so, in this paper, this concept is developed with a different approach, as we will elaborate in the following items.

1) The redundancy is defined as $R = \binom{N}{2}/\max(\mathbb{U})$, where N is the number of sensors and $\max(\mathbb{U})$ is the maximum element in the central ULA segment of the difference coarray [2], [25]. According to this definition, the redundancy R is a scalar attribute of an array. On the other hand, in our proposed theory, *the essentialness property is a binary attribute of each individual sensor* while *the maximal economy and the k-fragility are attributes of the entire array*.
2) The redundancy R explicitly takes the central ULA segment $\mathbb{U}$ of the difference coarray into consideration. The set $\mathbb{U}$ is usually important for MUSIC-like algorithms on the difference coarray [4], [7]–[9], [26]. Even though the proposed theory is motivated by the coarray MUSIC algorithm, the central ULA segment in the difference coarray is *not involved* in the essentialness property and other derived attributes.
3) The essentialness property can be interpreted as the importance of each physical sensor in an array. Based on this interpretation, we can use the essentialness property to determine the implementation cost of the physical sensing devices. On the contrary, the redundancy [2], [25] does not reveal the importance of each sensor.
4) The redundancy R and the k-fragility F_k are both quantities characterizing the arrays. By definition, $R \geq 1$ and $0 \leq F_k \leq 1$ for any array configurations. ***The redundancy R can be interpreted as how much the central ULA segment of the difference coarray is away from the largest number of positive lags $\binom{N}{2}$.*** In the proposed theory, ***the k-fragility can be viewed as the tendency that the difference coarray changes under sensor failures, where the sizes and the structures of the difference coarray are not of primary interest.*** Therefore, the redundancy and the k-fragility are different concepts.
5) In the literature, the redundancy is mainly used for designing arrays with minimum redundancy [2]. In this paper, the essentialness property and the proposed theory aim to *quantify the robustness of an arbitrary array configuration*. Based on this, it is possible to design novel arrays that are as robust as ULAs and have size $\mathcal{O}(N^2)$ in the difference coarray [27], [28].
6) Minimum redundancy implies maximal economy, but the converse is not necessarily true [29, Section III]. For

instance, the nested array is maximally economic but its redundancy is not minimized.

II. Review of Sparse Arrays

Assume that D monochromatic and far-field sources with wavelength λ impinge on a one-dimensional sensor array, where the sensor locations are $n\lambda/2$. Here n belongs to an integer set $\mathbb{S}$. Let $\theta_i \in [-\pi/2, \pi/2]$ and $A_i \in \mathbb{C}$ be the DOA and the complex amplitude of the ith source, respectively. The array output of the linear sensor array $\mathbb{S}$, denoted by $\mathbf{x}_{\mathbb{S}}$, is modeled as

$$\mathbf{x}_{\mathbb{S}} = \sum_{i=1}^{D} A_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) + \mathbf{n}_{\mathbb{S}} \quad \in \mathbb{C}^{|\mathbb{S}|}, \tag{1}$$

where $\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) \triangleq [e^{j2\pi\bar{\theta}_i n}]_{n\in\mathbb{S}}$ is the steering vector and $\mathbf{n}_{\mathbb{S}}$ is the noise term. The normalized DOA is defined as $\bar{\theta}_i \triangleq (\sin\theta_i)/2 \in [-1/2, 1/2]$. The notation $|\mathbb{S}|$ denotes the cardinality of the set $\mathbb{S}$. It is assumed that the sources and the noise are zero-mean and uncorrelated. Namely, if $\mathbf{s} \triangleq [A_1, \ldots, A_D, \mathbf{n}_{\mathbb{S}}^T]^T$, then we have $\mathbb{E}[\mathbf{s}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \operatorname{diag}(p_1, \ldots, p_D, p_n\mathbf{I})$, where p_i and p_n are the powers of the ith source and the noise, respectively. Under these assumptions, the covariance matrix of $\mathbf{x}_{\mathbb{S}}$ becomes [4]:

$$\mathbf{R}_{\mathbb{S}} = \mathbb{E}[\mathbf{x}_{\mathbb{S}}\mathbf{x}_{\mathbb{S}}^H] = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + p_n\mathbf{I}. \tag{2}$$

Next we will define the difference coarray $\mathbb{D}$ as follows:

Definition 1: The difference coarray of a linear array $\mathbb{S}$ is defined as $\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$.

Based on Definition 1, vectorizing (2) and averaging over duplicated entries lead to the autocorrelation vector $\mathbf{x}_{\mathbb{D}}$ on the difference coarray:

$$\mathbf{x}_{\mathbb{D}} = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{D}}(\bar{\theta}_i) + p_n\mathbf{e}_0 \quad \in \mathbb{C}^{|\mathbb{D}|}, \tag{3}$$

where $\mathbf{e}_0$ is a column vector with 1 in the middle (the $(|\mathbb{D}|+1)/2$-th element) and 0 elsewhere.

Note that (3) can be regarded as the output defined on the difference coarray, instead of that on the physical array (1). If sensor locations are designed properly, the size of the difference coarray can be much larger than the size of the physical array. In particular, $|\mathbb{D}| = \mathcal{O}(|\mathbb{S}|^2)$. This property makes it possible to develop coarray-based DOA estimators that resolve more uncorrelated sources than sensors and achieve higher spatial resolution [4], [5], [7], [8].

Next we will define some useful quantities regarding the difference coarray. The central ULA segment of $\mathbb{D}$, denoted by $\mathbb{U}$, is the longest ULA in $\mathbb{D}$ that includes the entry 0. In other words, $\mathbb{U} \triangleq \{m : \{0, \pm1, \ldots, \pm|m|\} \subseteq \mathbb{D}\}$. The smallest ULA containing $\mathbb{D}$ is denoted by $\mathbb{V} \triangleq \{m \in \mathbb{Z} : \min(\mathbb{D}) \leq m \leq \max(\mathbb{D})\}$. An integer h is said to be a hole in the difference coarray if $h \in \mathbb{V}$ but $h \notin \mathbb{D}$. A difference coarray is *hole-free* if $\mathbb{D} = \mathbb{U}$.

Definition 2: The weight function $w(m)$ of a linear array $\mathbb{S}$ is defined as the number of sensor pairs with coarray index m. That is, $w(m) = \left|\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\}\right|$.

It is known that the difference coarray plays a significant role in DOA estimation based on (3). For instance, the performance of coarray MUSIC relies on $\mathbb{U}$ [4], [8], [15], [26]. In addition, the performance of any unbiased DOA estimator using sparse arrays is known to be limited by the difference coarray [15], [30], [31].

Now let us review some existing array geometries and their difference coarrays. First, the ULA with N sensors [1] is denoted by the set $\mathbb{S}_{\text{ULA}} \triangleq \{0, 1, \ldots, N-1\}$. The difference coarray for ULA is $\mathbb{D}_{\text{ULA}} = \{\pm0, \pm1, \ldots, \pm(N-1)\}$. It can be shown that $|\mathbb{D}_{\text{ULA}}| = 2N - 1 = \mathcal{O}(N)$. Next, the nested array [4] is defined as

$$\begin{aligned}\mathbb{S}_{\text{nested}} \triangleq \{&1,\ 2,\ \ldots,\ N_1,\\ &(N_1+1),\ 2(N_1+1),\ \ldots,\ N_2(N_1+1)\},\end{aligned} \tag{4}$$

where N_1 and N_2 are positive integers. The difference coarray of the nested array is $\mathbb{D}_{\text{nested}} = \{0, \pm1, \ldots, \pm(N_2(N_1+1)-1)\}$. In particular, $\mathbb{D}_{\text{nested}}$ has no holes. Given N sensors, if N_1 and N_2 are approximately $N/2$, the size of the difference coarray can be shown to be $|\mathbb{D}_{\text{nested}}| = \mathcal{O}(N^2)$ [4]. Finally, the coprime array [5] is parameterized by a pair of integers (M, N) whose greatest common divisor is 1. The sensors for the coprime array are located at

$$\begin{aligned}\mathbb{S}_{\text{coprime}} \triangleq \{&0,\ M,\ \ldots,\ (N-1)M,\\ &N,\ 2N, \ldots,\ (2M-1)N\}.\end{aligned} \tag{5}$$

It can be shown that the difference coarray for the coprime array has holes [5] and the largest central ULA segment is $\mathbb{U}_{\text{coprime}} = \{0, \pm1, \ldots, \pm(MN + M - 1)\}$ [6]. Namely, $|\mathbb{U}_{\text{coprime}}| = 2MN + 2M - 1 = \mathcal{O}(MN)$, and there are $|\mathbb{S}_{\text{coprime}}| = N + 2M - 1 = \mathcal{O}(M + N)$ physical sensors.

For some sparse arrays, such as minimum redundancy arrays (MRA) [2], minimum hole arrays (MHA) [32], and Cantor arrays [33], the sensor locations cannot be readily expressed in closed-form. The MRA and MHA are typically constructed using integer programming [2], [32], whereas the Cantor arrays can be constructed recursively [23], [33]. For the details of these arrays, the interested readers are referred to [2], [23], [32] and the references therein.

III. The Essentialness Property

In this section, we will present the essentialness property, which is useful in studying the robustness of sparse arrays.

It is well-known that coarray MUSIC is applicable to the autocorrelation vector on $\mathbb{U}$ as long as $|\mathbb{U}| > 1$ (e.g., see [8]). However, it will be demonstrated in Example 1 that $\mathbb{U}$ is susceptible to sensor failure. For certain array geometries, even one damaged physical sensor could alter $\mathbb{U}$ significantly and coarray MUSIC may fail.

Example 1: In Fig. 1, Array #1 has $\mathbb{S}_1 = \{0, 1, 2, 4, 6\}$ and $\mathbb{D}_1 = \{0, \pm1, \ldots, \pm6\} = \mathbb{U}_1$. In this case, the coarray MUSIC algorithm may be used, since $|\mathbb{U}_1| = 13 > 1$. Now suppose the sensor located at 1 fails. The new array configuration (Array #2) and the associated difference coarray becomes $\mathbb{S}_2 = \{0, 2, 4, 6\}$ and $\mathbb{D}_2 = \{0, \pm2, \pm4, \pm6\}$, respectively. So the size of the ULA segment of $\mathbb{D}_2$ is 1 and the coarray MUSIC algorithm is not applicable. On the other hand, if the sensor at 2 fails, we have Array #3, which has $\mathbb{S}_3 = \{0, 1, 4, 6\}$ and $\mathbb{D}_3 = \{0, \pm 1, \ldots, \pm 6\}$. Since $|\mathbb{U}_3| = 13 > 1$, the coarray MUSIC algorithm may still be implemented.

Example 1 shows that, the location of the faulty sensors could modify the difference coarray, which affects the applicability of coarray MUSIC. Note that, even if the difference coarray changes, there might exist other DOA estimators, such as compressed sensing based methods [10], [11] and coarray interpolation [12], [34], that work on the new difference coarray. However, these approaches are typically computationally expensive and the exact conditions under which the method works, remain to be explored. For this reason, we only focus on coarray MUSIC and the integrity of the difference coarray in this paper and the companion paper [29]. Other scenarios are left for future work.

We begin with the following definition [23]:

Definition 3: The sensor located at $n \in \mathbb{S}$ is said to be essential with respect to $\mathbb{S}$ if the difference coarray changes when sensor at n is deleted from the array. That is, if $\overline{\mathbb{S}} = \mathbb{S}\backslash\{n\}$, then $\overline{\mathbb{D}} \neq \mathbb{D}$. Here $\mathbb{D}$ and $\overline{\mathbb{D}}$ are the difference coarrays for $\mathbb{S}$ and $\overline{\mathbb{S}}$, respectively.

The essentialness property was originally introduced in [23] to study symmetric arrays and Cantor arrays. The main focus in [23] was the economy of sensors.

In this paper, we focus on the fact that the removal of (or failure of) an essential sensor makes it difficult to apply coarray MUSIC. The removal of an inessential sensor, on the other hand, does not affect the applicability of coarray MUSIC at all. Our focus in this paper is a study of essentialness, and its generalization called k-essentialness for arbitrary array geometries. One potential use of this knowledge is that one can design essential sensors more carefully so they have smaller failure probability, although this is not the focus here.

Given an array $\mathbb{S}$, the essential sensors can be found by searching over all the sensors in $\mathbb{S}$, according to Definition 3. The knowledge of the weight function $w(m)$ also gives useful insights about this:

Lemma 1: Suppose that $w(m)$ is the weight function of $\mathbb{S}$. Let n_1 and n_2 belong to $\mathbb{S}$. If $w(n_1 - n_2) = 1$, then n_1 and n_2 are both essential [23].

The proof of Lemma 1 can be found in [23, Lemma 1]. Note that the condition that $w(n_1 - n_2) = 1$ is sufficient, but not necessary for the essentialness of both n_1 and n_2. For instance, if $\mathbb{S} = \{0, 1, 2\}$, then it can be shown that $n_1 = 1$ and $n_2 = 0$ are both essential with respect to $\mathbb{S}$, due to Definition 3. However, the weight function satisfies $w(n_1 - n_2) = w(1) = 2$.

Lemma 1 serves as a building block of many results in this paper and the companion paper [29], as we will develop later.

Due to Lemma 1 and the fact that $w(\max(\mathbb{S}) - \min(\mathbb{S})) = 1$ for any $\mathbb{S}$, we have the following lemma:

Lemma 2: For any array $\mathbb{S}$, the leftmost element $(\min(\mathbb{S}))$ and the rightmost element $(\max(\mathbb{S}))$ are both essential.

As a result, when studying the essentialness property, it suffices to consider the elements $\min(\mathbb{S}) < n < \max(\mathbb{S})$, which simplifies the discussion.

Next we will develop the concept of maximal economy, which was first presented in [23]. It is formally defined as

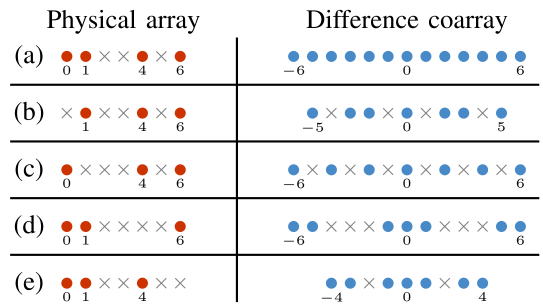

Fig. 2. An example of MESA. (a) The original array and its difference coarray. The array configurations and the difference coarrays after the deletion of (b) the sensor at 0, (c) the sensor at 1, (d) the sensor at 4, or (e) the sensor at 6, from the original array in (a). Here the sensors are denoted by dots while crosses denote empty space.

Definition 4: A sensor array $\mathbb{S}$ is said to be *maximally economic* if all the sensors in $\mathbb{S}$ are essential [23].

These arrays are also called maximally economic sparse arrays (MESA) [23]. By definition, none of the sensors in MESA can be removed without changing the difference coarray. For instance, the array $\mathbb{S} = \{0, 1, 4, 6\}$ in Fig. 2(a) is maximally economic, due to the results in Figs. 2(b) (c), (d), and (e).

Note that maximal economy is a property of the entire array, in contrast to the essentialness property, which is associated with sensors in an array, as in Definition 3. In this paper, the general properties will be discussed while in the companion paper [29, Section III], it will be proved that MESA includes MRA, MHA, nested arrays with $N_2 \geq 2$, and Cantor arrays.

A. The k-Essential Family

If there are *multiple sensor failures*, the influence of these faulty sensors on the difference coarray becomes more complicated. If two sensors are inessential, it means that either one of them can be removed without changing the coarray. But *if both sensors are removed, the difference coarray may change.*

In the following development, the essentialness property in Definition 3 will be generalized into the k-essentialness property to handle multiple sensor failures. To begin with, the family of all size-k subarrays over an integer set $\mathbb{S}$ is defined as

$$\mathcal{S}_k \triangleq \{\mathbb{A} \subseteq \mathbb{S} : |\mathbb{A}| = k\}. \tag{6}$$

Then the k-essentialness property is defined as

Definition 5: A subarray $\mathbb{A}$ of $\mathbb{S}$ is said to be k-essential with respect to array $\mathbb{S}$ if it has the following properties.

1) $\mathbb{A}$ has size exactly k. Namely, $\mathbb{A} \in \mathcal{S}_k$.
2) The difference coarray changes when $\mathbb{A}$ is removed from $\mathbb{S}$.

Note that essentialness, as defined in Definition 3, is equivalent to 1-essentialness ($k = 1$ in Definition 5). Namely, $n \in \mathbb{S}$ is essential if and only if $\{n\} \subseteq \mathbb{S}$ is 1-essential. For brevity, we will use these terms interchangeably.

Example 2: For instance, let us consider the ULA with 9 sensors, as depicted in Fig. 3(a). It can be shown that $\{1\}$ is not 1-essential, $\{2\}$ is not 1-essential, $\{7\}$ is not 1-essential, $\{1, 2\}$

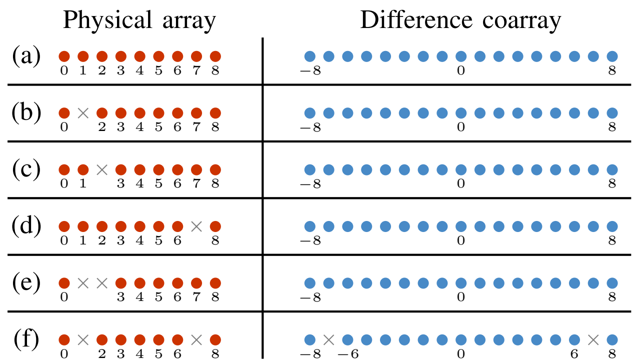

Fig. 3. Array configurations and the difference coarrays for (a) the ULA with 9 sensors, and the arrays after the removal of (b) 1, (c) 2, (d) 7, (e) $\{1, 2\}$, and (f) $\{1, 7\}$ from (a).

is not 2-essential, but $\{1, 7\}$ is 2-essential, all with respect to the array in Fig. 3(a).

It is useful to enumerate all the k-essential subarrays because these are the subarrays whose failure could make the coarray MUSIC fail. The collection of these subarrays is called the k-essential family:

Definition 6: The k-essential family $\mathcal{E}_k$ with respect to a sensor array $\mathbb{S}$ is defined as

$$\mathcal{E}_k \triangleq \{\mathbb{A} : \mathbb{A} \text{ is } k\text{-essential with respect to } \mathbb{S}\}. \tag{7}$$

Here $k \in \{1, 2, \ldots, |\mathbb{S}|\}$.

The implication of the k-essential family is as follows. If an array $\mathbb{S}$ and its k-essential family $\mathcal{E}_k$ are given, then for any subarray $\mathbb{A}$ of size k, it is possible to determine whether $\mathbb{S}$ and $\mathbb{S}\backslash\mathbb{A}$ share the same difference coarray, without actually computing the difference coarray. This can be done by searching for $\mathbb{A}$ in $\mathcal{E}_k$. Furthermore, the size of $\mathcal{E}_k$ (i.e., the number of k-essential subarrays) also quantifies the robustness of the system, as we shall elaborate in Section IV.

In general, given an array configuration $\mathbb{S}$, the k-essential family $\mathcal{E}_k$ can be uniquely determined, by examining all possible $\binom{|\mathbb{S}|}{k}$ subarrays, as in Definition 6. From the computational perspective, this task becomes intractable for large number of sensors. In addition, even if $\mathcal{E}_k$ can be enumerated, it remains difficult to retrieve information from $\mathcal{E}_k$, which might have size up to the order of $\binom{|\mathbb{S}|}{k}$.

These challenges will be addressed in two respects. First, the size of the k-essential family, namely $|\mathcal{E}_k|$, can be expressed or bounded in terms of simpler things like the number of sensors, the weight function, and the number of essential sensors, as presented in Theorem 1. These results lead to the robustness analysis of array configurations, as we will develop in Section IV. Second, the retrieval of the information in $\mathcal{E}_k$ could be accelerated by the k-essential Sperner family, which will be discussed in Section V in detail.

Next, some properties of $\mathcal{E}_k$ are discussed in Theorem 1, whose proof can be found in the next subsection.

Theorem 1: Let $\mathcal{E}_k$ be the k-essential family with respect to a nonempty integer set $\mathbb{S}$ (set of sensors), and let the family $\mathcal{S}_k$ be as defined in (6). Let $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ be the ceiling function and

$$\begin{aligned}
\mathcal{E}_1 &= \{\{0\}, \{5\}\},\\
\mathcal{E}_2 &= \{\{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \{1,5\},\\
&\quad\ \{2,5\}, \{3,5\}, \{4,5\}, \{1,4\}, \{2,3\}\},\\
\mathcal{E}_k &= \mathcal{S}_k, \qquad \text{for all } 3 \le k \le 6.
\end{aligned}$$

Fig. 4. The ULA with 6 physical sensors, where the essential sensors and the inessential sensors are denoted by diamonds and rectangles, respectively. The k-essential subarrays are also listed.

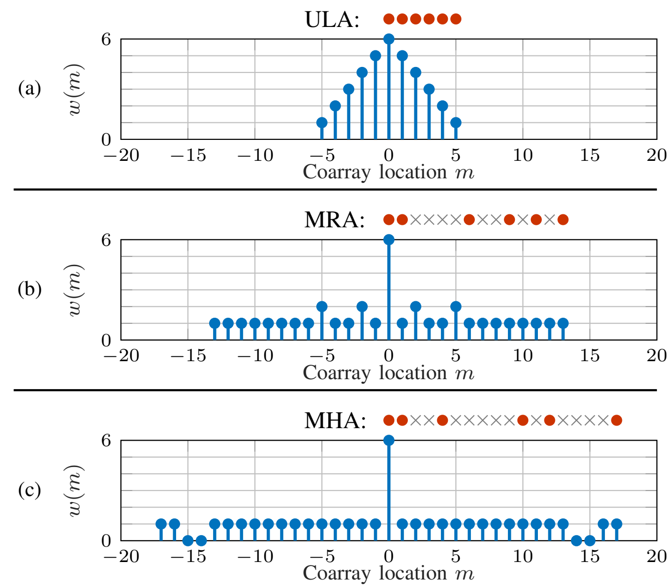

Fig. 5. The array geometries and the weight functions for (a) the ULA with 6 sensors, (b) the MRA with 6 sensors, and (c) the MHA with 6 sensors. The sensors are depicted in dots while the empty space is shown in crosses. The definition of M_q in Property 3 of Theorem 1 leads to $M_1 = 2, M_2 = 2$ for (a), $M_1 = 22, M_2 = 4$ for (b), and $M_1 = 30, M_2 = 0$ for (c).

the floor function, respectively. Then the following properties hold true:

1) $(|\mathbb{S}| - k)|\mathcal{E}_k| \le (k+1)|\mathcal{E}_{k+1}|$ for all $1 \le k \le |\mathbb{S}| - 1$. The equality holds if and only if $\mathcal{E}_k = \mathcal{S}_k$.
2) $\mathcal{E}_k = \mathcal{S}_k$ for all $Q \le k \le |\mathbb{S}|$, where $Q = \min\{Q_1, Q_2\}$. The parameters Q_1 and Q_2 are given by

$$Q_1 = |\mathbb{S}| - |\mathcal{E}_1| + 1, \tag{8}$$

$$Q_2 = \left\lceil |\mathbb{S}| - \frac{\sqrt{8|\mathbb{S}| - 11} + 1}{2} \right\rceil, \quad \text{for } |\mathbb{S}| \ge 2. \tag{9}$$

3) Let $M_q = |\{m \in \mathbb{D} : w(m) = q\}|$ be the number of elements in the difference coarray such that the associated weight function is q. If $|\mathbb{S}| \ge 2$, then

$$\left\lceil \frac{\sqrt{4M_1 + 1} + 1}{2} \right\rceil \le |\mathcal{E}_1| \le \min\left\{ M_1 + \left\lfloor \frac{M_2}{2} \right\rfloor, |\mathbb{S}| \right\}. \tag{10}$$

Example 3: Theorem 1 can be illustrated by the following concrete examples. Fig. 4 depicts the k-essential family of the ULA with 6 sensors. First, we obtain that $\mathcal{E}_1 \neq \mathcal{S}_1$, $|\mathcal{E}_1| = 2$, and $|\mathcal{E}_2| = 11$. If $k = 1$, then we have $(|\mathbb{S}| - 1)|\mathcal{E}_1| = 10$

and $(1+1)|\mathcal{E}_2| = 22$, which illustrates Property 1 of Theorem 1. Second, Fig. 4 shows that the ULA with 6 sensors has $|\mathcal{E}_1| = 2$ and $\mathcal{E}_k = \mathcal{S}_k$ for $3 \leq k \leq 6$. This is consistent with Property 2 of Theorem 1, since we have $Q_1 = 5$, $Q_2 = 3$, and $Q = 3$ in (8) and (9). Finally, let us demonstrate Property 3 of Theorem 1 using the ULA, the MRA, and the MHA. The array geometries and the weight functions for these arrays are depicted in Fig. 5. Furthermore, the parameters M_1 and M_2 can be found in the caption of Fig. 5. Substituting M_1 and M_2 into the lower bound and the upper bound in (10) leads to

$$\text{(a) ULA:} \quad \text{Lower bound} = 2, \quad \text{Upper bound} = 3, \tag{11}$$

$$\text{(b) MRA:} \quad \text{Lower bound} = 6, \quad \text{Upper bound} = 6, \tag{12}$$

$$\text{(c) MHA:} \quad \text{Lower bound} = 6, \quad \text{Upper bound} = 6. \tag{13}$$

Next let us consider the number of essential sensors ($|\mathcal{E}_1|$) for these arrays. According to Fig. 4, we have $|\mathcal{E}_1| = 2$ for the ULA with 6 sensors, which is in accordance with (11). For MRA and MHA with 6 sensors, using Definition 5, it can be numerically shown that they are maximally economic. This result is consistent with (12) and (13). Note that the maximal economy of MRA and MHA will be proved in the companion paper [29].

Remarks on Property 1 of Theorem 1 : This property says that the size of $\mathcal{E}_k$ cannot be arbitrary. In particular, if $\mathcal{E}_k$ becomes $\mathcal{S}_k$, we have the following corollary:

Corollary 1: For any k in $1 \leq k \leq |\mathbb{S}|-1$, if $\mathcal{E}_k = \mathcal{S}_k$, then $\mathcal{E}_{k+1} = \mathcal{S}_{k+1}$.

Note that Corollary 1 can be utilized to accelerate the computation of $\mathcal{E}_k$ for all k. Due to Definition 6, $\mathcal{E}_k$ can be evaluated numerically from $k = 1$, 2, and so on. If $\mathcal{E}_k$ is $\mathcal{S}_k$ for some particular k, then the algorithm stops, since it is guaranteed that $\mathcal{E}_\ell = \mathcal{S}_\ell$ for $k+1 \leq \ell \leq |\mathbb{S}|$. Another usage of Corollary 1 is to study the k-essential family of MESA. Due to Definition 4 ($\mathcal{E}_1 = \mathcal{S}_1$) and Corollary 1, we obtain

Corollary 2: If $\mathbb{S}$ is maximally economic, then $\mathcal{E}_k = \mathcal{S}_k$ for all $1 \leq k \leq |\mathbb{S}|$.

Implications of Property 2 of Theorem 1: If the number of faulty sensors k is sufficiently large ($\geq Q$ where Q is defined in Property 2 of Theorem 1), then the difference coarray is guaranteed to change. For instance, if $k = |\mathbb{S}|$, then all the sensors fail so the difference coarray changes from a nonempty set to the empty set. The parameter Q depends on Q_1 and Q_2, which can be readily computed given the array geometry. Q_1 is the number of inessential sensors plus one while Q_2 is purely a function of the number of sensors. In particular, assume that the number of sensors $|\mathbb{S}|$ is large enough. Based on (9), we have $Q_2 \approx |\mathbb{S}| - \sqrt{2|\mathbb{S}|}$, implying that, if the number of operational sensors ($|\mathbb{S}| - k$) is smaller than $\sqrt{2|\mathbb{S}|}$, then $\mathcal{E}_k = \mathcal{S}_k$, that is, any subset of k sensors is k-essential with respect to $\mathbb{S}$.

Note that the condition that $Q \leq k \leq |\mathbb{S}|$ is only sufficient but not necessary for $\mathcal{E}_k = \mathcal{S}_k$. For instance, if $\mathbb{S} = \{0, 1, \ldots, 15\}$, then (8) and (9) result in $Q_1 = 15$, $Q_2 = 11$, so $Q = 11$. However, in this case, it can be numerically shown that $\mathcal{E}_{10} = \mathcal{S}_{10}$.

Property 2 of Theorem 1 can also be utilized to characterize MESA, as in the following corollary, which is readily verified:

Corollary 3: Let $\mathbb{S}$ be a sensor array. If $1 \leq |\mathbb{S}| \leq 3$, then $\mathbb{S}$ is maximally economic.

Remarks on Property 3 of Theorem 1 : Eq. (10) is analogous to Cheeger inequalities in graph theory [35], where the Cheeger constant is bounded by the expressions based on the topology of graphs. Here in (10), the number of essential sensors is analogous to the Cheeger constant. The bounds in (10) also depend on the weight functions, which depend on the array geometry.

Another remark is the connection between the difference coarray and graph theory. It was shown in [36] that the difference coarray is closely related to *numbered undirected graphs*, where each vertex corresponds to a number and the differences of the numbers on the vertices are assigned to edges. Interested readers are referred to [36] and the references therein. In this paper, we will use this concept to prove Theorem 1.

B. Proof of Theorem 1

The following results are useful in proving Theorem 1:

Proposition 1: Let $\mathbb{D}$ and $\overline{\mathbb{D}}$ be the difference coarrays of $\mathbb{S}$ and $\overline{\mathbb{S}}$, respectively. If $\overline{\mathbb{S}} \subseteq \mathbb{S}$, then $\overline{\mathbb{D}} \subseteq \mathbb{D}$.

The proof of Proposition 1 can be readily seen.

Lemma 3: Assume that $\mathbb{A}$ and $\mathbb{B}$ are sets such that $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{S}$. If $\mathbb{A} \in \mathcal{E}_{|\mathbb{A}|}$, then $\mathbb{B} \in \mathcal{E}_{|\mathbb{B}|}$.

Proof: Assume that $\mathbb{S}_1 \triangleq \mathbb{S}\backslash\mathbb{A}$ and $\mathbb{S}_2 \triangleq \mathbb{S}\backslash\mathbb{B}$. The difference coarrays of $\mathbb{S}$, $\mathbb{S}_1$, and $\mathbb{S}_2$ are denoted by $\mathbb{D}$, $\mathbb{D}_1$, and $\mathbb{D}_2$, respectively. The notation $\mathbb{X} \subset \mathbb{Y}$ denotes that $\mathbb{X}$ is a subset of $\mathbb{Y}$ but $\mathbb{X} \neq \mathbb{Y}$. We will show that $\mathbb{D}_2 \subseteq \mathbb{D}_1 \subset \mathbb{D}$. First, since $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{S}$, we have $\mathbb{S}_2 \subseteq \mathbb{S}_1$, implying $\mathbb{D}_2 \subseteq \mathbb{D}_1$ due to Proposition 1. Second, due to the definition of the k-essential family, $\mathbb{A} \in \mathcal{E}_{|\mathbb{A}|}$ is equivalent to $\mathbb{D}_1 \subset \mathbb{D}$. Hence $\mathbb{D}_2 \subset \mathbb{D}$, which means $\mathbb{B} \in \mathcal{E}_{|\mathbb{B}|}$. ∎

Lemma 4: Assume that an array $\mathbb{S}$ has difference coarray $\mathbb{D}$. Then $\mathbb{D}$ satisfies $2|\mathbb{S}| - 1 \leq |\mathbb{D}| \leq |\mathbb{S}|^2 - |\mathbb{S}| + 1$.

Proof: Let $\mathbb{S}$ be $\{s_1, s_2, \ldots, s_N\}$ such that $s_1 < s_2 < \cdots < s_N$, where $N = |\mathbb{S}|$ is the number of sensors. If $N = 1$, then this lemma is trivially true. Next let us consider $N \geq 2$. Since the sensor locations $s_1, s_2, \ldots, s_N$ are distinct, the differences $0, \pm(s_2 - s_1), \pm(s_3 - s_1), \ldots, \pm(s_N - s_1)$ are all distinct, which proves the lower bound. For the upper bound, it is known that there are $\binom{N}{2}$ ways to choose two distinct numbers from N numbers and each choice leads to two differences. In addition, the difference 0 is obtained by choosing the same number twice. Hence $|\mathbb{D}|$ is at most $2\binom{N}{2} + 1 = N^2 - N + 1$. ∎

Now let us move on to the proof of Theorem 1:

1) Proof of Property 1 of Theorem 1: This proof technique can be found in [37]. Let us count the number of pairs $(\mathbb{A}, \mathbb{B}) \in \mathcal{E}_k \times \mathcal{E}_{k+1}$ such that $\mathbb{A} \subset \mathbb{B}$. Let L be the number of such pairs. For every $n_1 \in \mathbb{S}$ but $n_1 \notin \mathbb{A}$, it can be shown that $\mathbb{A} \subset \mathbb{A} \cup \{n_1\} \subseteq \mathbb{S}$ and therefore $\mathbb{A} \cup \{n_1\} \in \mathcal{E}_{k+1}$, due to Lemma 3. Since $(\mathbb{A}, \mathbb{A} \cup \{n_1\})$ has $|\mathcal{E}_k| \times |\mathbb{S}\backslash\mathbb{A}|$ choices, we have

$$L = (|\mathbb{S}| - k)|\mathcal{E}_k|. \tag{14}$$

Similarly, it can be shown that $\mathbb{B}\backslash\{n_2\} \subset \mathbb{B} \subseteq \mathbb{S}$, for all $\mathbb{B} \in \mathcal{E}_{k+1}$ and $n_2 \in \mathbb{B}$. However, the statement that $\mathbb{B}\backslash\{n_2\} \in \mathcal{E}_k$,

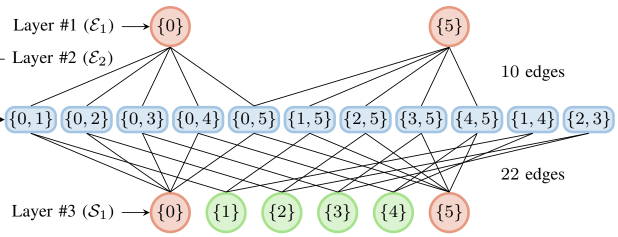

Fig. 6. An illustration for the main idea of the proof of Property 1 of Theorem 1. Here the array is the ULA with 6 sensors and the k-essential family $\mathcal{E}_1$ and $\mathcal{E}_2$ are depicted in Fig. 4.

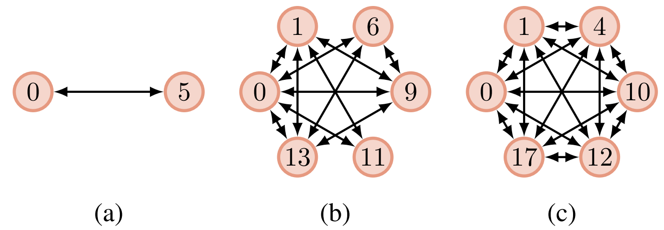

Fig. 7. The directed graph $\mathcal{G}$ in the proof of (20), for (a) the ULA with 6 sensors, (b) the MRA with 6 sensors, and (c) the MHA with 6 sensors. The number of directed edges is (a) $M_1 = 2$, (b) $M_1 = 22$, and (c) $M_1 = 30$.

(the converse of Lemma 3), is not necessarily true. Therefore, by counting the number of n_2 and $\mathbb{B}$, we have

$$L \le (k+1)|\mathcal{E}_{k+1}|, \tag{15}$$

with equality if and only if $\mathbb{B}\backslash\{n_2\} \in \mathcal{E}_k$ for all $\mathbb{B} \in \mathcal{E}_{k+1}$ and all $n_2 \in \mathbb{B}$. Combining (14) and (15) proves the inequality. The equality holds if and only if $(\mathbb{A}\cup\{n_1\})\backslash\{n_2\} \in \mathcal{E}_k$. Therefore $\mathcal{E}_k = \varnothing$ or $\mathcal{S}_k$, where $\varnothing$ is the empty set. Since $\min(\mathbb{S})$ and $\max(\mathbb{S})$ are both essential, $\mathcal{E}_k$ is not empty. This proves the condition for equality. ∎

Example 4: For clarity, the proof of Property 1 of Theorem 1 is demonstrated using an undirected graph in Fig. 6. We focus on the ULA with 6 sensors and $k = 1$. The array geometry and the k-essential family $\mathcal{E}_1$ and $\mathcal{E}_2$ are depicted in Fig. 4. The nodes in Fig. 6 are grouped into Layer #1 (elements in $\mathcal{E}_1$), Layer #2 (elements in $\mathcal{E}_2$), and Layer #3 (elements in $\mathcal{S}_1$). The numbers in the nodes denote subarrays of the ULA. We say that the node $\mathbb{A}$ in Layer #1 and the node $\mathbb{B}$ in Layer #2 are connected, if and only if $\mathbb{A}$ is a subset of $\mathbb{B}$. For instance, the node $\{0\}$ in Layer #1 and the node $\{0, 1\}$ in Layer #2 are connected. By definition, the parameter L is exactly the number of edges between Layer #1 and Layer #2. Since there are $|\mathcal{E}_1| = 2$ nodes in Layer #1 and each node contributes to $|\mathbb{S}| - 1 = 5$ edges, we have $L = 2 \times 5 = 10$, which is (14). Next, let us consider the number of edges between Layer #2 and Layer #3. Let $\mathbb{B}$ in Layer #2 and $\mathbb{C}$ in Layer #3. We say that there exists an edge between $\mathbb{B}$ and $\mathbb{C}$ if and only if $\mathbb{C}$ is a subset of $\mathbb{B}$. For example, the node $\{3, 5\}$ in Layer #2 is connected to both node $\{3\}$ and node $\{5\}$ in Layer #3. As a result, each node in Layer #2 corresponds to $k+1 = 2$ edges. Then the number of edges between Layer #2 and #3 becomes $11 \times 2 = 22$, which is indeed greater than or equal to $L = 10$.

2) Proof of Property 2 of Theorem 1: Let us consider any subarray $\mathbb{A} \subseteq \mathbb{S}$ such that $|\mathbb{A}| = k \ge |\mathbb{S}| - |\mathcal{E}_1| + 1 = Q_1$. The cardinality of $\mathbb{S}\backslash\mathbb{A}$ becomes $|\mathbb{S}| - k \le |\mathcal{E}_1| - 1 < |\mathcal{E}_1|$, implying that there is at least one essential element in $\mathbb{A}$. Due to Lemma 3, $\mathbb{A}$ is k-essential, which proves the lower bound Q_1.

The proof for the lower bound Q_2 is as follows. Let the difference coarray of an array $\mathbb{S}$ be denoted by $\mathbb{D}$. Suppose that $\mathbb{A} \subseteq \mathbb{S}$ and $|\mathbb{A}| = k$. Assume that $\overline{\mathbb{S}} \triangleq \mathbb{S}\backslash\mathbb{A}$ has difference coarray $\overline{\mathbb{D}}$. Due to Lemma 4, $\mathbb{D}$ and $\overline{\mathbb{D}}$ satisfy

$$2|\mathbb{S}| - 1 \le |\mathbb{D}| \le |\mathbb{S}|^2 - |\mathbb{S}| + 1, \tag{16}$$

$$2(|\mathbb{S}| - k) - 1 \le |\overline{\mathbb{D}}| \le (|\mathbb{S}| - k)^2 - (|\mathbb{S}| - k) + 1. \tag{17}$$

It is guaranteed that $\mathbb{D} \ne \overline{\mathbb{D}}$, if the range of $|\mathbb{D}|$ in (16) and that of $|\overline{\mathbb{D}}|$ in (17) are disjoint. Therefore, $\mathcal{E}_k = \mathcal{S}_k$ if

$$(|\mathbb{S}| - k)^2 - (|\mathbb{S}| - k) + 1 \le (2|\mathbb{S}| - 1) - 1. \tag{18}$$

If $|\mathbb{S}| \ge 2$, then the sufficient condition (18) leads to $|\mathbb{S}| - (\sqrt{8|\mathbb{S}| - 11} + 1)/2 \le k \le |\mathbb{S}| + (\sqrt{8|\mathbb{S}| - 11} - 1)/2$. Since k is an integer, we have $k \ge Q_2$. ∎

3) Proof of Property 3 of Theorem 1: Let $\mathbb{S}_q = \{n_1, n_2 : w(n_1 - n_2) = q\} \subseteq \mathbb{S}$ be the sensors such that the associated weight function is q. The set $\mathbb{G}_q$ collects the essential sensors in $\mathbb{S}_q$ but not in $\mathbb{S}_\ell$ for $1 \le \ell \le q - 1$. Namely,

$$\begin{aligned}\mathbb{G}_q = \{n : \{n\} \in \mathcal{E}_1,\ n \in \mathbb{S}_q,\\ n \notin \mathbb{S}_\ell,\ 1 \le \ell \le q - 1\}.\end{aligned} \tag{19}$$

By definition, the number of essential sensors is given by $|\mathcal{E}_1| = \sum_{q=1}^{|\mathbb{S}|} |\mathbb{G}_q|$. Next, it can be shown (see below) that the size of $\mathbb{G}_q$ satisfies:

$$(\sqrt{4M_1 + 1} + 1)/2 \le |\mathbb{G}_1| \le M_1, \tag{20}$$

$$0 \le |\mathbb{G}_2| \le M_2/2, \tag{21}$$

$$|\mathbb{G}_q| = 0, \qquad q \ge 3. \tag{22}$$

Since $|\mathcal{E}_1|$ is an integer, $|\mathcal{E}_1|$ is lower bounded by $\lceil(\sqrt{4M_1 + 1} + 1)/2\rceil$ and upper bounded by $M_1 + \lfloor M_2/2 \rfloor$, which proves this theorem. ∎

Proof of (20)*:* Consider a simple, directed graph $\mathcal{G}$ with vertices $\mathbb{G}_1$ and directed edges from n_1 to n_2 if $w(n_1 - n_2) = 1$ for all distinct $n_1, n_2 \in \mathbb{G}_1$. Due to $|\mathbb{S}| \ge 2$ and Lemma 2, both of the distinct elements $\min(\mathbb{S})$ and $\max(\mathbb{S})$ belong to $\mathbb{G}_1$. Therefore $|\mathbb{G}_1| \ge 2$. By definition, M_1 is the number of directed edges in $\mathcal{G}$. Next the range of M_1 is discussed. Due to (19), each vertex in $\mathcal{G}$ corresponds to at least one directed edge and hence $|\mathbb{G}_1| \le M_1$. On the other hand, the maximum number of edges in $\mathcal{G}$ is $2\binom{|\mathbb{G}_1|}{2} = |\mathbb{G}_1|(|\mathbb{G}_1| - 1)$ [38]. Rearranging $M_1 \le |\mathbb{G}_1|(|\mathbb{G}_1| - 1)$ proves the lower bound in (20). ∎

Example 5: Let us consider the arrays in Fig. 5 to elaborate the proof of (20). For instance, in Figs. 4 and 5(a), the ULA has $\mathcal{E}_1 = \{\{0\}, \{5\}\}$ and $w(5 - 0) = 1$. In this case, we have $\mathbb{G}_1 = \{0, 5\}$, due to (19), and the number of directed edges is $M_1 = 2$, as in Fig. 7(a), which is in accordance with (20). For the MRA with 6 sensors, Example 3 and Fig. 5(b) show that all sensors are essential and $w(13 - 0) = w(11 - 1) = w(9 - 6) = 1$. Therefore, we obtain $\mathbb{G}_1 = \{0, 1, 6, 9, 11, 13\}$ and $M_1 = 22$, as depicted in Fig. 7(b). These quantities also confirm (20). Finally, as in Example 3 and Fig. 5(c), the MHA with 6 sensors has $\mathcal{E}_1 = \mathcal{S}_1$ and $w(1-0) = w(12-10) = w(17-4) = 1$. Hence $\mathbb{G}_1 = \{0, 1, 4, 10, 12, 17\}$ and $M_1 = 30$, which are consistent with (20). Note that, in this case, the associated graph $\mathcal{G}$ is a complete directed graph, as illustrated in Fig. 7(c).

Proof of (21): First, it will be shown that, each case of $w(m) = w(-m) = 2$ corresponds to at most one element in $\mathbb{G}_2$. Then the upper bound in (21) can be proved since there are at most $M_2/2$ such cases.

Let $(n_1, n_2), (n_1', n_2') \in \mathbb{S}^2$ be the only two sensor pairs such that $(n_1, n_2) \neq (n_1', n_2')$ and $n_1 - n_2 = n_1' - n_2'$. We have $w(n_1 - n_2) = w(n_2 - n_1) = 2$. Without loss of generality, the pair (n_1, n_2) is considered in the following. If $n_1 \in \mathbb{G}_2$, then the sensor failure at n_1 leads to failure of the pairs (n_1, n_2) and (n_1', n_2') at the same time. Since $n_1 \neq n_1'$, we have $n_2' = n_1 \in \mathbb{G}_2$ and $n_2 + n_1' = 2n_1$. Similarly, if $n_2 \in \mathbb{G}_2$, then $n_1' = n_2 \in \mathbb{G}_2$ and $n_1 + n_2' = 2n_2$. If both n_1 and n_2 belong to $\mathbb{G}_2$, then $n_1 = n_2 = n_1' = n_2'$. These arguments show that, among n_1, n_2, n_1', and n_2', there is at most one element in $\mathbb{G}_2$. Therefore, each case of $w(m) = w(-m) = 2$ leads to at most one element in $\mathbb{G}_2$. ∎

Proof of (22): Let $n_1 \in \mathbb{G}_q$. Since $3 \leq q \leq |\mathbb{S}|$, there exist three distinct pairs $(n_1, n_2), (n_1', n_2'), (n_1'', n_2'') \in \mathbb{S}^2$ such that $n_1 - n_2 = n_1' - n_2' = n_1'' - n_2''$. The essentialness property of n_1 indicates that, the sensor failure at n_1 removes these three pairs simultaneously. Since $n_1 \neq n_1'$ and $n_1 \neq n_1''$, we have $n_1 = n_2' = n_2''$ so $n_1' = n_1''$, which disagrees with the assumption of distinct pairs. Hence $|\mathbb{G}_q| = 0$. ∎

IV. The k-Fragility

After studying the general properties of the k-essential family $\mathcal{E}_k$, in this section, we will focus on the size of the k-essential family. Larger the size, higher is the likelihood that the difference coarray changes due to failure of k sensors. For instance, if $\mathcal{E}_k = \mathcal{S}_k$, it means that any k faulty sensors shrink the difference coarray. The notion of *fragility* is useful to capture this idea.

Definition 7: The fragility or k-fragility of a sensor array $\mathbb{S}$ is defined as

$$F_k \triangleq \frac{|\mathcal{E}_k|}{|\mathcal{S}_k|} = \frac{|\mathcal{E}_k|}{\binom{|\mathbb{S}|}{k}}, \tag{23}$$

where $k = 1, 2, \ldots, |\mathbb{S}|$.

F_k can also be regraded as the probability that the difference coarray changes, if all failure patterns of size k are equiprobable. *Larger F_k indicates that this array configuration is less robust, or more fragile to sensor failure, in the sense of changing the difference coarray.*

With these physical interpretations, next we will move on to some properties of the k-fragility F_k:

Theorem 2: Let $\mathbb{S}$ be an integer set denoting the sensor locations. The k-fragility F_k with respect to $\mathbb{S}$ has the following properties:

1) $F_k \leq F_{k+1}$ for all $1 \leq k \leq |\mathbb{S}| - 1$. The equality holds if and only if $F_k = 1$.
2) $F_k = 1$ for all k such that $Q \leq k \leq |\mathbb{S}|$, where Q is defined in Property 2 of Theorem 1.

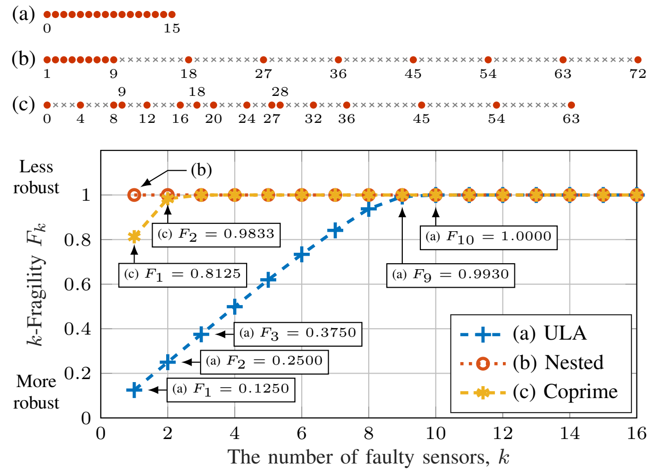

Fig. 8. The array geometries (top) and the k-fragility F_k (bottom) for (a) the ULA with 16 sensors, (b) the nested array with $N_1 = N_2 = 8$, and (c) the coprime array with $M = 4$ and $N = 9$.

3) $\min\{1, 2/|\mathbb{S}|\} \leq F_k \leq 1$ for all $1 \leq k \leq |\mathbb{S}|$.

Proof: Properties 1 and 2 of Theorem 2 follow from Properties 1 and 2 of Theorem 1, respectively. The lower bound in Property 3 of Theorem 2 is due to Definition 7, Lemma 2, and Property 1 of Theorem 2. ∎

Example 6: Fig. 8 demonstrates the k-fragility F_k for (a) the ULA with 16 sensors, (b) the nested array with $N_1 = N_2 = 8$, as in (4), and (c) the coprime array with $M = 4$ and $N = 9$, (5). All these arrays have 16 physical sensors. The array geometries for these arrays are depicted on the top of Fig. 8. On the bottom of Fig. 8, the data points of F_k are computed numerically using Definitions 6 and 7. For all these arrays, the k-fragility F_k is increasing in k (Property 1 of Theorem 2) and F_k is bounded between $2/|\mathbb{S}| = 0.125$ and 1 (Property 3 of Theorem 2). As an example, the ULA has $|\mathbb{S}| = 16$, $Q_1 = 15$, $Q_2 = 11$, and $Q = \min\{Q_1, Q_2\} = 11$. Hence we obtain $F_k = 1$ for all $11 \leq k \leq 16$ (Property 2 of Theorem 2), which is consistent with Fig. 8.

Furthermore, smaller F_k indicates that the array configuration tends to be more robust to sensor failures. Among the arrays considered in Fig. 8, the most robust array in terms of F_1, is the ULA, followed by the coprime array, and finally the nested array.

Next we will present the k-fragility for MESA. According to Definition 4, an array $\mathbb{S}$ being a MESA is equivalent to $F_1 = 1$, implying the following corollary due to Theorem 2:

Corollary 4: If $\mathbb{S}$ is maximally economic, then $F_k = 1$ for all $1 \leq k \leq |\mathbb{S}|$.

For instance, for the nested array with $N_1 = N_2 = 8$, the k-fragility $F_k = 1$ for all k, as shown in Fig. 8. This numerical result is consistent with the fact that the nested array with $N_2 \geq 2$ is a MESA, as proved in the companion paper [29, Theorem 1].

As another remark, Theorem 1 of the companion paper [29] indicates that MRA, MHA, and Cantor arrays are all maximally economic. Therefore they have $F_k = 1$ for all k, like the nested array.

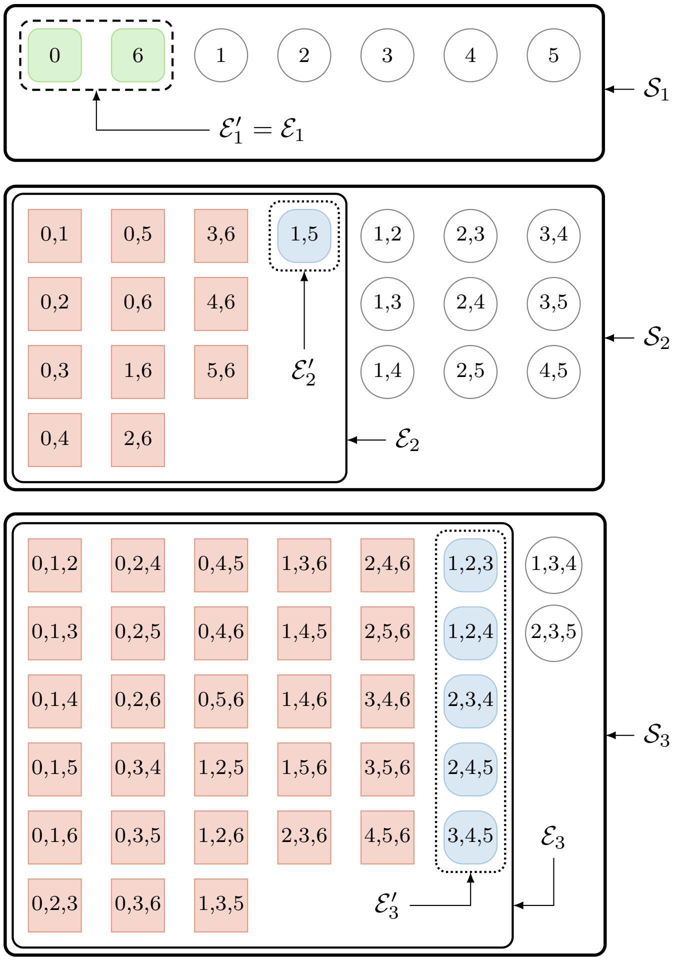

Fig. 9. An example of the underlying structure of k-essential family $\mathcal{E}_k$. Here the ULA with 7 sensors, $\mathbb{S} = \{0, 1, \ldots, 6\}$, is considered while the numbers in each small box denote a subarray. For instance, "$0, 1, 2$" represents the subarray $\{0, 1, 2\}$.

V. The k-Essential Sperner Family

The concept of the k-essentialness property makes it possible to investigate the failure patterns that modify the difference coarray. However the k-essential family $\mathcal{E}_k$ may contain as many as $\binom{|\mathbb{S}|}{k}$ subarrays of size k. Hence, in general, it is challenging to retrieve information from $\mathcal{E}_k$, for large number of sensors and k. It will be demonstrated through the following example that there exist simple and compact representations of the k-essential family:

Example 7: Here we consider the ULA with 7 physical sensors $\mathbb{S} = \{0, 1, 2, 3, 4, 5, 6\}$. All of the subarrays over $\mathbb{S}$ with size 1, 2, and 3 are depicted in small boxes in Fig. 9. The numbers in the small box denote the contents of the subarray. For instance, "$0, 1, 2$" represent the subarray $\{0, 1, 2\}$. Among these, the subarrays in $\mathcal{E}_k$ are enumerated and shown in shaded boxes. For example, the boxes within $\mathcal{S}_1$ show that 0 and 6 are both essential while 1, 2, 3, 4, and 5 are all inessential. Next, let us focus on the 12 subarrays in the family $\mathcal{E}_2$. It can be observed that $\mathcal{E}_2$ can be partitioned into two parts:

1) *Subarrays that contain essential sensors*. For instance, the subarray $\{0, 1\} \in \mathcal{E}_2$ satisfies $0 \in \{0, 1\}$, where 0 is essential. These subarrays are illustrated in light red rectangles with sharp corners.
2) *Subarrays that do not contain essential sensors*. For example, $\{1, 5\} \in \mathcal{E}_2$ but 1 and 5 are both inessential. This subarray is depicted in a light blue rectangle with rounded corners.

Furthermore, every subarray in Part 1 of $\mathcal{E}_2$ can be obtained by combining an essential sensor and another sensor in $\mathbb{S}$. For instance, $\{0, 1\}$ is constructed from the essential element 0 and the inessential element 1. As another example, the subarray $\{0, 6\} \in \mathcal{E}_2$ is composed of two essential elements 0 and 6.

The above discussion indicates that $\mathcal{E}_2$ can be characterized by

1) $\{0\}, \{6\} \in \mathcal{E}_1$ (essential sensors), and
2) $\{1, 5\}$ (those belonging to $\mathcal{E}_2$ but not containing essential sensors),

without listing all the 12 subarrays in $\mathcal{E}_2$. This decomposition results in a compact representation of $\mathcal{E}_2$, where only 3 subarrays ($\{0\}, \{6\}, \{1, 5\}$) are recorded.

Similarly, in Fig. 9, the same technique can be utilized in $\mathcal{E}_3$, which is decomposed into 1) subarrays that include the elements in $\mathcal{E}_2$, as depicted in light red rectangles with sharp corners, and 2) those that do not, as illustrated in light blue rectangles with rounded corners. In particular, the second part of $\mathcal{E}_3$ is grouped by a dashed box and denoted by the family $\mathcal{E}_3'$. This second part of $\mathcal{E}_k$, called the *k-essential Sperner family*, is formally defined next. The name comes from Sperner theory in discrete mathematics [37], [39] as elaborated later.

Definition 8: Let $\mathcal{E}_k$ be the k-essential family with respect to the array $\mathbb{S}$, where the integer k satisfies $1 \leq k \leq |\mathbb{S}|$. The k-essential Sperner family $\mathcal{E}_k'$ is defined as follows:

$$\mathcal{E}_k' \triangleq \begin{cases} \mathcal{E}_1, & \text{if } k = 1, \quad \text{(24a)} \\ \{\mathbb{A} \in \mathcal{E}_k : \forall \mathbb{B} \in \mathcal{E}_{k-1},\ \mathbb{B} \not\subset \mathbb{A}\}, & \text{otherwise}, \quad \text{(24b)} \end{cases}$$

where $\mathbb{B} \not\subset \mathbb{A}$ denotes that $\mathbb{B}$ is not a proper subset of $\mathbb{A}$.

Note that the definition $\mathcal{E}_1' = \mathcal{E}_1$ is introduced such that $\mathcal{E}_k'$ is well-defined for all $1 \leq k \leq |\mathbb{S}|$.

As one of the advantages, the k-essential Sperner family $\mathcal{E}_k'$ could compress $\mathcal{E}_k$ significantly, which would be quite useful especially when the size of $\mathcal{E}_k$ is huge. The example in Fig. 9 displays the k-essential Sperner family $\mathcal{E}_1'$, $\mathcal{E}_2'$, and $\mathcal{E}_3'$. It can be deduced that the sizes of the k-essential Sperner family $|\mathcal{E}_1'| = 2$, $|\mathcal{E}_2'| = 1$, and $|\mathcal{E}_3'| = 5$ are much smaller than those of the k-essential family $|\mathcal{E}_1| = 2$, $|\mathcal{E}_2| = 12$, and $|\mathcal{E}_3| = 33$.

Definition 9 shows that $\{\mathcal{E}_1', \mathcal{E}_2', \ldots, \mathcal{E}_{|\mathbb{S}|}'\}$ can be uniquely determined from $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{|\mathbb{S}|}\}$. Conversely, if $\{\mathcal{E}_1', \mathcal{E}_2', \ldots, \mathcal{E}_{|\mathbb{S}|}'\}$ is given, then $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_{|\mathbb{S}|}\}$ can be perfectly reconstructed due to the following lemma:

Lemma 5: Let $\mathcal{E}_k'$ be the k-essential Sperner family of $\mathbb{S}$ with $1 \leq k \leq |\mathbb{S}|$. Then the k-essential family $\mathcal{E}_k$ satisfies

$$\mathcal{E}_k = \begin{cases} \mathcal{E}_1', & \text{if } k = 1, \quad \text{(25a)} \\ \{\mathbb{A} \cup \mathbb{B} : \mathbb{A} \in \mathcal{E}_\ell',\ 1 \leq \ell \leq k, \\ \qquad \mathbb{B} \subseteq \mathbb{S} \backslash \mathbb{A},\ |\mathbb{B}| = k - \ell\}, & \text{otherwise}. \quad \text{(25b)} \end{cases}$$

For instance, as in Fig. 9, the 3-essential subarray $\{1, 2, 5\}$ can be decomposed into $\mathbb{A} \cup \mathbb{B}$, where $\mathbb{A} = \{1, 5\} \in \mathcal{E}_2'$ and $\mathbb{B} = \{2\} \subseteq \mathbb{S} \backslash \mathbb{A} = \{0, 2, 3, 4, 6\}$. Another example is

$\{0,3,6\}$, which corresponds to either $\mathbb{A} = \{0\} \in \mathcal{E}'_1, \mathbb{B} = \{3,6\} \subseteq \mathbb{S}\backslash\mathbb{A}$ or $\mathbb{A} = \{6\} \in \mathcal{E}'_1, \mathbb{B} = \{0,3\} \subseteq \mathbb{S}\backslash\mathbb{A}$.

Proof of Lemma 5: Eq. (25a) follows from (24a) directly, so it suffices to prove (25b). Let $\mathbb{C}_0 \in \mathcal{E}_k$. If $\mathbb{C}_0 \in \mathcal{E}'_k$, then $\mathbb{C}_0$ is trivially included in (25b). If $\mathbb{C}_0 \notin \mathcal{E}'_k$, due to Definition 8, there exists $\mathbb{C}_1 \in \mathcal{E}_{k-1}$ such that $\mathbb{C}_1 \subset \mathbb{C}_0$. The same argument for $\mathbb{C}_1$ and $\mathcal{E}_{k-1}$ shows that either $\mathbb{C}_1 \in \mathcal{E}'_{k-1}$ or $\mathbb{C}_1$ is a superset of some $\mathbb{C}_2 \in \mathcal{E}_{k-2}$. Repeating these steps show that $\mathbb{C}_0$ is a superset of some elements in $\mathcal{E}'_\ell$. Next, let us consider the right-hand side of (25b). Since $\mathbb{A} \subseteq \mathbb{A} \cup \mathbb{B} \subseteq \mathbb{S}$ and $\mathbb{A} \in \mathcal{E}'_\ell \subseteq \mathcal{E}_\ell$, we have $\mathbb{A} \cup \mathbb{B} \in \mathcal{E}_{|\mathbb{A}\cup\mathbb{B}|} = \mathcal{E}_k$, due to Lemma 3. ■

Another advantage of the k-essential Sperner family is that the k-essentialness property of a given subarray $\mathbb{A} \subseteq \mathbb{S}$, can be readily determined from the k-essential Sperner family, without computing the difference coarray or searching within $\mathcal{E}_k$. This can be done by iterating over the elements in $\mathcal{E}'_k$ from $k = 1$ to $k = |\mathbb{S}|$. The subarray $\mathbb{A}$ is reported to be k-essential if there exists $\mathbb{B} \subseteq \mathbb{A}$ for some $\mathbb{B} \in \mathcal{E}'_\ell$ and $1 \le \ell \le k$. As an example, we know that $\{4,5,6\}$ in Fig. 9 is 3-essential since $\{0\} \not\subseteq \{4,5,6\}$ and $\{6\} \subseteq \{4,5,6\}$, where only two comparisons are needed. On the other hand, if we search for $\{4,5,6\}$ within the box of $\mathcal{E}_3$ from top to bottom and then from left to right, then 28 comparisons are required. As another example, $\{1,3,4\}$ can be concluded not to be 3-essential with 8 comparisons using $\mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}'_3$, but with 33 comparisons using $\mathcal{E}_3$. Empirically, the reduction in the number of comparisons is huge especially for large number of sensors and large k. However, the precise analysis of the complexity is beyond the scope of this paper and is left for future work.

The term *Sperner* originates from the Sperner theory in discrete mathematics [37], [39]. A Sperner family is a family of sets in which none of the elements is a subset of the other, which is formally defined as

Definition 9: A family of sets $\mathcal{F}$ is a Sperner family if $\mathbb{A} \not\subset \mathbb{B}$ for all $\mathbb{A}, \mathbb{B} \in \mathcal{F}$ [37].

With Definition 9, we will show an explicit connection between the k-essential Sperner family and the Sperner family, as indicated in Lemma 6:

Lemma 6: The union of any selection of the k-essential Sperner family $\{\mathcal{E}'_1, \mathcal{E}'_2, \ldots, \mathcal{E}'_{|\mathbb{S}|}\}$ is a Sperner family. Namely, $\bigcup_{k\in\mathbb{I}} \mathcal{E}'_k$ is a Sperner family, where $\mathbb{I} \subseteq \{1,2,\ldots,|\mathbb{S}|\}$.

Proof: Let $\mathbb{A}, \mathbb{B} \in \bigcup_{k\in\mathbb{I}} \mathcal{E}'_k$ such that $\mathbb{A} \subset \mathbb{B}$. Here $\mathbb{A} \subset \mathbb{B}$ indicates that $\mathbb{A}$ is a subset of $\mathbb{B}$ and $\mathbb{A} \neq \mathbb{B}$. If $\mathbb{A}, \mathbb{B} \in \mathcal{E}'_k$ for some $k \in \mathbb{I}$, then $|\mathbb{A}| = |\mathbb{B}| = k$, violating $\mathbb{A} \subset \mathbb{B}$. Assume that $\mathbb{A} \in \mathcal{E}'_{k_1}$ and $\mathbb{B} \in \mathcal{E}'_{k_2}$ for some $k_1, k_2 \in \mathbb{I}$ and $k_1 < k_2$. Let $\mathbb{C}$ be a subset of $\mathbb{B}\backslash\mathbb{A}$ with size $|\mathbb{C}| = k_2 - k_1 - 1$. Since $\mathbb{A} \in \mathcal{E}'_{k_1} \subseteq \mathcal{E}_{k_1}$ and $\mathbb{A} \subseteq \mathbb{A} \cup \mathbb{C} \subseteq \mathbb{S}$, Lemma 3 indicates that, $\mathbb{A} \cup \mathbb{C} \in \mathcal{E}_{k_2-1}$. However $\mathbb{A} \cup \mathbb{C} \subset \mathbb{B}$, contradicting (24b). ■

As an example of Lemma 6, if $\mathbb{I} = \{2,3\}$ and $\mathcal{E}'_k$ is given in Fig. 9, then $\mathcal{E}'_2 \cup \mathcal{E}'_3$ contains $\{1,5\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{2,3,4\}$, $\{2,4,5\}$, and $\{3,4,5\}$, where none of the elements in $\mathcal{E}'_2 \cup \mathcal{E}'_3$ is a superset of another. Hence $\mathcal{E}'_2 \cup \mathcal{E}'_3$ is a Sperner family.

Furthermore, Lemma 6 connects the essentialness property, the fragility, and the k-essential (Sperner) family, with the well-established results in Sperner theory, such as Sperner's theorem [39], the Lubell-Yamamoto-Meshalkin inequality (the LYM inequality) [40]–[43], and the Ahlswede-Zhang identity (the AZ identity) [44]. Interested readers are referred to [37] for more details.

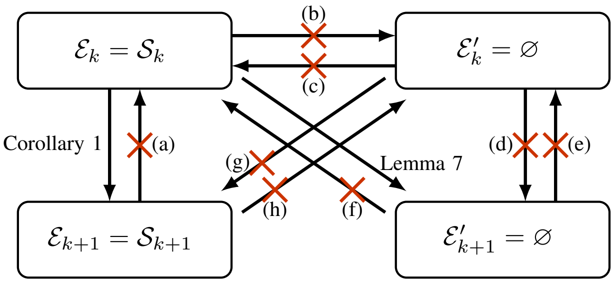

Fig. 10. The relation between $\mathcal{E}_k = \mathcal{S}_k$ and $\mathcal{E}'_k = \varnothing$. Here solid arrows represent logical implication while arrows with red crosses mean that one condition does not necessarily imply the other.

Similar to Corollary 1, the following show the relations between the equality $\mathcal{E}_k = \mathcal{S}_k$ and the emptiness of $\mathcal{E}'_k$. These results will be quite useful in studying the probability that the difference coarray changes in Section VI and the k-essentialness property for several array configurations in the companion paper [29].

Lemma 7: Let $\varnothing$ denote the empty set. Assume that the integer k satisfies $1 \le k \le |\mathbb{S}|-1$. If $\mathcal{E}_k = \mathcal{S}_k$, then $\mathcal{E}'_{k+1} = \varnothing$.

Lemma 8: Let $\mathcal{E}'_k$ be the k-essential Sperner family of an array $\mathbb{S}$. Then $\mathcal{E}'_k = \varnothing$ for all $Q + 1 \le k \le |\mathbb{S}|$, where Q is defined in Property 2 of Theorem 1.

These lemmas can be proved readily according to Definition 8, Property 2 of Theorem 1, and Lemma 7.

Fig. 10 summarizes the logical relation between $\mathcal{E}_k = \mathcal{S}_k$ and $\mathcal{E}'_k = \varnothing$ in detail. Here Corollary 1 and Lemma 7 are denoted by solid arrows while arrows with red crosses (Cases (a) to (h)) indicate that one condition does not imply the other. The counter examples for Cases (a) to (h) are listed as follows. If $\mathbb{S} = \{0,1,3,4,5,6,7,8,10\}$, then the k-essential family and the k-essential Sperner family become

$$\mathcal{E}_1 \neq \mathcal{S}_1, \qquad \mathcal{E}'_1 = \{\{0\},\{1\},\{8\},\{10\}\} \neq \varnothing, \tag{26}$$
$$\mathcal{E}_2 \neq \mathcal{S}_2, \qquad \mathcal{E}'_2 = \varnothing, \tag{27}$$
$$\mathcal{E}_3 \neq \mathcal{S}_3, \qquad \mathcal{E}'_3 = \{\{3,5,6\},\{4,6,7\}\} \neq \varnothing, \tag{28}$$
$$\mathcal{E}_4 = \mathcal{S}_4, \qquad \mathcal{E}'_4 = \{\{3,4,5,7\}\} \neq \varnothing, \tag{29}$$
$$\mathcal{E}_5 = \mathcal{S}_5, \qquad \mathcal{E}'_5 = \varnothing. \tag{30}$$

Counter examples for Cases (a) to (h) can be found in (26) to (30). For instance, $\mathcal{E}_4 = \mathcal{S}_4$ but $\mathcal{E}_3 \neq \mathcal{S}_3$, which contradicts (a). Furthermore, the case of $\mathcal{E}_4$ and $\mathcal{E}'_4$ contradicts (b); the instance of $\mathcal{E}_2$ and $\mathcal{E}'_2$ contradicts (c). The example of $\mathcal{E}'_1$, $\mathcal{E}'_2$, and $\mathcal{E}'_3$ disapproves Cases (d) and (e) while $\mathcal{E}_1$, $\mathcal{E}'_2$, and $\mathcal{E}_3$ contradict Cases (f) and (g). Case (h) has a counter example of $\mathcal{E}'_4$ and $\mathcal{E}_5$. These examples confirm that Cases (a) to (h) are not necessarily true.

VI. Robustness Analysis for Random Sensor Failures

In this section, we assume that the sensors in an array have a certain probability of failure, and derive an expression for the probability that the difference coarray will change due to this failure. We will show that the concepts introduced in this paper, such as k-essentialness and fragility, play a crucial role in this analysis. As explained earlier, the importance of this analysis arises from the fact that the robustness of the difference coarray (to sensor failures) is crucial for the success of algorithms such as coarray MUSIC.

Assumptions: In this section, let a sensor array be $\mathbb{S}$ and the difference coarray be $\mathbb{D}$. Assume that ***each sensor fails independently with probability*** p. After the removal of faulty sensors, the array and the difference coarray are denoted by $\overline{\mathbb{S}}$ and $\overline{\mathbb{D}}$, respectively. Then the probability that $\overline{\mathbb{D}} \neq \mathbb{D}$ is denoted by

$$P_c \triangleq \Pr[\overline{\mathbb{D}} \neq \mathbb{D}]. \tag{31}$$

An array is more robust, as P_c is close to 0. This property can also be used in comparing the robustness among several array configurations.

Note that P_c is different from the k-fragility F_k, even though they both correspond to the concept of probability. As presented in Section IV, if there are k faulty sensors in the array and all possible failure patterns are equiprobable, the k-fragility F_k can be interpreted as the probability that the difference coarray changes. On the other hand, P_c denotes the probability that the difference coarray changes, due to any possible sensor failure pattern. Furthermore, F_k depends purely on the array geometry while P_c depends on the array geometry and the failure probability of each sensor. In practice, P_c is more useful since 1) it does not require the information of the number of faulty sensors and 2) the parameter p, which determines the quality and the cost of the sensing device, can be designed based on the budget.

Next, we will present a closed-form relation between P_c and F_k. Let $\mathbb{A} \subseteq \mathbb{S}$ be the set of faulty sensors. Assume that $\overline{\mathbb{S}} \triangleq \mathbb{S}\backslash\mathbb{A}$ and the associated difference coarray $\overline{\mathbb{D}}$. Due to Definition 5, the difference coarray changes ($\overline{\mathbb{D}} \neq \mathbb{D}$) if and only if there exist $1 \leq k \leq |\mathbb{S}|$ and $\mathbb{A} \in \mathcal{E}_k$ such that 1) all the elements in $\mathbb{A} \in \mathcal{E}_k$ fail and 2) all the elements in $\overline{\mathbb{S}}$ are operational. Summing over all possible k and $\mathbb{A}$ leads to the following expression of P_c:

$$\begin{aligned} P_c &= \sum_{k=1}^{|\mathbb{S}|} \sum_{\mathbb{A}\in\mathcal{E}_k} \Pr\left[\left(\bigcap_{n_1\in\mathbb{A}} (n_1 \text{ fails})\right) \cap \left(\bigcap_{n_2\in\overline{\mathbb{S}}} (n_2 \text{ fails})^c\right)\right] \\ &= \sum_{k=1}^{|\mathbb{S}|} \sum_{\mathbb{A}\in\mathcal{E}_k} \left[\prod_{n_1\in\mathbb{A}} \Pr[n_1 \text{ fails}]\right]\left[\prod_{n_2\in\overline{\mathbb{S}}} (1-\Pr[n_2 \text{ fails}])\right] \\ &= \sum_{k=1}^{|\mathbb{S}|} |\mathcal{E}_k| p^k (1-p)^{|\mathbb{S}|-k}, \end{aligned} \tag{32}$$

where the second equation is due to the independence of sensor failures. The complement of an event $\mathfrak{F}$ is denoted by $\mathfrak{F}^c$. Substituting Definition 7 into (32) leads to

$$P_c = \sum_{k=1}^{|\mathbb{S}|} \binom{|\mathbb{S}|}{k} F_k p^k (1-p)^{|\mathbb{S}|-k}, \tag{33}$$

where F_k is the k-fragility of $\mathbb{S}$.

Note that (33) shows the explicit relation between F_k and P_c, which holds for any array configuration $\mathbb{S}$. Here each term in (33) has two contributions: F_k and $\binom{|\mathbb{S}|}{k}p^k(1-p)^{|\mathbb{S}|-k}$. F_k depends purely on the array geometry while $\binom{|\mathbb{S}|}{k}p^k(1-p)^{|\mathbb{S}|-k}$ relies on k, the number of sensors $|\mathbb{S}|$, and p. This observation means that, for a fixed number of sensors and a fixed p, it is possible to reduce P_c by designing new array geometries with reduced F_k. On the other hand, for a fixed array configuration, F_k is uniquely determined. In this case, it can be shown that P_c decreases with p, as p is sufficiently small. Namely, to reduce P_c, we can deploy sensing devices with small p.

However, the right-hand side of (33) is not computationally tractable. For instance, if k is approximately $|\mathbb{S}|/2$, the complexity for evaluating F_k is around $\binom{|\mathbb{S}|}{|\mathbb{S}|/2}$, which becomes computationally expensive for large $|\mathbb{S}|$. Even so, the behavior of P_c can still be analyzed based on the following theorem:

Theorem 3: The probability that the difference coarray changes satisfies $\max\{L_1, L_2\} \leq P_c \leq \min\{U_1, U_2, 1\}$, where L_1, U_1, L_2, and U_2 are given by

$$L_1 = 1-(1-p)^{|\mathbb{S}|} - \left(1-\frac{2}{|\mathbb{S}|}\right)\sum_{k=1}^{Q-1}\binom{|\mathbb{S}|}{k}p^k(1-p)^{|\mathbb{S}|-k}, \tag{34}$$

$$U_1 = 1-(1-p)^{|\mathbb{S}|}, \tag{35}$$

$$L_2 = 1-(1-p)^{|\mathcal{E}_1|}, \tag{36}$$

$$U_2 = 1-(1-p)^{|\mathcal{E}_1|} + (1-p)^{|\mathcal{E}_1|}\sum_{k=2}^{Q}|\mathcal{E}'_k|p^k. \tag{37}$$

Here the parameter Q is given in Property 2 of Theorem 1. The notation $\mathcal{E}_k$ and $\mathcal{E}'_k$ represent the k-essential family and the k-essential Sperner family for the sensor array $\mathbb{S}$.

Proof: First we will show that $L_1 \leq P_c \leq U_1$. Property 3 of Theorem 2 indicates that P_c is upper bounded by $\sum_{k=1}^{|\mathbb{S}|}\binom{|\mathbb{S}|}{k}p^k(1-p)^{|\mathbb{S}|-k} = 1-(1-p)^{|\mathbb{S}|} = U_1$, which proves (35). For the lower bound, if $|\mathbb{S}| = 1$, then it can be shown that $P_c = p = L_1$. If $|\mathbb{S}| \geq 2$, then Properties 2 and 3 of Theorem 2 imply that $F_k \geq 2/|\mathbb{S}|$ for $k = 1, 2, \ldots, Q-1$ and $F_k = 1$ otherwise. Substituting these relations into (33) proves (34).

The proof of Eqs. (36) and (37) is as follows. Let the sensor array be $\mathbb{S}$ and the k-essential Sperner family be $\mathcal{E}'_k$. Assume that each sensor fails independently with probability p. Let $\mathbb{B}$ be the set of faulty sensors. Assume that $\overline{\mathbb{S}} \triangleq \mathbb{S}\backslash\mathbb{B}$ and its difference coarray is denoted by $\overline{\mathbb{D}}$. Since a subarray $\mathbb{B}$ is k-essential if and only if $\mathbb{B}$ is a superset of some elements in $\mathcal{E}'_\ell$ for some $1 \leq \ell \leq k$, as in (25b), it suffices to consider all elements in $\mathcal{E}'_1, \mathcal{E}'_2, \ldots, \mathcal{E}'_{|\mathbb{S}|}$ and the probability that $\overline{\mathbb{D}} \neq \mathbb{D}$ becomes

$$\begin{aligned} P_c &= \Pr\left[\overline{\mathbb{D}} \neq \mathbb{D}\right] = \Pr\left[\bigcup_{k=1}^{|\mathbb{S}|}\bigcup_{\mathbb{A}_k\in\mathcal{E}'_k}\mathfrak{F}(\mathbb{A}_k)\right] \\ &= \Pr\Bigg[\underbrace{\left(\bigcup_{\mathbb{A}_1\in\mathcal{E}'_1}\mathfrak{F}(\mathbb{A}_1)\right)}_{\text{Event } \mathfrak{G}_1} \cup \underbrace{\left(\bigcup_{k=2}^{|\mathbb{S}|}\bigcup_{\mathbb{A}_k\in\mathcal{E}'_k}\mathfrak{F}(\mathbb{A}_k)\right)}_{\text{Event } \mathfrak{G}_2}\Bigg], \end{aligned} \tag{38}$$

where $\mathfrak{F}(\mathbb{A}_k) \triangleq \cap_{n\in\mathbb{A}_k}(n \text{ fails})$ denotes the event in which all the elements in $\mathbb{A}_k$ fail. Since the event $\mathfrak{G}_1$ involves

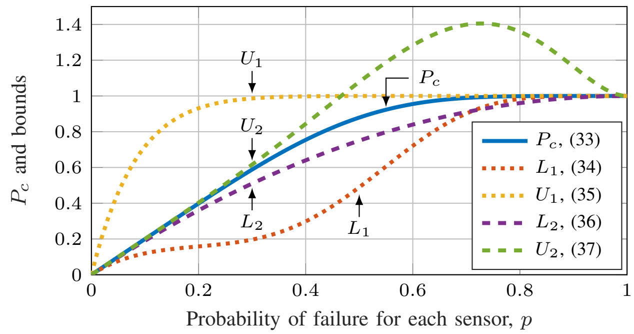

Fig. 11. The probability that the difference coarray changes P_c and its lower bounds and upper bounds for the ULA with 12 sensors.

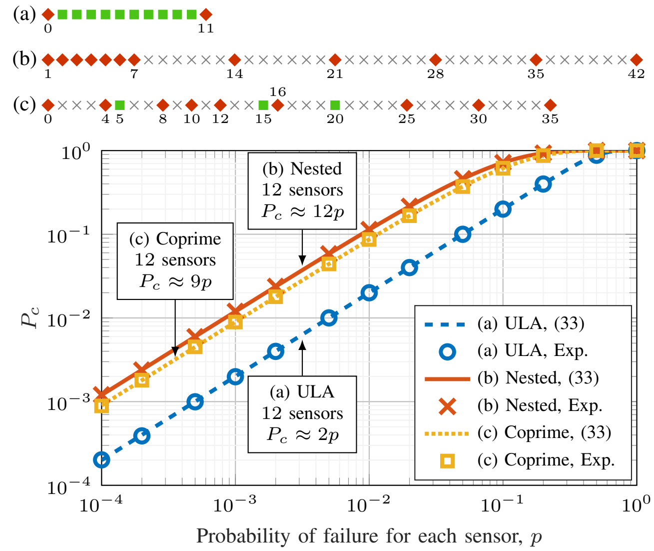

Fig. 12. The dependence of the probability P_c that the difference coarray changes, on the probability of single sensor failure p for (a) the ULA with 12 sensors, (b) the nested array with $N_1 = N_2 = 6$, and (c) the coprime array with $M = 4$ and $N = 5$. Here the essential sensors (diamonds) and the inessential sensors (squares) are depicted on the top of this figure. Experimental data points (Exp.) are averaged from 10^7 Monte-Carlo runs. The approximations of P_c are valid for $p \ll 1/12$ due to (41).

only the essential elements and $\mathfrak{G}_2$ are associated with inessential sensors, $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are independent. Namely, $\Pr[\mathfrak{G}_1 \cap \mathfrak{G}_2] = \Pr[\mathfrak{G}_1]\Pr[\mathfrak{G}_2]$. Hence $\Pr[\mathfrak{G}_1 \cup \mathfrak{G}_2] = 1 - \Pr[\mathfrak{G}_1^c] + \Pr[\mathfrak{G}_1^c]\Pr[\mathfrak{G}_2]$, where $\mathfrak{G}_1^c$ is the complement of the event $\mathfrak{G}_1$. The probability $\Pr[\mathfrak{G}_1^c]$ can be simplified as

$$\Pr[\mathfrak{G}_1^c] = \Pr\left[\bigcap_{\mathbb{A}_1 \in \mathcal{E}_1'} (\mathfrak{F}(\mathbb{A}_1))^c\right] = (1-p)^{|\mathcal{E}_1|}. \tag{39}$$

Applying the union bound of $\Pr[\mathfrak{G}_2]$ leads to

$$0 \leq \Pr[\mathfrak{G}_2] \leq \sum_{k=2}^{|\mathbb{S}|} \sum_{\mathbb{A}_k \in \mathcal{E}_k'} \Pr[\mathfrak{F}(\mathbb{A}_k)] = \sum_{k=2}^{|\mathbb{S}|} |\mathcal{E}_k'| p^k. \tag{40}$$

Substituting (39), (40), and Lemma 8 into $P_c = 1 - \Pr[\mathfrak{G}_1^c] + \Pr[\mathfrak{G}_1^c]\Pr[\mathfrak{G}_2]$ proves (36) and (37). ■

It can be observed that all these expressions (34) to (37) do not require the complete knowledge of F_k. For instance, U_1 depends only on the probability of single sensor failure p and the number of sensors, while L_2 requires p and the size of $\mathcal{E}_1$. The bounds L_1 and U_2 are functions of the parameter Q, as given in Property 2 of Theorem 1. If Q is much smaller than the number of sensors, then U_2 can be evaluated efficiently with the first few $\mathcal{E}_k'$.

Example 8: Next we will demonstrate an example for the bounds in Theorem 3. Fig. 11 shows the curves of P_c, L_1, U_1, L_2, and U_2 for the ULA with $N = 12$ physical sensors, as a function of p. First it can be observed that the bounds L_1 and U_1 are close to P_c for $p \geq 0.8$ while for small p, the bounds L_2 and U_2 are tighter than L_1 and U_1. Second, in this example, the bound U_2 is greater than 1 for $p \geq 0.5$, which becomes a trivial upper bound for P_c. This is because the term $\sum_{k=2}^{Q} |\mathcal{E}_k'| p^k$ in (37) is derived from the union bound of the probability, which could be greater than 1.

The bounds in Theorem 3 also makes it possible to derive approximations for P_c. For fixed number of sensors, if $p \ll 1/|\mathbb{S}|$, then the high-order terms $\sum_{k=2}^{Q} |\mathcal{E}_k'| p^k$ in (37) become negligible, since $|\mathcal{E}_k'| \leq \binom{|\mathbb{S}|}{k} = \mathcal{O}(|\mathbb{S}|^k)$. Then we have $L_2 \leq P_c \leq U_2 \approx L_2$. Therefore, for any array geometry $\mathbb{S}$ and $p \ll 1/|\mathbb{S}|$, the probability that the difference coarray changes can be approximated by

$$P_c \approx L_2 = 1 - (1-p)^{|\mathcal{E}_1|} \approx |\mathcal{E}_1| p, \tag{41}$$

since $(1+x)^N \approx 1 + Nx$ for $|x| \ll 1$. Eq. (41) shows that, for small p, the probability P_c is approximately linear in p with slope $|\mathcal{E}_1|$. This result can be verified through the curve of P_c in Fig. 11, where the ULA has $|\mathcal{E}_1| = 2$, as proved in the companion paper [29, (26)].

Note that (41) holds for any array configuration $\mathbb{S}$, which indicates that for the same $p \ll 1/|\mathbb{S}|$, smaller $|\mathcal{E}_1|$ leads to smaller P_c. For instance, due to [29, (26)], the ULA with $N \geq 4$ physical sensors always has $P_c \approx 2p$, even for large N. However this does not hold for MESA, since MESA with N sensors own $P_c \approx Np$, which grows linearly with N. Eq. (41) can also be expressed as $P_c \approx (|\mathbb{S}|p)F_1$. This indicates that, if the number of sensors $|\mathbb{S}|$ and the sensor failure probability p are fixed, then P_c is proportional to fragility F_1.

Example 9: Fig. 12 demonstrates a numerical example for P_c across various array configurations with 12 sensors, such as the ULA with 12 sensors, the nested array with $N_1 = N_2 = 6$, as in (4), and the coprime array with $M = 4$ and $N = 5$, as in (5). The probability that the difference coarray changes is first evaluated based on (33), as depicted in solid, dashed, and dotted curves on the bottom of Fig. 12. Next, these probabilities are also averaged empirically from 10^7 Monte-Carlo runs and each run corresponds to an independent realization of the array geometry with sensor failure probability p. The results based on Monte-Carlo runs are marked in empty circles, crosses, and empty squares on the bottom of Fig. 12.

First, it can be deduced that the experimental results match (33) for all these array configurations. For the same p and the same number of physical sensors, by comparing the values of P_c, the most robust array geometry is the ULA, followed by the coprime array, and finally the nested array. Furthermore, for $p \ll 1/|\mathbb{S}| = 1/12$, the approximations for P_c in Fig. 12 become $P_c \approx 2p$ for ULA, $P_c \approx 12p$ for the nested array, and $P_c \approx 9p$ for the coprime array. These results are consistent with the approximations in (41).

The results in Fig. 12 and the approximations in (41) hold true for more sensors. For instance, let us consider the ULA with 24 sensors, the nested array with $N_1 = N_2 = 12$, and the coprime array with $M = 7$ and $N = 11$. All these arrays have 24 physical sensors, and their P_c can be shown to behave similarly to those in Fig. 12. In particular, P_c can be approximated by $2p$ for ULA, $24p$ for the nested array, and $17p$ for the coprime array, when $p \ll 1/24$.

VII. Concluding Remarks

In this paper, we presented a theory to quantify the robustness of difference coarrays with respect to sensor failures. We began by defining the (k-)essentialness property and the k-essential family. Based on these, the k-fragility characterizes the likelihood that the difference coarray changes, while the k-essential Sperner family offers a compact representation of the k-essential family. Under mild assumptions, the proposed theory explained the behavior of the probability that the difference coarray changes, which is crucial for the functionality of coarray MUSIC.

In the companion paper [29], we will concentrate on the relation between the presented theory and the array geometry. The closed-form expressions of the k-essential Sperner family for ULA, MRA, MHA, Cantor arrays, nested arrays, and coprime arrays, will be derived to provide insights into the importance of each sensor and the robustness of these arrays.

In the future, it is of considerable interest to investigate the interplay between the DOA estimation performance and coarray robustness, which may find applications in practical systems using sparse arrays. Another future topic is to quantify the robustness of sparse arrays with respect to the central ULA segment in the difference coarray, which affects the applicability of DOA estimators such as coarray MUSIC.

As a final remark, the essentialness property can be reformulated to study the robustness of sparse arrays in various problems. For instance, the performance of MIMO radar [45] depends on the sum coarray while the $2q$th-order difference coarray [46] plays a critical role in DOA estimation with $2q$th-order cumulants. In addition, the proposed theory can be extended to two-dimensional sparse arrays and their difference coarrays, which are capable of resolving both the azimuth and the elevation of the source. It will be interesting to investigate the robustness of the coarray in these scenarios.

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