# Robustness of Difference Coarrays of Sparse Arrays to Sensor Failures – Part II: Array Geometries

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***Abstract*—In array processing, sparse arrays are capable of resolving $\mathcal{O}(N^2)$ uncorrelated sources with $N$ sensors. Sparse arrays have this property because they possess uniform linear array (ULA) segments of size $\mathcal{O}(N^2)$ in the difference coarray, defined as the differences between sensor locations. However, the coarray structure of sparse arrays is susceptible to sensor failures and the reliability of sparse arrays remains a significant but challenging topic for investigation. In the companion paper, a theory of the $k$-essential family, the $k$-fragility, and the $k$-essential Sperner family were presented not only to characterize the patterns of $k$ faulty sensors that shrink the difference coarray, but also to provide a number of insights into the robustness of arrays. This paper derives closed-form characterizations of the $k$-essential Sperner family for several commonly used array geometries, such as ULA, minimum redundancy arrays (MRA), minimum holes arrays (MHA), Cantor arrays, nested arrays, and coprime arrays. These results lead to many insights into the relative importance of each sensor, the robustness of these arrays, and the DOA estimation performance in the presence of sensor failure. Broadly speaking, ULAs are more robust than coprime arrays, while coprime arrays are more robust than maximally economic sparse arrays, such as MRA, MHA, Cantor arrays, and nested arrays.**

***Index Terms*—Sparse arrays, difference coarrays, the $k$-essentialness property, the $k$-fragility, the $k$-essential Sperner family.**

## I. Introduction

Sparse arrays [1]–[4], such as minimum redundancy arrays (MRA) [2], nested arrays [3], coprime arrays [4], and their generalizations [5], can resolve $\mathcal{O}(N^2)$ uncorrelated sources using $N$ physical elements. This $\mathcal{O}(N^2)$ property arises because the difference coarray, defined as the differences between the sensor locations, possesses an $\mathcal{O}(N^2)$-long central uniform linear array (ULA) segment. However, as far as the system reliability is concerned [6], [7], in the past, sparse arrays were considered not to be robust to sensor failures, due to empirical observations. More details on this argument can be found in [8], [9] and the references therein.

In the companion paper [9], the concepts such as the $k$-essential family, the $k$-fragility, and the $k$-essential Sperner family were proposed to assess the robustness of difference coarrays of sparse arrays to sensor failures. A subarray of size $k$ is said to be $k$-essential if its deletion changes the difference coarray. All these $k$-essential subarrays constitute the $k$-essential family. With this tool, the robustness can be quantified by the $k$-fragility, or simply *fragility*, which ranges from 0 to 1. An array is more robust or less fragile if the fragility is closer to 0. However, from the computational perspective, the size of the $k$-essential family can be as large as $\binom{N}{k}$, where $N$ is the number of physical elements. It was shown in the companion paper [9] that the $k$-essential family can be compactly represented by the $k$-essential Sperner family. With these tools, the system reliability can be quantified by the probability that the difference coarray changes, $P_c$, under the assumption of random sensor failures. Many insights into the interplay between the proposed theory and $P_c$ were also offered in the companion paper [9].

The main contribution of this paper is to analyze the robustness of several commonly used array configurations, such as the ULA [1], MRA [2], minimum hole arrays (MHA) [10], Cantor arrays [11], [12], nested arrays [3], and coprime arrays [4], based on the theory in the companion paper [9]. These arrays are widely used in various topics of array signal processing, such as beamforming [1], [3], [11], [13], [14] and DOA estimation [3]–[5], [15]. However the robustness of the difference coarrays of these arrays to sensor failures remains an open but significant topic in this field. It will be shown in this paper that MRA, MHA, Cantor arrays, and nested arrays are maximally economic, that is, any sensor failure changes the difference coarray. It will also be proved that the fragility and $P_c$ for maximally economic sparse arrays (MESA) are the largest among all arrays with a fixed number of sensors. These theoretical results confirm the empirical observation that MRA are not robust to sensor failures, in terms of the preservation of the difference coarray.

In this paper, the closed-form expressions of the $k$-essential Sperner family for ULA and coprime arrays are also established with detailed derivations. These expressions lead to a number of contributions. First, it can be proved that, for sufficiently large number of sensors, ULA are more robust than MESA and coprime arrays (in terms of the fragility), which is in accordance with the observation that sparse arrays are in general less robust than ULA. Furthermore, the explicit expressions of the $k$-essential Sperner family for the coprime array *allow one to construct arrays with fewer sensors but with the same difference coarray as the coprime array.* Note that this topic was previously addressed in the thinned coprime array [16], where a specific selection of sensors is removed from the coprime array. Using the expressions we propose in this paper, it can be shown that there exist other array geometries that achieve the same difference coarray as the thinned coprime array.

It is also demonstrated through numerical examples that,

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the DOA estimation performance of arrays is influenced by the trade-offs between the size and the robustness of the difference coarray. For this, a number of sparse arrays are compared, with a fixed failure probability $p$ for each sensor, and fixed number of sensors. It will be deduced in the examples that, for small $p$, the MRA has the best DOA estimation performance, due to the largest difference coarray, while for large $p$, the ULA owns the best performance because of its robustness. An interesting observation is that, for moderate $p$, the coprime array could outperform the ULA, the MRA, and the nested array, since the coprime array strikes a balance between the size and the robustness of the difference coarray.

In the literature, more general sparse array configurations have been reported. For instance, the generalized coprime arrays [5] have recently received considerable attention. They extend coprime arrays by two operations: compressions and displacements. In principle, the robustness of other array configurations could be analyzed using the theory in the companion paper [9], but the details would be very involved. Due to page limitations, the robustness analysis of these arrays is left for future.

*Paper outline:* Section II gives a quick review of some of the key results from the companion paper [9]. Sections III, IV, and V study the $k$-essential Sperner family for MESA, ULA, and coprime arrays, respectively, along with examples, discussions, and proofs. The performance of these arrays in the presence of sensor failure is demonstrated in Section VI while Section VII concludes this paper. Parts of the results were presented in a conference paper [12].

## II. Review of the Essentialness Property

Consider a linear array whose sensors are located at $nd$. Here $n$ belongs to an integer set $\mathbb{S}$ and $d$ is half of the wavelength of the incoming monochromatic, far-field, and uncorrelated sources. Under these assumptions, the source directions can be resolved according to the difference coarray and the weight function [3], [4], [17]–[19]:

*Definition 1:* The difference coarray of the sensor array $\mathbb{S}$ is defined as $\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$.

*Definition 2:* The weight function $w(m)$ of a linear array $\mathbb{S}$ is defined as the number of sensor pairs with coarray index $m$. That is, $w(m) = \left|\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\}\right|$.

Furthermore, the central ULA segment of the difference coarray, denoted by $\mathbb{U}$, is defined as the largest ULA in $\mathbb{D}$ that contains the element 0, i.e., $\mathbb{U} \triangleq \{m : \{0, 1, \dots, |m|\} \subseteq \mathbb{D}\}$.

Based on this concept, the companion paper [9] studies the influence of sensor failures on the difference coarray $\mathbb{D}$, as we will review next:

*Definition 3:* The sensor located at $n \in \mathbb{S}$ is said to be essential with respect to $\mathbb{S}$ if the difference coarray changes when sensor at $n$ is deleted from the array. That is, if $\overline{\mathbb{S}} = \mathbb{S}\backslash\{n\}$, then $\overline{\mathbb{D}} \neq \mathbb{D}$. Here $\mathbb{D}$ and $\overline{\mathbb{D}}$ are the difference coarrays for $\mathbb{S}$ and $\overline{\mathbb{S}}$, respectively.

Note that the essentialness property in Definition 3 assumes one faulty element at a time. A more realistic situation is the case of *multiple sensor failures*.

*Definition 4:* The subarray of size $k$ over an integer set $\mathbb{S}$ is defined as $\mathcal{S}_k \triangleq \{\mathbb{A} \subseteq \mathbb{S} : |\mathbb{A}| = k\}$.

*Definition 5:* A subarray $\mathbb{A}$ is said to be $k$-essential if 1) $\mathbb{A} \in \mathcal{S}_k$, and 2) the difference coarray changes when $\mathbb{A}$ is removed from $\mathbb{S}$.

*Definition 6:* The $k$-essential family $\mathcal{E}_k$ with respect to a sensor array $\mathbb{S}$ is defined as

$$\mathcal{E}_k \triangleq \{\mathbb{A} : \mathbb{A} \text{ is } k\text{-essential with respect to } \mathbb{S}\}, \tag{1}$$

where $k = 1, 2, \dots, |\mathbb{S}|$.

With this tool, the robustness of a linear array can be quantified by the $k$-fragility, or simply the fragility:

*Definition 7:* The fragility or $k$-fragility of a sensor array $\mathbb{S}$ is defined as

$$F_k \triangleq \frac{|\mathcal{E}_k|}{|\mathcal{S}_k|} = \frac{|\mathcal{E}_k|}{\binom{|\mathbb{S}|}{k}}, \tag{2}$$

where $k = 1, 2, \dots, |\mathbb{S}|$.

It was shown in the companion paper [9] that $F_k$ is an increasing function of $k$ and $\min\{1, 2/|\mathbb{S}|\} \leq F_k \leq 1$. As $F_k$ becomes closer to 1, the array $\mathbb{S}$ is less robust (or more fragile) to sensor failures in the sense of changing the difference coarray.

Based on the underlying structure of $\mathcal{E}_k$, the $k$-essential family $\mathcal{E}_k$ can be compactly represented by the $k$-essential Sperner family $\mathcal{E}'_k$ [9]:

*Definition 8:* Let $\mathcal{E}_k$ be the $k$-essential family with respect to the array $\mathbb{S}$, where the integer $k$ satisfies $1 \leq k \leq |\mathbb{S}|$. The $k$-essential Sperner family $\mathcal{E}'_k$ is defined as follows:

$$\mathcal{E}'_k \triangleq \begin{cases} \mathcal{E}_1, & \text{if } k = 1, \quad \text{(3a)} \\ \{\mathbb{A} \in \mathcal{E}_k : \forall \mathbb{B} \in \mathcal{E}_{k-1},\ \mathbb{B} \not\subset \mathbb{A}\}, & \text{otherwise}, \quad \text{(3b)} \end{cases}$$

where $\mathbb{B} \not\subset \mathbb{A}$ denotes that $\mathbb{B}$ is not a proper subset of $\mathbb{A}$. Here $\mathbb{P}$ being a proper subset of $\mathbb{Q}$ means that $\mathbb{P}$ is a subset of $\mathbb{Q}$ and $\mathbb{P} \neq \mathbb{Q}$.

Finally, let us consider the case where each element in a linear array $\mathbb{S}$ fails independently with probability $p$ [9]. Assume that the faulty sensors constitute a set $\mathbb{A}$. The set $\overline{\mathbb{S}} \triangleq \mathbb{S}\backslash\mathbb{A}$ denotes the set of the operational sensors. The difference coarrays of $\mathbb{S}$ and $\overline{\mathbb{S}}$ are expressed as $\mathbb{D}$ and $\overline{\mathbb{D}}$, respectively. As discussed in the companion paper [9], the system reliability can be studied through the probability that the difference coarray changes, namely, $P_c \triangleq \Pr[\overline{\mathbb{D}} \neq \mathbb{D}]$. It was shown in [9, (33)] that $P_c$ can be expressed in terms of the number of sensors, the probability $p$, and the $k$-fragility $F_k$.

The main contribution of this paper is to apply the above-mentioned theory to several commonly used array geometries, such as minimum redundancy arrays (MRA), minimum hole arrays (MHA), nested arrays, Cantor arrays, uniform linear arrays (ULA), and coprime arrays, to assess the robustness. In what follows, the closed-form expressions of the $k$-essential Sperner family $\mathcal{E}'_k$, the $k$-fragility $F_k$, and the probability $P_c$ that the difference coarray changes will be investigated comprehensively.

## III. Maximally Economic Sparse Arrays

We begin with the definition of maximally economic sparse arrays (MESA):

*Definition 9:* A sensor array $\mathbb{S}$ is said to be *maximally economic* if all the sensors in $\mathbb{S}$ are essential [12].

The definition was introduced in [12] to study the economy of the number of sensors in $\mathbb{S}$. However, this paper and the companion paper [9] concentrate on the robustness analysis of MESA with respect to the difference coarray.

Definition 9 is actually equivalent to the statement $\mathcal{E}_1 = \mathcal{S}_1$. This result leads to the following lemma [9, Corollary 2, Corollary 4, Lemma 7]:

*Lemma 1:* Let $\mathbb{S}$ be a MESA, as defined in Definition 9. Then the $k$-essential family $\mathcal{E}_k$, the $k$-fragility $F_k$, and the $k$-essential Sperner family $\mathcal{E}'_k$ for $\mathbb{S}$ are given by

$$\mathcal{E}_k = \mathcal{S}_k, \qquad F_k = 1, \qquad k = 1, 2, \ldots, |\mathbb{S}|, \tag{4}$$

$$\mathcal{E}'_1 = \mathcal{S}_1, \qquad \mathcal{E}'_k = \varnothing, \qquad k = 2, 3, \ldots, |\mathbb{S}|, \tag{5}$$

where $\mathcal{S}_k$ is defined in Definition 4 and $\varnothing$ denotes the empty set.

Due to Lemma 1, MESA are the least robust arrays in terms of the $k$-fragility $F_k$, since they own the largest $k$-fragility $F_k$ among all array configurations. Furthermore, according to Lemma 1, the condition that $\mathcal{E}'_k = \varnothing$ for all $2 \leq k \leq N$ is necessary, but not sufficient for $\mathbb{S}$ being maximally economic. As an example, the array $\mathbb{S} = \{0, 1, 2, 3, 4, 12, 14\}$ has $\mathcal{E}'_k = \varnothing$ for all $2 \leq k \leq 7$. But $\mathcal{E}'_1 = \mathcal{S}_1 \backslash \{\{2\}\}$ and $\mathbb{S}$ is not maximally economic.

The probability $P_c$ that the difference coarray changes can also be characterized in closed form. In view of Lemma 1, Eq. [9, (33)] simplifies to

$$P_c = 1 - (1-p)^{|\mathbb{S}|} \qquad \text{for MESA.} \tag{6}$$

Eq. (6) depends only on the number of sensors in MESA, instead of the sensor locations. It was shown in [9, Theorem 3] that, for a fixed number of sensors, MESA has the largest $P_c$. This observation is in accordance with the statement that MESA are the least robust or the most fragile arrays among all possible array geometries, as seen from our earlier discussion on $k$-fragility.

The above discussions do not assume a specific array geometry. One of the main contributions of this paper is the following theorem:

*Theorem 1:* The MESA family includes minimum redundancy arrays (MRA), minimum hole arrays (MHA), nested arrays with $N_2 \geq 2$, and Cantor arrays.

*Example 1:* The definitions of these arrays and the proofs can be found later in Sections III-A to III-D. In this example, let us consider the geometries and the weight functions of MRA, MHA, nested arrays, and Cantor arrays with 8 physical sensors, as illustrated in Fig. 1. Here the essential sensors are depicted in red diamonds, empty space is shown in crosses, and the weight functions are illustrated in blue dots. Due to the symmetry of the difference coarray, only the nonnegative portion of the weight function is depicted. By definition, the difference coarray is the support of the weight function, as in Definition 1 and 2.

It can be observed that the size of the nonnegative part of the difference coarray, as given by the number of $m$ such that $w(m) \geq 1$ in Fig. 1, is 24 for the MRA, 29 for the MHA, 20 for the nested array, and finally 14 for the Cantor array. This is because Cantor arrays only have $\mathcal{O}(|\mathbb{S}|^{1.585})$ elements in the difference coarray [12] while the remaining arrays have $\mathcal{O}(|\mathbb{S}|^2)$ elements in $\mathbb{D}$ [2], [3], [10]. Furthermore, the MHA has holes in the difference coarray. That is, there are some missing elements, such as 26, 27, and 29 in Fig. 1(b), which cannot be obtained from the pairwise differences of the sensor locations. The remaining arrays have hole-free difference coarrays, i.e., the difference coarray is composed of consecutive integers ($\mathbb{D} = \mathbb{U}$). Theorem 1 indicates that none of the physical elements (as the diamonds in Fig. 1) in these arrays can be removed without changing the difference coarray.

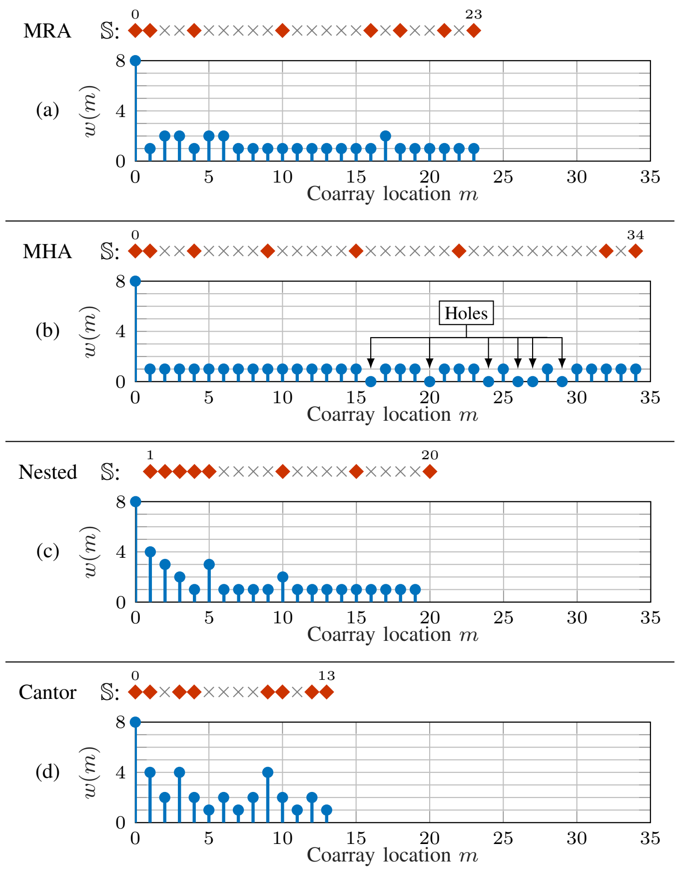


Fig. 1. The array geometry ($\mathbb{S}$, in diamonds) and the nonnegative part of the weight function ($w(m)$, in dots) for (a) the MRA with 8 elements, (b) the MHA with 8 elements, (c) the nested array with $N_1 = N_2 = 4$ (8 elements), and (d) the Cantor array with 8 elements. Here crosses denote empty space.

In Sections III-A to III-D, the details of Theorem 1 will be clarified, including the definition of these arrays and the claims of the theorem will be proved.

## A. Minimum Redundancy Arrays

Minimum redundancy arrays (MRA) were first proposed by Moffet [2]. These minimize the so-call *redundancy* $R$, defined as

$$R \triangleq \frac{\binom{|\mathbb{S}|}{2}}{(|\mathbb{U}|-1)/2} = \frac{\binom{|\mathbb{S}|}{2}}{\max(\mathbb{U})}, \tag{7}$$

subject to the hole-free constraint on the difference coarray.

Next, the definition of the MRA is given as follows:

*Definition 10:* The MRA with $N$ physical elements can be defined as [2]:

$$\mathbb{S}_{\mathrm{MRA}} \triangleq \arg\max_{\mathbb{S}} |\mathbb{D}| \text{ subject to } |\mathbb{S}| = N,\ \mathbb{D} = \mathbb{U}. \tag{8}$$

Namely, Eq. (8) indicates that the MRA has the largest hole-free difference coarray for a given number of sensors. For a fixed number of sensors, it can be shown that Moffet's definition is equivalent to Definition 10. However, this paper considers Definition 10 to facilitate the proof of Theorem 1, as presented below.

*Proof of the maximal economy of MRA:* Definition 10 implies that the MRA has the largest hole-free difference coarray $\mathbb{D}_{\mathrm{MRA}} \triangleq \{0, \pm1, \pm2, \cdots \pm (\max(\mathbb{S}_{\mathrm{MRA}}) - \min(\mathbb{S}_{\mathrm{MRA}}))\}$, among all array configurations with $N$ elements. Due to [9, Corollary 3], the MRA is maximally economic for $1 \leq N \leq 3$. If $N \geq 4$, then we have the following chain of arguments. Assume that $n \in \mathbb{S}_{\mathrm{MRA}}$ is inessential. It can be shown that 1) $n \neq \min(\mathbb{S}_{\mathrm{MRA}})$ [9, Lemma 2] and 2) the difference coarray of $\mathbb{S}_{\mathrm{MRA}}\backslash\{n\}$ is also $\mathbb{D}_{\mathrm{MRA}}$. Now we construct a new array geometry

$$\overline{\mathbb{S}} \triangleq (\mathbb{S}_{\mathrm{MRA}}\backslash\{n\}) \cup \{\max(\mathbb{S}_{\mathrm{MRA}}) + 1\}, \tag{9}$$

which has difference coarray $\overline{\mathbb{D}}$. Based on (9), the following properties can be shown to be true

1) $|\overline{\mathbb{S}}| = N$.
2) $\overline{\mathbb{D}} = \mathbb{D}_{\mathrm{MRA}} \cup \{\pm(\max(\mathbb{S}_{\mathrm{MRA}}) - \min(\mathbb{S}_{\mathrm{MRA}}) + 1)\}$.

Hence $\overline{\mathbb{D}}$ is hole-free. However, we have $|\overline{\mathbb{D}}| = |\mathbb{D}_{\mathrm{MRA}}| + 2$, which contradicts (8). Therefore all elements in $\mathbb{S}_{\mathrm{MRA}}$ are essential. ■

### B. Minimum Hole Arrays

Minimum hole arrays (MHA) are also called Golomb arrays or minimum gap arrays [10], [20]. These arrays are defined to minimize the number of holes, such that each nonzero element in the difference coarray results from a unique sensor pair. Formally:

*Definition 11:* The MHA with $N$ physical elements can be defined as [10]

$$\mathbb{S}_{\mathrm{MHA}} \triangleq \arg\min_{\mathbb{S}} |\mathbb{H}| \quad \text{subject to } |\mathbb{S}| = N,\ w(m) = 1 \text{ for } m \in \mathbb{D}\backslash\{0\}, \tag{10}$$

where $\mathbb{H} \triangleq \{m : \min(\mathbb{D}) \leq m \leq \max(\mathbb{D}), m \notin \mathbb{D}\}$ are the holes in $\mathbb{D}$.

More details on MHA can be found in [21] and the references therein. In this paper, the main focus of MHA is to prove their maximal economy, as presented below:

*Proof of the maximal economy of MHA:* Let $\mathbb{S}_{\mathrm{MHA}} = \{s_1, s_2, \ldots, s_N\}$ be a MHA with $N$ elements such that $s_1 < s_2 < \cdots < s_N$. Due to [9, Corollary 3], it suffices to consider MHA with $N \geq 4$. Next, Definition 11 indicates that the weight function of $\mathbb{S}_{\mathrm{MHA}}$ satisfies $w(s_2 - s_1) = w(s_3 - s_1) = \cdots = w(s_N - s_1) = 1$, This relation proves the maximal economy of MHA owing to [9, Lemma 1] and Definition 9. ■

*Example 2:* Consider Fig. 1(b), where the MHA has sensor locations $\mathbb{S}_{\mathrm{MHA}} = \{0, 1, 4, 9, 15, 22, 32, 34\}$. It can be observed that the weight function satisfies $w(1-0) = w(4-0) = w(9-0) = w(15-0) = w(22-0) = w(32-0) = w(34-0) = 1$. As a result, the MHA with 8 sensors is maximally economic.

### C. Nested Arrays with $N_2 \geq 2$

A downside for MRA and MHA is the lack of computationally efficient algorithms or closed-form solutions for the sensor locations [2], [10]. By contrast, the sensor locations of nested arrays are expressed in closed-form [3]:

*Definition 12:* Assume that $N_1$ and $N_2$ are positive integers. The nested array with $N_1$ and $N_2$ is specified by the set $\mathbb{S}_{\mathrm{nested}} \triangleq \mathbb{T}_1 \cup \mathbb{T}_2$, where $\mathbb{T}_1$ and $\mathbb{T}_2$ are defined as

$$\mathbb{T}_1 = \{1, 2, \ldots, N_1\}, \tag{11}$$

$$\mathbb{T}_2 = \{n(N_1 + 1) : n = 1, 2, \ldots, N_2\}. \tag{12}$$

Here $\mathbb{T}_1$ denotes a dense ULA with interelement spacing 1 (in unit of half of the wavelength) while $\mathbb{T}_2$ represents a sparse ULA with spacing $N_1+1$. For instance, in Fig. 1(c), the nested array has $\mathbb{T}_1 = \{1, 2, 3, 4\}$ and $\mathbb{T}_2 = \{5, 10, 15, 20\}$.

Nested arrays possess hole-free difference coarrays [3]. Furthermore, if $N_1$ and $N_2$ are approximately $N/2$, then the size of the difference coarray of the nested array becomes $\mathcal{O}(N^2)$, which is as many as that of MRA and MHA [2], [3], [10]. For brevity, other properties of the nested array are skipped in this paper and interested readers are referred to [3] and the references therein.

Next, as one of the contributions of this paper, the maximal economy of nested arrays with $N_2 \geq 2$ will be proved. As a remark, if $N_2 = 1$, then the nested array becomes the ULA with $N_1 + 1$ elements, which is, in general, not maximally economic, as we will show in Theorem 2.

*Proof of the maximal economy of nested arrays with $N_2 \geq 2$:* First, the weight function for $\mathbb{S}_{\mathrm{nested}}$ is denoted by $w_{\mathrm{nested}}(m)$. Then, we invoke the following two lemmas, whose proofs can be found at the end of this subsection.

*Lemma 2:* Assume that $N_2 \geq 2$. If $n_1 = N_2(N_1 + 1)$ and $n_2 \in \mathbb{T}_1$, then $w_{\mathrm{nested}}(n_1 - n_2) = 1$.

*Lemma 3:* If $n_1 \in \mathbb{T}_2$ and $n_2 = 1$, then $w_{\mathrm{nested}}(n_1 - n_2) = 1$.

Finally, combining [9, Lemma 1], Lemma 2, Lemma 3, and Definition 9 proves the maximal economy of the nested array with $N_2 \geq 2$. ■

*Example 3:* Let us verify Lemmas 2 and 3 using the nested array with $N_1 = N_2 = 4$ in Fig. 1(c). Assume that $n_1 = N_2(N_1 + 1) = 20$ and $n_2 = 3 \in \mathbb{T}_1$. Due to Fig. 1(c), the weight function of the nested array satisfies $w(n_1 - n_2) = w(17) = 1$, which confirms Lemma 2. Next, suppose that $n_1 = 15 \in \mathbb{T}_2$ and $n_2 = 1$. We obtain $w(n_1 - n_2) = w(14) = 1$ based on Fig. 1(c). The above example is also consistent with Lemma 3.

Finally, the proofs of Lemmas 2 and 3 are given as follows:

*Proof of Lemma 2:* In this case, we have $n_1 - n_2 \geq N_2(N_1 + 1) - N_1$. Assume that there exist $n_1', n_2' \in \mathbb{S}_{\mathrm{nested}}$ such that the pair $(n_1', n_2') \neq (n_1, n_2)$ and $n_1' - n_2' = n_1 - n_2$. If $n_1'$ is not the rightmost element in $\mathbb{S}_{\mathrm{nested}}$, namely, $n_1' \neq N_2(N_1 + 1)$, then $n_1' \leq (N_2 - 1)(N_1 + 1)$, because $N_2 \geq$

2. Furthermore, since $n_2' \geq 1$, we have $n_1' - n_2' \leq (N_2 - 1)(N_1+1) - 1 = N_2(N_1+1) - N_1 - 2$, which disagrees with $n_1' - n_2' = n_1 - n_2 \geq N_2(N_1+1) - N_1$. Therefore $n_1' = n_1 = N_2(N_1+1)$, $n_2' = n_2$, and $w_{\text{nested}}(n_1 - n_2) = 1$. ■

*Proof of Lemma 3:* Since $n_1 \in \mathbb{T}_2$ and $n_2 = 1$, we have

$$n_1 - n_2 \equiv N_1 \mod (N_1+1), \tag{13}$$

$$n_1 - n_2 \geq N_1, \tag{14}$$

where $\bmod N$ denotes the modulo-$N$ operation. Suppose that there exist $n_1', n_2' \in \mathbb{S}_{\text{nested}}$ such that the pair $(n_1, n_2) \neq (n_1', n_2')$ and $n_1' - n_2' = n_1 - n_2$. The parameters $n_1'$ and $n_2'$ can be divided into four cases. If $n_1', n_2' \in \mathbb{T}_1$, then $|n_1' - n_2'| \leq N_1 - 1$, which contradicts (14). If $n_1', n_2' \in \mathbb{T}_2$, then $n_1' - n_2'$ is divisible by $N_1 + 1$, which violates (13). If $n_1' \in \mathbb{T}_1$ and $n_2' \in \mathbb{T}_2$, then $n_1' - n_2' \leq -1$, which disagrees with (14). Finally, if $n_1' \in \mathbb{T}_2$ and $n_2' \in \mathbb{T}_1$, then we obtain

$$n_2' = n_1' - (n_1 - n_2) \equiv 1 \mod (N_1+1),$$

due to (13) and $n_1' \in \mathbb{T}_2$. Therefore $n_2' = n_2 = 1$ and $n_1' = n_1$, which proves this lemma. ■

### D. Cantor Arrays

In this subsection, we will concentrate on *Cantor arrays*, which first appeared in the context of fractal array design [11], [22], [23]. These arrays originated from the Cantor set in fractal theory [24], [25]. Previous research on Cantor arrays was mainly conducted towards the relationships between fractal geometries and the beampatterns of the arrays [11], [22], [23]. A recent study focused on the difference coarray of Cantor arrays [12], including the weight function, the size and the structure of the difference coarray, and its maximal economy, as we will present next.

First, the definition of the Cantor array $\mathbb{S}_r$ is parameterized by a nonnegative integer $r$. The translated array of $\mathbb{S}_r$ is defined as $\mathbb{T}_r \triangleq \{n + D_r : \forall n \in \mathbb{S}_r\}$, where $D_r \triangleq 2A_r + 1$, with $A_r$ denoting the aperture of $\mathbb{S}_r$, that is, $A_r \triangleq \max(\mathbb{S}_r) - \min(\mathbb{S}_r)$. With this, we are ready to define a Cantor array:

*Definition 13:* The Cantor array $\mathbb{S}_r$ is defined as

$$\mathbb{S}_r \triangleq \begin{cases} \{0\} & \text{if } r = 0, \\ \mathbb{S}_{r-1} \cup \mathbb{T}_{r-1}, & \text{if } r \geq 1. \end{cases} \tag{15}$$

Notice that $\mathbb{S}_r$ has $N = 2^r$ sensors. So, Cantor arrays are defined only for the case that the number of sensors is a power of two. Furthermore, it was shown in [12] that Cantor arrays are symmetric arrays, i.e. $n \in \mathbb{S}_r$ if and only if $A_r - n \in \mathbb{S}_r$.

For instance, let us consider the Cantor arrays for $r = 0, 1, 2, 3$. According to Definition 13, these arrays become

$$\mathbb{S}_0 = \{0\}, \quad A_0 = 0, \quad D_0 = 1, \tag{16}$$

$$\mathbb{S}_1 = \{0, 1\}, \quad A_1 = 1, \quad D_1 = 3, \tag{17}$$

$$\mathbb{S}_2 = \{0, 1, 3, 4\}, \quad A_2 = 4, \quad D_2 = 9 \tag{18}$$

$$\mathbb{S}_3 = \{0, 1, 3, 4, 9, 10, 12, 13\}, \quad A_3 = 13, \quad D_3 = 27, \tag{19}$$

where (19) is depicted in Fig. 1(d). It can also be deduced from Fig. 1(d) that $\mathbb{S}_3$ is symmetric.

The arrays in Definition 13 are equivalent to the Cantor array proposed in [11], [22], [23], with proper amount of translation and scaling. The Cantor arrays in [11], [22], [23] are built upon the Cantor sets in fractal theory [26], [27]. But here we start with a different definition (Definition 13), which will facilitate the discussion on its coarray properties. We begin by proving:

*Lemma 4:* For the Cantor array (15) with parameter $r \geq 1$ in Definition 13, the weight function $w_r(m)$ satisfies

$$w_r(m) = \begin{cases} 2w_{r-1}(m), & \text{if } |m| \leq A_{r-1}, \\ w_{r-1}(m \pm D_{r-1}), & \text{if } |m \pm D_{r-1}| \leq A_{r-1}, \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

where $A_r$ and $D_r$ are defined as in Definition 13.

*Proof:* The weight function $w_r(m)$ can be expressed as

$$\begin{aligned} w_r(m) &= \left|\left\{(n_1, n_2) \in \mathbb{S}_r^2 : n_1 - n_2 = m\right\}\right| \\ &= \left|\left\{(n_1, n_2) \in \mathbb{S}_{r-1}^2 : n_1 - n_2 = m\right\}\right| \\ &\quad + \left|\left\{(n_1, n_2) \in \mathbb{T}_{r-1}^2 : n_1 - n_2 = m\right\}\right| \\ &\quad + \left|\{(n_1, n_2) \in \mathbb{S}_{r-1} \times \mathbb{T}_{r-1} : n_1 - n_2 = m\}\right| \\ &\quad + \left|\{(n_1, n_2) \in \mathbb{T}_{r-1} \times \mathbb{S}_{r-1} : n_1 - n_2 = m\}\right|, \end{aligned} \tag{21}$$

which is due to $\mathbb{S}_r = \mathbb{S}_{r-1} \cup \mathbb{T}_{r-1}$ for $r \geq 1$ in Definition 13. Since every element in $\mathbb{T}_{r-1}$ can be expressed as $n' + D_{r-1}$, where $n' \in \mathbb{S}_{r-1}$, (21) can be written as

$$\begin{aligned} w_r(m) &= \left|\left\{(n_1, n_2) \in \mathbb{S}_{r-1}^2 : n_1 - n_2 = m\right\}\right| \\ &\quad + \left|\left\{(n_1', n_2') \in \mathbb{S}_{r-1}^2 : n_1' - n_2' = m\right\}\right| \\ &\quad + \left|\left\{(n_1, n_2') \in \mathbb{S}_{r-1}^2 : n_1 - n_2' = m + D_{r-1}\right\}\right| \\ &\quad + \left|\left\{(n_1', n_2) \in \mathbb{S}_{r-1}^2 : n_1' - n_2 = m - D_{r-1}\right\}\right| \\ &= 2w_{r-1}(m) + w_{r-1}(m + D_{r-1}) \\ &\quad + w_{r-1}(m - D_{r-1}). \end{aligned} \tag{22}$$

Equation (22) simplifies to (20) in the following cases:

1) Suppose that $|m| \leq A_{r-1}$, which is equivalent to the condition that $-A_{r-1} \leq m \leq A_{r-1}$. Since $D_{r-1} = 2A_{r-1} + 1$, we have

$$|m + D_{r-1}| \geq -A_{r-1} + D_{r-1} = A_{r-1} + 1 > A_{r-1},$$
$$|m - D_{r-1}| \geq |A_{r-1} - D_{r-1}| = A_{r-1} + 1 > A_{r-1}.$$

Since the aperture of the Cantor array with parameter $r - 1$ is $A_{r-1}$, we have, by definition, $w_{r-1}(p) = 0$ for any $|p| > A_{r-1}$. This property indicates that $w_{r-1}(m \pm D_{r-1}) = 0$ if we set $p = m \pm D_{r-1}$. Therefore, (22) becomes $w_r(m) = 2w_{r-1}(m)$ in this case.

2) Assume that $|m + D_{r-1}| \leq A_{r-1}$. This condition can be rewritten as $-A_{r-1} \leq m + D_{r-1} \leq A_{r-1}$ so $-3A_{r-1} - 1 \leq m \leq -A_{r-1} - 1$. As a result, $|m|$ and $|m - D_{r-1}|$ satisfy

$$|m| \geq A_{r-1} + 1 > A_{r-1}, \tag{23}$$

$$|m - D_{r-1}| \geq |-A_{r-1} - 1 - D_{r-1}| > A_{r-1}. \tag{24}$$

Therefore, we have $w_{r-1}(m) = 0$ and $w_{r-1}(m - D_{r-1}) = 0$. Using (22), we obtain that $w_r(m) = w_{r-1}(m + D_{r-1})$ in this case.

3) If $|m - D_{r-1}| \le A_{r-1}$, then due to similar arguments as the case of $|m + D_{r-1}| \le A_{r-1}$, we have $w_r(m) = w_{r-1}(m - D_{r-1})$. ∎

Lemma 4 shows that the weight function for the Cantor array $\mathbb{S}_r$ can be recursively constructed from the weight function for $\mathbb{S}_{r-1}$. To give some feelings for Lemma 4, the following numerical example is considered. Due to Lemma 4 and (16) to (19), the weight function becomes $w_3(6) = w_2(6 - D_2) = w_2(3) = w_1(3 - D_1) = 2w_0(0) = 2$, which is consistent with the weight function in Fig. 1(d).

Furthermore, based on Lemma 4, it can be proved that Cantor arrays have hole-free difference coarrays of size $\mathcal{O}(|\mathbb{S}|^{\log_2 3}) \approx \mathcal{O}(|\mathbb{S}|^{1.585})$. This result is distinct from the MRA (hole-free difference coarray of size $\mathcal{O}(|\mathbb{S}|^2)$) and the ULA (hole-free difference coarray of size $\mathcal{O}(|\mathbb{S}|)$). The detailed proofs are skipped here and can be found in [12].

*Proof of the maximal economy of Cantor arrays:* Finally the maximal economy of Cantor arrays will be proved at the end of this subsection. First we prove:

*Lemma 5:* Let the Cantor array with parameter $r$ be denoted by $\mathbb{S}_r = \{s_1, s_2, \ldots, s_N\}$, where $0 = s_1 < s_2 < \cdots < s_N$ and $N = 2^r$. Then the weight function of $\mathbb{S}_r$ satisfies $w_r(s_{N+1-k} - s_k) = 1$ for all $k = 1, 2, \ldots, N$.

*Proof:* First, if $r = 0$, then $\mathbb{S}_0 = \{0\}$ and $w_0(0) = 1$, which holds trivially. Assume $w_r(s_{N+1-k} - s_k) = 1$ holds true for all $k = 1, 2, \ldots, N$. Then the sensor locations for $\mathbb{S}_{r+1}$ are given by

$$\mathbb{S}_{r+1} = \{s_1, s_2, \ldots, s_N, s_1 + D_r, s_2 + D_r, \ldots, s_N + D_r\}.$$

It can be shown that $s_N < s_1 + D_r < s_2 + D_r < \cdots < s_N + D_r$. Due to Lemma 4, the weight function of $\mathbb{S}_{r+1}$ satisfies $w_{r+1}((s_{N+1-k} + D_r) - s_k) = w_r(s_{N+1-k} - s_k) = 1$ for all $k = 1, 2, \ldots, N$. Furthermore, the symmetry of the difference coarray shows that $w_{r+1}(s_k - (s_{N+1-k} + D_r)) = 1$. These arguments complete the proof. ∎

Due to [9, Lemma 1] and Lemma 5, $s_k$ and $s_{N+1-k}$ are both essential for all $k = 1, 2, \ldots, N$, which proves the maximal economy of Cantor arrays. ∎

For clarity, Fig. 1(d) demonstrates the weight function of $\mathbb{S}_3$, where $w(13-0) = w(12-1) = w(10-3) = w(9-4) = 1$. Due to [9, Lemma 1], this result implies that the sensors at 13, 0, 12, 1, 10, 3, 9, and 4 are all essential, which confirms the maximal economy of $\mathbb{S}_3$.

## IV. Uniform Linear Arrays

In what follows, two commonly used array geometries, the ULA and the coprime array, will be discussed in Sections IV and V, respectively. Among the arrays considered in this paper, it will be shown that the most robust arrays are ULA, followed by coprime arrays, and finally MESA.

The ULA with $N$ physical elements is defined as [1]:

$$\mathbb{S}_{\text{ULA}} \triangleq \{0, 1, \ldots, N-1\}. \tag{25}$$

It can be shown that the difference coarray of the ULA is $\{0, \pm 1, \ldots, \pm(N-1)\}$, whose size is $2N - 1$. This property indicates that the ULA resolves at most $N - 1$ uncorrelated sources, unlike sparse arrays such as MRA or nested arrays ($\mathcal{O}(N^2)$ uncorrelated sources) [3]. However, in the past, ULA are regarded as more robust than sparse arrays. In this section, this observation will be confirmed using the theory in the companion paper [9]. Using (25) and Definition 8, the $k$-essential Sperner family of the ULA can be shown to have the following closed-form expressions:

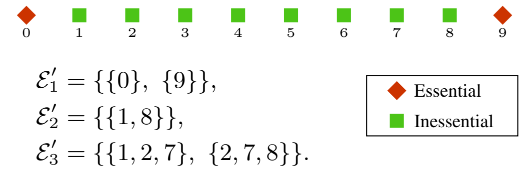


Fig. 2. The ULA with $N = 10$ elements and the $k$-essential Sperner family $\mathcal{E}'_1$, $\mathcal{E}'_2$, and $\mathcal{E}'_3$.

*Theorem 2:* The $k$-essential Sperner family of $\mathbb{S}_{\text{ULA}}$ satisfies

$$\mathcal{E}'_1 = \begin{cases} \mathcal{S}_{1,\text{ULA}}, & \text{if } 1 \le N \le 3, \\ \{\{0\},\ \{N-1\}\}, & \text{if } N \ge 4, \end{cases} \tag{26}$$

$$\mathcal{E}'_2 = \begin{cases} \varnothing, & \text{if } 1 \le N \le 3, \\ \{\{1,2\}\}, & \text{if } N = 4, \\ \{\{1,2\},\ \{1,3\},\ \{2,3\}\}, & \text{if } N = 5, \\ \{\{1,4\},\ \{2,3\}\}, & \text{if } N = 6, \\ \{\{1, N-2\}\}, & \text{if } N \ge 7, \end{cases} \tag{27}$$

$$\mathcal{E}'_3 = \begin{cases} \varnothing, & \text{if } N \le 6, \\ \{\{1,2,3\},\ \{1,2,4\},\ \{2,3,4\}, \\ \quad \{2,4,5\},\ \{3,4,5\}\}, & \text{if } N = 7, \\ \{\{1,2,5\},\ \{2,3,4\}, \\ \quad \{2,5,6\},\ \{3,4,5\}\}, & \text{if } N = 8, \\ \{\{1,2,6\},\ \{2,6,7\},\ \{3,4,5\}\}, & \text{if } N = 9, \\ \{\{1,2,N-3\}, \\ \quad \{2,N-3,N-2\}\}, & \text{if } N \ge 10. \end{cases} \tag{28}$$

Here $\mathcal{S}_{1,\text{ULA}} \triangleq \{\{n\} : n \in \mathbb{S}_{\text{ULA}}\}$ denotes all the subarrays of size 1 over $\mathbb{S}_{\text{ULA}}$.

The derivation of the expressions in Theorem 2 is quite involved, and it can be found in Section IV-A. Next the expressions in Theorem 2 are demonstrated through the following numerical example:

*Example 4:* Consider the ULA with $N = 10$ elements. Fig. 2 depicts the $k$-essential Sperner family $\mathcal{E}'_1$, $\mathcal{E}'_2$, and $\mathcal{E}'_3$. Since $N \ge 3k + 1$ for $k = 1, 2, 3$, the last equations in (26) to (28) are used. First, some of the subarrays in $\mathcal{E}'_k$ are mirror images of each other, with respect to the center of $\mathbb{S}_{\text{ULA}}$, like $\{1, 2, 7\}$ and $\{2, 7, 8\}$. This phenomenon is because the difference coarray is invariant to the reversal of the array configuration [12]. Second, using Fig. 2, given any subarray of size $k \le 3$, its $k$-essentialness property can be readily examined by the contents of $\mathcal{E}'_k$, as presented in the companion paper [9, Section V]. For instance, since $\{1, 2, 8\}$ is a superset of $\{1, 8\} \in \mathcal{E}'_2$, we have $\{1, 2, 8\} \in \mathcal{E}_3$, so removing $\{1, 2, 8\}$ from $\mathbb{S}_{\text{ULA}}$ alters the difference coarray, as depicted later in Fig. 3(c). As another example, deleting $\{3, 5, 8\}$ from $\mathbb{S}_{\text{ULA}}$ preserves the difference coarray, as illustrated later in Fig. 3(d).

This observation is consistent with Fig. 2 since $\{3,5,8\}$ is not a superset of any elements in $\mathcal{E}'_k$ for $k=1,2,3$ and hence $\{3,5,8\} \notin \mathcal{E}_3$.

Theorem 2 also shows that ***the elements at both ends of*** $\mathbb{S}_{\mathrm{ULA}}$ ***are more important than others.*** It was reported in [28] that for the ULA with 6 elements ($\mathbb{S}_{\mathrm{ULA}}=\{0,1,2,3,4,5\}$), the elements at 0 and 5 are the most important ones while the elements $1,2,3,4$ are less important. On the other hand, as presented in Theorem 2, for $\mathbb{S}_{\mathrm{ULA}}=\{0,1,2,3,4,5\}$, *the elements* 0 *and* 5 *are essential* while *the elements* $1,2,3,4$ *are inessential*, which is in accordance with [28]. Our contribution here is to utilize the essentialness property as another notion of the importance of elements in arrays. Unlike the previous work [28], our approach depends purely on the array geometry, rather than other factors such as source directions and source powers.

Next, the closed-form expressions of the $k$-fragility for the ULA will be derived based on Theorem 2. The main focus would be $F_1$, $F_2$, and $F_3$, for $N \geq 4$, $N \geq 7$, and $N \geq 10$, respectively. If $N \geq 4$, then $|\mathcal{E}'_1| = |\mathcal{E}_1| = 2$ so $F_1 = 2/N$. If $N \geq 7$, then due to [9, Lemma 5], the cardinality of $\mathcal{E}_2$ can be computed as $|\mathcal{E}_2| = |\{\{0,n\},\ \{n,N-1\},\ \{0,N-1\},\ \{1,N-2\} : 1 \leq n \leq N-2\}| = 2(N-1)$. Hence $F_2 = 2(N-1)/\binom{N}{2} = 4/N$. Finally, the 3-essential family for the ULA with $N \geq 10$ is given by

$$\mathcal{E}_3 = \underbrace{\{\mathbb{A} \in \mathcal{S}_{3,\mathrm{ULA}} : 0 \in \mathbb{A}\}}_{\mathcal{G}_1} \cup \underbrace{\{\mathbb{A} \in \mathcal{S}_{3,\mathrm{ULA}} : N-1 \in \mathbb{A}\}}_{\mathcal{G}_2} \cup \underbrace{\{\mathbb{A} \in \mathcal{S}_{3,\mathrm{ULA}} : \{1,N-2\} \subset \mathbb{A}\}}_{\mathcal{G}_3} \cup \mathcal{E}'_3, \tag{29}$$

where $\mathcal{S}_{3,\mathrm{ULA}} \triangleq \{\mathbb{A} \subseteq \mathbb{S}_{\mathrm{ULA}} : |\mathbb{A}| = 3\}$ represents all the subarrays of size 3 over $\mathbb{S}_{\mathrm{ULA}}$. Substituting $|\mathcal{G}_1| = |\mathcal{G}_2| = \binom{N-1}{2}$, $|\mathcal{G}_3| = |\mathcal{G}_1 \cap \mathcal{G}_2| = N-2$, $|\mathcal{G}_1 \cap \mathcal{G}_3| = |\mathcal{G}_2 \cap \mathcal{G}_3| = 1$, and $|\mathcal{G}_1 \cap \mathcal{E}'_3| = |\mathcal{G}_2 \cap \mathcal{E}'_3| = |\mathcal{G}_3 \cap \mathcal{E}'_3| = 0$, into (29) leads to $|\mathcal{E}_3| = (N-1)(N-2)$ so that fragility $F_3 = 6/N$. Summarizing, the $k$-fragility $F_k$ for the ULA with $N$ elements satisfies

$$\text{ULA:} \qquad F_1 = \frac{2}{N}, \qquad F_2 = \frac{4}{N}, \qquad F_3 = \frac{6}{N}, \tag{30}$$

where these expressions are valid for $N \geq 4$, $N \geq 7$, and $N \geq 10$, respectively. For instance, for the ULA with $N=16$ elements, (30) leads to $F_1 = 0.125$, $F_2 = 0.25$, and $F_3 = 0.375$, which are consistent with the numerical example in [9, Fig. 8].

*Failure probabilities.* Finally, here are some remarks on the probability that the difference coarray changes, $P_c$, for ULA. Even though $P_c$ has closed-form expressions associated with the fragility $F_k$, as in [9, (33)], it remains challenging to derive closed-form expressions of $P_c$ for ULA, due to the lack of closed forms of $\mathcal{E}'_k$ and $F_k$, for all $k$. Even so, $P_c$ for the ULA can still be analyzed either numerically using [9, (33)], or analytically using the bounds of $P_c$, as in the companion paper [9, Theorem 3]. For instance, as discussed in [9, Section VI], if the probability of element failure $p$ is sufficiently small, then $P_c$ is approximately $|\mathcal{E}_1|p$. This approximation indicates that, for ULA with $N \geq 4$ elements, $P_c$ has an asymptotic expression of $2p$. Namely, the probability that the difference coarray changes is around $2p$. This is the smallest among all possible array configurations with fixed $N$, due to [9, Lemma 2].

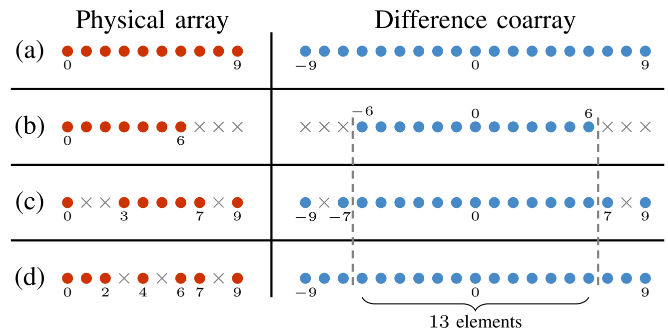


Fig. 3. (a) The ULA with 10 physical elements $\mathbb{S}_{\mathrm{ULA}}$ and its difference coarray. The physical array (left) and the difference coarray (right) after removing (b) $\{7,8,9\}$, (c) $\{1,2,8\}$, and (d) $\{3,5,8\}$, from $\mathbb{S}_{\mathrm{ULA}}$, respectively. Here bullets denote elements and crosses represent empty space. It can be observed that the difference coarrays of (b), (c), and (d) contain $\{0,\pm 1,\ldots,\pm 6\}$.

## A. Derivation of the Expressions in Theorem 2

Before deriving the expressions in Theorem 2, we first invoke Lemma 6 to describe the difference coarray after removing $k$ physical sensors.

*Lemma 6:* Let $\mathbb{A} \subseteq \mathbb{S}_{\mathrm{ULA}}$ satisfy $|\mathbb{A}| = k$. Assume that $\overline{\mathbb{S}} \triangleq \mathbb{S}_{\mathrm{ULA}} \backslash \mathbb{A}$ and its difference coarray is denoted by $\overline{\mathbb{D}}$. If $N \geq 3k+1$, then $\{0,\pm 1,\ldots,\pm(N-k-1)\} \subseteq \overline{\mathbb{D}}$.

Lemma 6 implies that, if $N$ is sufficiently large, then even though $k$ elements are removed from $\mathbb{S}_{\mathrm{ULA}}$, the difference coarray $\overline{\mathbb{D}}$ still possesses a central ULA segment of at least $2(N-k-1)+1$ elements. The detailed proof of Lemma 6 will be given after Example 5:

*Example 5:* Fig. 3 demonstrates an example of Lemma 6. We consider the ULA with $N=10$ elements and its difference coarray, as depicted in Fig. 3(a). In Figs. 3(b), (c), and (d), we remove $k=3$ physical elements from $\mathbb{S}_{\mathrm{ULA}}$ and evaluate their difference coarrays. Regardless of the locations of the removed elements, all these difference coarrays possess a central ULA segment, whose size is at least $2(N-k-1)+1 = 13$, as claimed by Lemma 6.

*Proof of Lemma 6:* First let us consider several useful results for the proof [13]:

*Definition 14:* Let $\mathbb{S}$ be an integer set. The discrete sequence $c(n)$ is 1 if $n \in \mathbb{S}$ and 0 otherwise.

*Proposition 1:* Let $c(n)$ and $w(m)$ be the discrete sequence and the weight function for $\mathbb{S}$, respectively. Then $w(m)$ satisfies

$$w(m) = \sum_{n=-\infty}^{\infty} c(n+m)c(n), \tag{31}$$

for any integer $m$.

Furthermore, the difference coarray can be expressed as the support of the weight function. Namely, $\mathbb{D} = \{m : w(m) \neq 0\}$.

Next it will be proved that $\{0,\pm 1,\pm 2,\ldots,\pm(N-k-1)\} \subseteq \overline{\mathbb{D}}$. It suffices to consider the nonnegative part of the set, due to the symmetry of the difference coarray. Assume that there exists some $\widehat{m} \in \{0, 1, 2, \ldots, N-k-1\}$ such that $\widehat{m} \notin \overline{\mathbb{D}}$. The discrete sequence and the weight function of $\overline{\mathbb{S}} \triangleq \mathbb{S}_{\mathrm{ULA}} \backslash \mathbb{A}$ are denoted by $\overline{c}(n)$ and $\overline{w}(m)$, respectively. Since $\widehat{m} \notin \overline{\mathbb{D}}$, we have $\overline{w}(\widehat{m}) = 0$, implying that

$$\overline{c}(n+\widehat{m})\overline{c}(n) = 0, \tag{32}$$

for all $n = 0, 1, \ldots, N-\widehat{m}-1$, due to Definition 14 and (31). Eq. (32) indicates that, $n+\widehat{m} \in \mathbb{A}$ or $n \in \mathbb{A}$. This condition implies

$$(\mathbb{A}-\widehat{m}) \cup \mathbb{A} \supseteq \mathbb{O} \triangleq \{0, 1, \ldots, N-\widehat{m}-1\}. \tag{33}$$

Here the notation $\mathbb{A} \pm \widehat{m} \triangleq \{a \pm \widehat{m} : a \in \mathbb{A}\}$.

According to (33), the size of $\mathbb{O}$ satisfies

$$\begin{aligned} |\mathbb{O}| &= |((\mathbb{A}-\widehat{m}) \cup \mathbb{A}) \cap \mathbb{O}| = |((\mathbb{A}-\widehat{m}) \cap \mathbb{O}) \cup (\mathbb{A} \cap \mathbb{O})| \\ &\leq |(\mathbb{A}-\widehat{m}) \cap \mathbb{O}| + |\mathbb{A} \cap \mathbb{O}| = |\mathbb{A} \cap (\mathbb{O}+\widehat{m})| + |\mathbb{A} \cap \mathbb{O}|, \end{aligned} \tag{34}$$

where the inequality is due to the union bound between sets. In what follows, (34) will be analyzed in detail. First, the set $\mathbb{S}_{\mathrm{ULA}}$ is partitioned into three subsets $\mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_3$:

$$\mathbb{L}_1 = \{0,\ 1,\ \ldots,\ P-1\}, \tag{35}$$

$$\mathbb{L}_2 = \{P,\ P+1,\ \ldots,\ N-P-1\}, \tag{36}$$

$$\mathbb{L}_3 = \{N-P,\ N-P+1,\ \ldots,\ N-1\}, \tag{37}$$

where $P \triangleq \min\{\widehat{m}, N-\widehat{m}\}$. We also define $\mathbb{A}_\ell \triangleq \mathbb{A} \cap \mathbb{L}_\ell$ and $k_\ell \triangleq |\mathbb{A}_\ell|$ for $\ell = 1, 2, 3$. It can be shown that

$$k = k_1 + k_2 + k_3, \qquad 0 \leq k_\ell \leq \min\{k,\ |\mathbb{L}_\ell|\}, \tag{38}$$

for $\ell = 1, 2, 3$.

According to $\widehat{m}$, Eq. (34) can be analyzed in two cases:

1) If $\widehat{m} \leq N/2$, then we obtain $P = \widehat{m}$. The sets $\mathbb{O}$ and $\mathbb{O}+\widehat{m}$ can be expressed as $\mathbb{O} = \mathbb{L}_1 \cup \mathbb{L}_2$ and $\mathbb{O}+\widehat{m} = \mathbb{L}_2 \cup \mathbb{L}_3$, respectively. Combining (34) and (38) leads to

$$N - \widehat{m} \leq (k_2 + k_3) + (k_1 + k_2) = k + k_2. \tag{39}$$

Now let us consider the upper bounds of $k+k_2$ for two cases of $\widehat{m}$. First, if $0 \leq \widehat{m} \leq N/3$, then using (38) and $N \geq 3k+1$, we obtain $k+k_2 \leq 2k < 2k+\frac{2}{3} \leq \frac{2}{3}N \leq N-\widehat{m}$. Therefore $k+k_2 < N-\widehat{m}$, which contradicts (39). On the other hand, if $N/3 < \widehat{m} \leq N/2$, then we have $\widehat{m} > N/3 \geq k+\frac{1}{3}$ so $k-\widehat{m} < 0$. In addition, the size of $\mathbb{L}_2$ is given by $N-2P = N-2\widehat{m}$. In this case, we have $k+k_2 \leq k+|\mathbb{L}_2| = k+(N-2\widehat{m}) = (N-\widehat{m})+(k-\widehat{m}) < N-\widehat{m}$, disagreeing with (39).

2) If $N/2 < \widehat{m} \leq N-k-1$, then $P = N-\widehat{m}$. In this case, we have $\mathbb{O} = \mathbb{L}_1$ and $\mathbb{O}+\widehat{m} = \mathbb{L}_3$. Hence (34) becomes

$$N - \widehat{m} \leq k_3 + k_1 = k - k_2. \tag{40}$$

However, the right-hand side of (40) satisfies $k-k_2 \leq k \leq N-\widehat{m}-1$, due to (38) and $\widehat{m} \leq N-k-1$. This result contradicts (40).

These arguments complete the proof of Lemma 6. ■

Next, the expressions in Theorem 2 will be derived. Here we will skip the expressions of $\mathcal{E}'_k$ for $N \leq 3k$ and $k = 1, 2, 3$, since they can be readily verified by enumerating all subarrays with size $k$. The main focus here would be the case of $N \geq 3k+1$. In what follows, the sensor locations, the difference coarray, the discrete sequence (Definition 14), and the weight function after the removal of $k$ elements will be denoted by $\overline{\mathbb{S}}$, $\overline{\mathbb{D}}$, $\overline{c}(n)$, and $\overline{w}(m)$, respectively. We will study the circumstances under which the difference coarray changes, namely $\overline{\mathbb{D}} \neq \mathbb{D}_{\mathrm{ULA}}$, where $\mathbb{D}_{\mathrm{ULA}}$ is the difference coarray of $\mathbb{S}_{\mathrm{ULA}}$.

1) $\mathcal{E}'_1$ for $N \geq 4$: Due to Lemma 6, the difference coarray $\overline{\mathbb{D}}$ contains $\{0, \pm1, \pm2, \ldots, \pm(N-2)\}$ for $k = 1$. If $\overline{\mathbb{D}} \neq \mathbb{D}_{\mathrm{ULA}}$, then $\overline{w}(N-1) = 0$. This implies

$$\overline{w}(N-1) = \overline{c}(N-1)\overline{c}(0) = 0, \tag{41}$$

due to Proposition 1. Eq. (41) shows that removing either 0 or $N-1$ leads to $\overline{\mathbb{D}} \neq \mathbb{D}_{\mathrm{ULA}}$. Hence $\mathcal{E}'_1 = \{\{0\}, \{N-1\}\}$ for $N \geq 4$.

2) $\mathcal{E}'_2$ for $N \geq 7$: Lemma 6 indicates that it suffices to consider (a) $\overline{w}(N-1) = 0$ and (b) $\overline{w}(N-2) = 0$. Let $\mathbb{A}$ be a subarray of size 2 over $\mathbb{S}_{\mathrm{ULA}}$. First, assume that $\overline{w}(N-1) = 0$. The argument of (41) shows that $0 \in \mathbb{A}$ or $N-1 \in \mathbb{A}$. Therefore $\mathbb{A}$ does not belong to $\mathcal{E}'_2$. Second, if $\overline{w}(N-2) = 0$, then we obtain

$$\overline{w}(N-2) = \overline{c}(N-2)\overline{c}(0) + \overline{c}(N-1)\overline{c}(1) = 0. \tag{42}$$

There are four choices of $\mathbb{A}$ satisfying (42): $\{0, 1\}$, $\{0, N-1\}$, $\{N-1, N-2\}$, and $\{1, N-2\}$. Since the first three subarrays contain either 0 or $N-1$, we have $\mathcal{E}'_2 = \{\{1, N-2\}\}$ for $N \geq 7$.

3) $\mathcal{E}'_3$ for $N \geq 10$: The arguments in $\mathcal{E}'_2$ indicates that, it suffices to consider $\overline{w}(N-3) = 0$ in this case. Hence we have

$$\overline{c}(N-3)\overline{c}(0) + \overline{c}(N-2)\overline{c}(1) + \overline{c}(N-1)\overline{c}(2) = 0.$$

Since the the elements in $\mathcal{E}'_3$ do not contain 0 or $N-1$, we have $\mathcal{E}'_3 = \{\{1, 2, N-3\}, \{2, N-3, N-2\}\}$, which proves Theorem 2. ■

## V. Coprime Arrays

In this section, we will move on to coprime arrays, which have recently attracted considerable attention in sparse array signal processing [4], [5], [14], [15]. These arrays are defined as:

*Definition 15:* Let $M$ and $N$ be a coprime pair of positive integers. A coprime array $\mathbb{S}_{\mathrm{coprime}}$ with parameters $M$ and $N$ can be defined as

$$\mathbb{S}_{\mathrm{coprime}} = \{0\} \cup \mathbb{P}_1 \cup \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3, \tag{43}$$

where the sets $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ are given by

$$\mathbb{P}_1 = \{p_1 M : 1 \leq p_1 \leq N-1\}, \tag{44}$$

$$\mathbb{P}_2 = \{p_2 N : 1 \leq p_2 \leq M-1\}, \tag{45}$$

$$\mathbb{P}_3 = \{p_3 N : M+1 \leq p_3 \leq 2M-1\}. \tag{46}$$

Coprime arrays are composed of two sparse ULAs. The first sparse ULA ($\{0\} \cup \mathbb{P}_1$) has $N$ elements with interelement spacing $M$ (in unit of half of the wavelength) while the

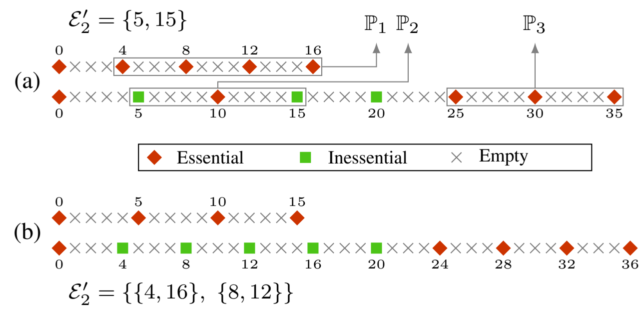


Fig. 4. An illustration for the $k$-essential Sperner family of the coprime arrays with (a) $M = 4, N = 5$ and (b) $M = 5, N = 4$. In these figures, the coprime arrays are split into two sparse ULAs for clarity.

second sparse ULA ($\{0\} \cup \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$) owns $2M$ elements with separation $N$. It can be shown that the difference coarray of $\mathbb{S}_{\text{coprime}}$ has a central ULA segment $\mathbb{U}_{\text{coprime}} = \{0, \pm 1, \ldots, \pm(MN + M - 1)\}$ and holes at $\pm(MN + M)$ [4], [5].

*Example 6:* Fig. 4(a) demonstrates the geometry of coprime arrays. For clarity, the first ULA with separation $M$ is depicted on the top while on the bottom is shown the second ULA with separation $N$. The physical sensors are denoted by diamonds or rectangles and the empty space is marked by crosses. If $M = 4$ and $N = 5$, then we have $\mathbb{P}_1 = \{4, 8, 12, 16\}$, $\mathbb{P}_2 = \{5, 10, 15\}$, and $\mathbb{P}_3 = \{25, 30, 35\}$, which are also illustrated in Fig. 4(a).

In the following development, the robustness of coprime arrays will be investigated based on the theory in the companion paper [9]. To begin with, the closed-form expressions of $\mathcal{E}'_k$ for coprime arrays will be presented in Theorem 3, whose proof can be found in Section V-C.

*Theorem 3:* Let $\mathbb{S}_{\text{coprime}}$ be a coprime array with a coprime pair of integers $M$ and $N$, as defined in Definition 15. Assume that $M, N \geq 2$. Then the $k$-essential Sperner family can be expressed as

$$\mathcal{E}'_1 = \begin{cases} \mathcal{A} \cup \mathcal{B}, & \text{if } M \text{ is odd,} \\ \mathcal{A} \cup \mathcal{B} \cup \{\{MN/2\}\}, & \text{if } M \text{ is even,} \end{cases} \tag{47}$$

$$\mathcal{E}'_2 = \begin{cases} \varnothing, & \text{if } M = 2, \\ \{\{N, 2N\},\ \{2N, 3N\}\}, & \text{if } M = 3, \\ \mathcal{C}, & \text{otherwise,} \end{cases} \tag{48}$$

$$\mathcal{E}'_k = \varnothing, \qquad 3 \leq k \leq |\mathbb{S}|, \tag{49}$$

where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are given by

$$\mathcal{A} \triangleq \{\{nM\} : 0 \leq n \leq N - 1\}, \tag{50}$$

$$\mathcal{B} \triangleq \{\{mN\} : M + 1 \leq m \leq 2M - 1\}, \tag{51}$$

$$\mathcal{C} \triangleq \{\{mN, (M - m)N\} : 1 \leq m \leq \lfloor (M - 1)/2 \rfloor\}. \tag{52}$$

*Example 7:* The implications of Theorem 3 are exemplified by Fig. 4, where the essential sensors (diamonds in Fig. 4), the inessential sensors (rectangles in Fig. 4), and $\mathcal{E}'_2$, are enumerated. Here the coprime arrays have parameters (a) $M = 4, N = 5$ and (b) $M = 5, N = 4$. In Fig. 4(a), the essential elements $0, 4, 8, 12, 16$ are associated with $\mathcal{A}$, as in (50), or $\{0\} \cup \mathbb{P}_1$, as in (44), while the elements $25, 30, 35$ are related to $\mathcal{B}$ in (51), or equivalently $\mathbb{P}_3$ in (46). Furthermore, in Fig. 4(a), the element $MN/2 = 10$ is also essential, which is consistent with (47). The sets in $\mathcal{E}'_2$ are also depicted in Fig. 4. For instance, in Fig. 4(b), both $\{8, 12\}$ and $\{4, 16\}$ belong to $\mathcal{E}'_2$, as described in (48) and (52). Note that the elements in these sets are symmetric with respect to the location $MN/2 = 10$. Another interesting observation is that, among the inessential sensors in Fig. 4(b), some are related to $\mathcal{E}'_2$, such as 4 and 8, but the inessential sensor $MN = 20$ does not belong to any elements in $\mathcal{E}'_k$ for all $k$. In fact, if $M \geq 4$ and $N \geq 2$, it can be shown that $MN$ does not belong to the elements in $\mathcal{E}'_k$ for all $k$, due to Theorem 3.

Theorem 3 can be interpreted as a generalization of the thinned coprime array [16]. For sufficiently large $M$ and $N$, it was shown in [16] that removing the elements at $(\lfloor M/2 \rfloor + 1)N, (\lfloor M/2 \rfloor + 2)N, \ldots, MN$ in a coprime array preserves the difference coarray and the new array geometry is called the thinned coprime array. The above statement is equivalent to the condition that $\{(\lfloor M/2 \rfloor + 1)N, (\lfloor M/2 \rfloor + 2)N, \ldots, MN\}$ is not $\lceil M/2 \rceil$-essential with respect to $\mathbb{S}_{\text{coprime}}$. For instance, in Fig. 4(a), removing $\{15, 20\}$ from $\mathbb{S}_{\text{coprime}}$ does not alter the difference coarray, since $\{15, 20\}$ is not 2-essential. *Furthermore, Theorem 3 makes it possible to create other arrays than thinned coprime arrays but with the same difference coarray.* For example, in Fig. 4(b), deleting either $\{8, 16, 20\}$, $\{4, 8, 20\}$, or $\{4, 12, 20\}$ from $\mathbb{S}_{\text{coprime}}$ does not alter the difference coarray, while none of them is identical to thinned coprime arrays.

### *A. The $k$-Fragility of Coprime Arrays*

In the following development, closed-from expressions for the $k$-fragility of the coprime array will be derived. It is first assumed that $M$ is an even number and $M \geq 4$. In this case, we have $|\mathcal{E}_1| = |\mathcal{E}'_1| = M + N$ so the fragility $F_1 = (N + M)/(N + 2M - 1)$. Next, due to [9, Lemma 5], the 2-essential family $\mathcal{E}_2$ can be expressed as

$$\mathcal{E}_2 = \underbrace{\{\{n_1, n_2\} : \{n_1\} \in \mathcal{E}'_1,\ \{n_2\} \notin \mathcal{E}'_1,\ n_2 \in \mathbb{S}_{\text{coprime}}\}}_{\mathcal{H}_1} \cup \underbrace{\{\{n_1, n_2\} : \{n_1\}, \{n_2\} \in \mathcal{E}'_1\}}_{\mathcal{H}_2} \cup \mathcal{E}'_2.$$

Since $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{E}'_2$ are disjoint, the size of $\mathcal{E}_2$ is given by $|\mathcal{E}_2| = |\mathcal{H}_1| + |\mathcal{H}_2| + |\mathcal{E}'_2| = (N + M)(M - 1) + \binom{N+M}{2} + (M/2 - 1)$ so that fragility $F_2$ becomes $F_2 = (3M^2 + 4MN - 2M + N^2 - 3N - 2)/((N + 2M - 1)(N + 2M - 2))$. Repeating similar arguments for odd $M$ leads to these expressions

$$F_1 = \begin{cases} \frac{N+M-1}{N+2M-1}, & \text{if } M \text{ is odd,} \\ \frac{N+M}{N+2M-1}, & \text{if } M \text{ is even,} \end{cases} \tag{53}$$

$$F_2 = \begin{cases} \frac{3M^2+4MN-4M+N^2-3N+1}{(N+2M-1)(N+2M-2)}, & \text{if } M \text{ is odd,} \\ \frac{3M^2+4MN-2M+N^2-3N-2}{(N+2M-1)(N+2M-2)}, & \text{if } M \text{ is even,} \end{cases} \tag{54}$$

where $M \geq 4$.

As $k$ increases, the closed-form expressions of $F_k$ can be derived but the details become more involved. Here these expressions are omitted in this paper. However, if $k$ is sufficiently large, then $F_k$ can still be characterized by the following proposition:

*Proposition 2:* For the coprime array with a coprime pair of integers $M \geq 4$ and $N \geq 2$, the $k$-fragility satisfies $F_k = 1$ for all $\lceil M/2 \rceil + 1 \leq k \leq N + 2M - 1$.

*Proof:* It follows from Item 3d in Section V-C (before Section VI). ∎

For example, let $M = 4$ and $N = 9$. Using (53), (54), and Proposition 2, it can be shown that $F_1 = 0.8125$, $F_2 = 0.9833$, and $F_k = 1$ for all $3 \leq k \leq 16$. These results are in accordance with the numerical values in the companion paper [9, Fig. 8].

### B. The Probability that the Difference Coarray Changes

In this subsection, the closed-form expressions of $P_c$ for the coprime array are characterized by the following theorem:

*Theorem 4:* Let $\mathbb{S}_{\text{coprime}}$ be the coprime array with a coprime pair of integers $M, N$, as in Definition 15. Assume that $M, N \geq 2$. Then the probability that the difference coarray changes is

$$P_c = \begin{cases} 1 - (1-p)^{|\mathcal{E}_1'|}(1 - 2p^2 + p^3), & \text{if } M = 3, \\ 1 - (1-p)^{|\mathcal{E}_1'|}(1 - p^2)^{|\mathcal{E}_2'|}, & \text{otherwise.} \end{cases} \tag{55}$$

Here $\mathcal{E}_1'$ and $\mathcal{E}_2'$ are the $k$-essential Sperner family of $\mathbb{S}_{\text{coprime}}$, whose expressions are given in Theorem 3.

*Proof:* According to the proof of [9, Theorem 3], the probability $P_c$ can be expressed as $1 - \Pr(\mathfrak{G}_1^c) + \Pr(\mathfrak{G}_1^c)\Pr(\mathfrak{G}_2)$, where $\mathfrak{G}_1^c$ denotes the complement of the event $\mathfrak{G}_1$. The events $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are defined as

$$\mathfrak{G}_1 \triangleq \bigcup_{\mathbb{A}_1 \in \mathcal{E}_1'} \mathfrak{F}(\mathbb{A}_1), \qquad \mathfrak{G}_2 \triangleq \bigcup_{k=2}^{|\mathbb{S}|} \bigcup_{\mathbb{A}_k \in \mathcal{E}_k'} \mathfrak{F}(\mathbb{A}_k). \tag{56}$$

Here $\mathfrak{F}(\mathbb{A}_k) \triangleq \cap_{n \in \mathbb{A}_k}(n \text{ fails})$ denotes the event in which all the elements in $\mathbb{A}_k$ fail. It was proved in [9, (39)] that $\Pr(\mathfrak{G}_1^c) = (1-p)^{|\mathcal{E}_1|}$ for any array geometry. Next we will simplify $\Pr(\mathfrak{G}_2)$. If $M = 2$, then $\Pr(\mathfrak{G}_2) = \Pr(\mathfrak{F}(\varnothing)) = 0$. If $M = 3$, then we obtain

$$\begin{aligned} \Pr(\mathfrak{G}_2) &= \Pr(\mathfrak{F}(\{N, 2N\}) \cup \mathfrak{F}(\{2N, 3N\})) \\ &= \Pr(\mathfrak{F}(\{N, 2N\})) + \Pr(\mathfrak{F}(\{2N, 3N\})) \\ &\qquad - \Pr(\mathfrak{F}(\{N, 2N, 3N\})) \\ &= 2p^2 - p^3. \end{aligned} \tag{57}$$

If $M \geq 4$, then $\Pr(\mathfrak{G}_2)$ can be simplified as

$$\Pr(\mathfrak{G}_2) = 1 - \Pr(\mathfrak{G}_2^c) = 1 - \Pr\left( \bigcap_{\mathbb{A}_2 \in \mathcal{E}_2'} (\mathfrak{F}(\mathbb{A}_2))^c \right). \tag{58}$$

Due to (48), all the elements in $\mathcal{E}_2'$ are disjoint, so all the events $\mathfrak{F}(\mathbb{A}_2)$ are mutually independent. Hence (58) becomes

$$\begin{aligned} \Pr(\mathfrak{G}_2) &= 1 - \prod_{\mathbb{A}_2 \in \mathcal{E}_2'} \Pr((\mathfrak{F}(\mathbb{A}_2))^c) = 1 - \prod_{\mathbb{A}_2 \in \mathcal{E}_2'} (1 - p^2) \\ &= 1 - (1-p^2)^{|\mathcal{E}_2'|}. \end{aligned} \tag{59}$$

Substituting (57), (59), and $\Pr(\mathfrak{G}_1^c) = (1-p)^{|\mathcal{E}_1|}$, into $P_c = 1 - \Pr(\mathfrak{G}_1^c) + \Pr(\mathfrak{G}_1^c)\Pr(\mathfrak{G}_2)$ proves this theorem. ∎

The closed-form expressions of $P_c$ for MESA (6) and coprime arrays (55) can be validated by Monte-Carlo simulations, as in [9, Fig. 12]. It is also deduced from [9, Fig. 12] that the smallest $P_c$ is exhibited by the ULA, followed by the coprime array, and finally the nested array. This observation is also consistent with the conclusion drawn from the fragility $F_k$ of these arrays.

### C. Derivation of the Expressions in Theorem 3

*Example 8:* To begin with, let us demonstrate the main concept of the derivation. Fig. 5(a) shows the coprime array with $M = 7, N = 8$ and its nonnegative part of the difference coarray. Here the elements are depicted in dots while empty space is denoted by crosses. The elements $0, 7, 14, 21, 28, 35, 42, 49, 64, 72, 80, 88, 96, 104$ can be shown to be essential ([9, Lemma 2] and Lemma 8). Therefore, for the elements in $\mathcal{E}_k'$ and $k \geq 2$, it suffices to consider the subarrays $\mathbb{A} \subseteq \{8, 16, 24, 32, 40, 48, 56\}$, as marked in Fig. 5(a). The remaining part of the derivation is to identify the constraints on $\mathbb{A}$ such that $\overline{\mathbb{D}}$ (the difference coarray after the removal of $\mathbb{A}$ from $\mathbb{S}_{\text{coprime}}$) is distinct from $\mathbb{D}$ (the difference coarray of $\mathbb{S}_{\text{coprime}}$). To identify these constraints, we will state and prove three lemmas in this section (Lemmas 10 to 12). The brief implications of these lemmas are as follows

| | | | |
|---|---|---|---|
| Lemma 10: | $\boxed{|\mathbb{A}| \leq M - 2}$ | $\Rightarrow$ | $\boxed{\overline{\mathbb{D}}_1 = \mathbb{D}_1,}$ |
| Lemma 11: | $\boxed{\mathbb{A} \text{ and } \mathbb{A}_R \text{ are disjoint}}$ | $\Leftrightarrow$ | $\boxed{\overline{\mathbb{D}}_3 = \mathbb{D}_3,}$ |
| Lemma 12: | $\boxed{\overline{\mathbb{D}}_1 = \mathbb{D}_1 \text{ and } \overline{\mathbb{D}}_3 = \mathbb{D}_3}$ | $\Leftrightarrow$ | $\boxed{\overline{\mathbb{D}} = \mathbb{D},}$ |

where $\overline{\mathbb{D}}_1$ $\mathbb{D}_1$, $\overline{\mathbb{D}}_3$, $\mathbb{D}_3$, and $\mathbb{A}_R$ will be defined shortly. These results can be applied to Fig. 5(b), where $\mathbb{A} = \{16, 32, 56\}$, $\mathbb{A}_R = \{0, 24, 40\}$, and $|\mathbb{A}| = 3$. It can be readily shown that $\overline{\mathbb{D}} = \mathbb{D}$ using Lemmas 10 to 12 without actually computing $\overline{\mathbb{D}}$. As a result, $\mathbb{A}$ does not belong to $\mathcal{E}_3'$.

Next we will proceed to the rigorous derivation of the expressions in Theorem 3. In what follows, it is assumed that the coprime array, as defined in Definition 15, satisfies $M, N \geq 2$. The self difference of a set $\mathbb{S}$ is denoted by $\text{SD}(\mathbb{S}) \triangleq \{n - n' : n, n' \in \mathbb{S}\}$ and the cross difference between $\mathbb{S}_1$ and $\mathbb{S}_2$ are given by $\text{CD}(\mathbb{S}_1, \mathbb{S}_2) \triangleq \{\pm(n_1 - n_2) : n_1 \in \mathbb{S}_1, n_2 \in \mathbb{S}_2\}$. The following lemmas are useful in proving Theorem 3:

*Lemma 7:* Assume that $n_1, n_2 \in \mathbb{S}_{\text{coprime}}$ and $1 \leq u \leq N - 1$ and $1 \leq v \leq M - 1$. Then $n_1 - n_2 = uM - vN$ if and only if the pair $(n_1, n_2)$ is $(uM, vN)$ or $((M - v)N, (N - u)M)$.

*Proof:* The proof consists of four cases of $n_1$ and $n_2$:

1) $n_1, n_2 \in \{0\} \cup \mathbb{P}_1$: Let $n_1 = q_1 M$ and $n_2 = q_2 M$ for $0 \leq q_1, q_2 \leq N - 1$. The equation $n_1 - n_2 = uM - vN$ can be rearranged as $(u - q_1 + q_2)M = vN$. Since $M$ and $N$ are coprime, $v$ is an integer multiple of $M$, which contradicts $1 \leq v \leq M - 1$.
2) $n_1, n_2 \in \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$: Assume that $n_1 = q_1 N$ and $n_2 = q_2 N$ for $1 \leq q_1, q_2 \leq 2M - 1$. Then $n_1 - n_2 = uM - vN$ gives $(v + q_1 - q_2)N = uM$. Hence $u$ is divisible by $N$, which disagrees with $1 \leq u \leq N - 1$.
3) $n_1 \in \{0\} \cup \mathbb{P}_1$ and $n_2 \in \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$: Suppose $n_1 = q_1 M$ and $n_2 = q_2 N$ for $0 \leq q_1 \leq N - 1$ and

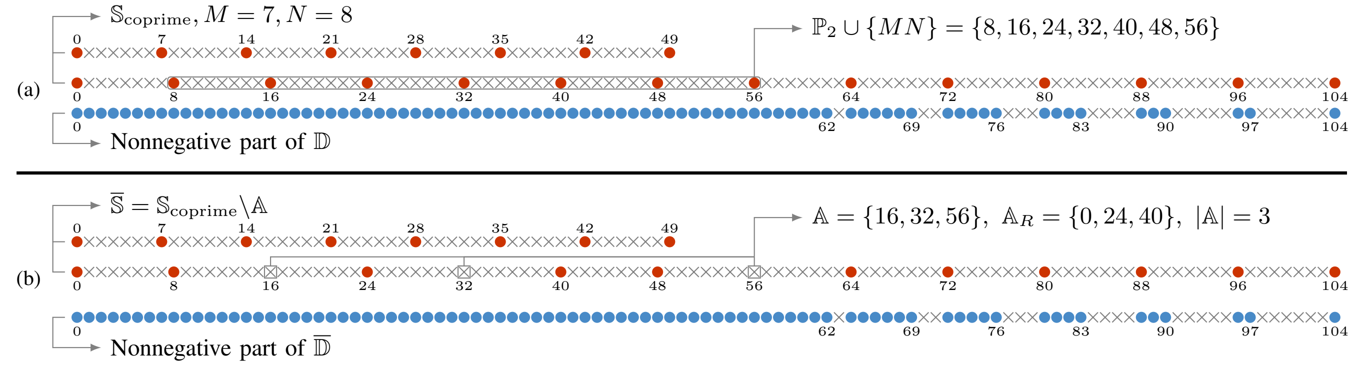


Fig. 5. (a) the coprime array $\mathbb{S}_{\text{coprime}}$ with $M=7, N=8$ and the nonnegative part of the difference coarray $\mathbb{D}$. (b) The array $\overline{\mathbb{S}}$, where the elements in $\mathbb{A}=\{16,32,56\}$ are removed from $\mathbb{S}_{\text{coprime}}$, and the nonnegative part of its difference coarray $\overline{\mathbb{D}}$.

$1 \le q_2 \le 2M-1$. Rearranging $n_1 - n_2 = uM - vN$ leads to $(u-q_1)M = (v-q_2)N$. Since $M$ and $N$ are coprime and $-N+2 \le u - q_1 \le N-1$, we obtain $q_1 = u$ and $q_2 = v$. Hence $(n_1, n_2) = (uM, vN)$.

4) $n_1 \in \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$ and $n_2 \in \{0\} \cup \mathbb{P}_1$: Consider $n_1 = q_1 N$ and $n_2 = q_2 M$ for $1 \le q_1 \le 2M-1$ and $0 \le q_2 \le N-1$. The equation $n_1 - n_2 = uM - vN$ can be rearranged as $(u+q_2)M = (v+q_1)N$. Then we obtain $u + q_2 = N$ and $v + q_1 = M$ because $M$ and $N$ are coprime and $1 \le u + q_2 \le 2N-2$. Therefore $(n_1, n_2) = ((M-v)N, (N-u)M)$. ■

*Lemma 8:* If $n \in \mathbb{P}_1$ or $n \in \mathbb{P}_3$, then $n$ is essential with respect to $\mathbb{S}_{\text{coprime}}$.

*Proof:* Due to [9, Lemma 1], it suffices to show that, if $n_1 = p_1 M \in \mathbb{P}_1$ and $n_3 = p_3 N \in \mathbb{P}_3$, then $w(n_1 - n_3) = 1$. Namely, $(n_1, n_3)$ is the only sensor pair of $\mathbb{S}_{\text{coprime}}$ with difference $n_1 - n_3$.

Assume that there exists another pair $(s_1, s_2) \in \mathbb{S}^2_{\text{coprime}}$ such that $(s_1, s_2) \ne (n_1, n_3)$, and $s_1 - s_2 = n_1 - n_3$. According to $(s_1, s_2)$, we have the following cases:

1) $s_1, s_2 \in \{0\} \cup \mathbb{P}_1$: Assume that $s_1 = q_1 M$ and $s_2 = q_2 M$ for $0 \le q_1, q_2 \le N-1$. The condition $s_1 - s_2 = n_1 - n_3$ can be rearranged as $(p_1 - q_1 + q_2)M = p_3 N$. Since $M$ and $N$ are coprime, the parameter $p_3$ is an integer multiple of $M$, which contradicts (46).
2) $s_1, s_2 \in \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$: Let $s_1 = q_1 N$ and $s_2 = q_2 N$ for $1 \le q_1, q_2 \le 2M-1$. The condition $s_1 - s_2 = n_1 - n_3$ becomes $(p_3 + q_1 - q_2)N = p_1 M$. Due to the coprimeness of $M$ and $N$, the parameter $p_1$ is divisible by $N$, causing a contradiction with (44).
3) $s_1 \in \{0\} \cup \mathbb{P}_1$ and $s_2 \in \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$: Suppose that $s_1 = q_1 M$ and $s_2 = q_2 N$ for $0 \le q_1 \le N-1$ and $1 \le q_2 \le 2M-1$. If $s_1 - s_2 = n_1 - n_3$, then $(p_1 - q_1)M = (p_3 - q_2)N$. The coprimeness of $M$ and $N$ indicates that $N$ divides $p_1 - q_1$. Since $-N+2 \le p_1 - q_1 \le N-1$, we have $p_1 = q_1$, $s_1 = n_1$, and $s_2 = n_3$, which contradicts $(s_1, s_2) \ne (n_1, n_3)$.
4) $s_1 \in \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$ and $s_2 \in \{0\} \cup \mathbb{P}_1$: We assume that $s_1 = q_1 N$ and $s_2 = q_2 M$ for $1 \le q_1 \le 2M-1$ and $0 \le q_2 \le N-1$. The condition $s_1 - s_2 = n_1 - n_3$ becomes $(p_3 + q_1)N = (p_1 + q_2)M$. We have $p_1 + q_2 = N$ because $M$ and $N$ are coprime and $1 \le p_1 + q_2 \le 2N-2$. Hence $p_3 + q_1 = M$, which contradicts the range of $p_3 + q_1$ $(M+2 \le p_3 + q_1 \le 4M-2)$. ■

*Lemma 9:* $\mathrm{SD}(\{0\} \cup \mathbb{P}_1) \cup \mathrm{CD}(\mathbb{P}_1, \{MN\}) = \mathrm{SD}(\{0\} \cup \mathbb{P}_1)$.

*Proof:* The elements in $\mathrm{CD}(\mathbb{P}_1, \{MN\})$ can be expressed as $\pm(MN - p_1 M)$ for $1 \le p_1 \le N-1$, which is equivalent to $\pm((N-p_1)M - 0)$. Since $1 \le N - p_1 \le N-1$, we have $\pm(MN - p_1 M) \in \mathrm{SD}(\{0\} \cup \mathbb{P}_1)$. ■

Next we move on to the main argument. Due to [9, Lemma 2] and Lemma 8, the family $\mathcal{E}'_1$ contains $\mathcal{A}$ and $\mathcal{B}$. For the remaining elements in $\mathbb{S}_{\text{coprime}}$, it is assumed that $\mathbb{A} \subseteq \mathbb{P}_2 \cup \{MN\}$ and $|\mathbb{A}| = k$. Let $\overline{\mathbb{S}}$ be $\mathbb{S}_{\text{coprime}} \backslash \mathbb{A}$ and $\overline{\mathbb{D}}$ be the difference coarray of $\overline{\mathbb{S}}$. The sets $\overline{\mathbb{D}}_1$, $\overline{\mathbb{D}}_2$, and $\overline{\mathbb{D}}_3$ are defined as

$$\overline{\mathbb{D}}_1 \triangleq \mathrm{SD}((\{0\} \cup \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3) \backslash \mathbb{A}), \tag{60}$$

$$\overline{\mathbb{D}}_2 \triangleq \mathrm{CD}(\mathbb{P}_1, (\mathbb{P}_2 \cup \{MN\}) \backslash \mathbb{A}), \tag{61}$$

$$\overline{\mathbb{D}}_3 \triangleq \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_2 \backslash \mathbb{A}). \tag{62}$$

Furthermore, the sets $\mathbb{D}_1 \triangleq \mathrm{SD}(\{0\} \cup \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3)$, $\mathbb{D}_2 \triangleq \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_2 \cup \{MN\})$, and $\mathbb{D}_3 \triangleq \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_2)$. Under these assumptions, $\overline{\mathbb{D}}$ can be expressed as

$$\begin{aligned} \overline{\mathbb{D}} &= \mathrm{SD}(\{0\} \cup \mathbb{P}_1) \cup \mathrm{SD}((\{0\} \cup \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3) \backslash \mathbb{A}) \\ &\quad \cup \mathrm{CD}(\{0\} \cup \mathbb{P}_1, (\{0\} \cup \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3) \backslash \mathbb{A}) \quad (63) \\ &= \mathrm{SD}(\{0\} \cup \mathbb{P}_1) \cup \overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_2 \cup \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_3). \quad (64) \end{aligned}$$

The term $\{0\}$ in the cross difference of (63) can be removed since $\mathrm{CD}(\mathbb{B}, \{0\})$ is a subset of $\mathrm{SD}(\{0\} \cup \mathbb{B})$ for any set $\mathbb{B}$. According to the relation between $MN$ and $\mathbb{A}$, the set $\overline{\mathbb{D}}_2$ can be expressed as

$$\overline{\mathbb{D}}_2 = \begin{cases} \overline{\mathbb{D}}_3, & \text{if } MN \in \mathbb{A}, \\ \mathrm{CD}(\mathbb{P}_1, \{MN\}) \cup \overline{\mathbb{D}}_3, & \text{if } MN \notin \mathbb{A}, \end{cases} \tag{65}$$

where $\overline{\mathbb{D}}_3$ is given by (62). Substituting (65) into (64) and using Lemma 9 result in

$$\overline{\mathbb{D}} = \mathrm{SD}(\{0\} \cup \mathbb{P}_1) \cup \overline{\mathbb{D}}_1 \cup \overline{\mathbb{D}}_3 \cup \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_3). \tag{66}$$

The following lemmas characterize the difference coarray $\overline{\mathbb{D}}$ in terms of $\overline{\mathbb{D}}_1$ and $\overline{\mathbb{D}}_3$. Here $\mathbb{D}$ is the difference coarray of the coprime array $\mathbb{S}_{\text{coprime}}$.

*Lemma 10:* Assume that $\mathbb{A} \subseteq \mathbb{P}_2 \cup \{MN\}$. If $|\mathbb{A}| = k \leq M-2$, then $\overline{\mathbb{D}}_1 = \mathbb{D}_1$.

*Proof:* First, it can be shown that $\mathrm{SD}(\{0\} \cup \mathbb{P}_3) = \mathbb{D}_1 \backslash \{\pm(M-1)N, \pm MN\}$. It suffices to show that $(M-1)N$ and $MN$ belong to $\overline{\mathbb{D}}_1$ if $k \leq M-2$. In this case, since $|(\mathbb{P}_2 \cup \{MN\}) \backslash \mathbb{A}| = M-k \geq 2$, there exists $n = qN \in (\mathbb{P}_2 \cup \{MN\}) \backslash \mathbb{A}$ such that $2 \leq q \leq M$, If $q = M$, then the differences $(M-1)N$ and $MN$ reside in $\overline{\mathbb{D}}_1$ since $(M-1)N = (2M-1)N - MN$ and $MN = MN - 0$. If $2 \leq q \leq M-1$, then the differences $(M-1)N$ and $MN$ live in $\overline{\mathbb{D}}_1$ since $(M-1)N = (M-1+q)N - qN, MN = (M+q)N - qN$, and $(M-1+q)N, (M+q)N \in \mathbb{P}_3$. ■

*Lemma 11:* Let $\mathbb{A} \subseteq \mathbb{P}_2 \cup \{MN\}$ and $\mathbb{A}_R \triangleq \{MN - a : a \in \mathbb{A}\}$. Then $\overline{\mathbb{D}}_3 = \mathbb{D}_3$ if and only if $\mathbb{A}$ and $\mathbb{A}_R$ are disjoint.

*Proof:* First, it is assumed that $MN$ does not belong to $\mathbb{A}$. We have $\mathbb{D}_3 = \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_2) = \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_2 \backslash \mathbb{A}) \cup \mathrm{CD}(\mathbb{P}_1, \mathbb{A})$. Therefore, the statement that $\overline{\mathbb{D}}_3 = \mathbb{D}_3$ is equivalent to $\mathrm{CD}(\mathbb{P}_1, \mathbb{A}) \subseteq \overline{\mathbb{D}}_3$.

If $\mathbb{A}$ and $\mathbb{A}_R$ are disjoint, then for every $n \in \mathbb{A}$, the location $MN - n \in \mathbb{P}_2 \backslash \mathbb{A}$. Due to Lemma 7, we have $\mathrm{CD}(\mathbb{P}_1, \mathbb{A}) \subseteq \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_2 \backslash \mathbb{A}) = \overline{\mathbb{D}}_3$. If $\mathbb{A}$ and $\mathbb{A}_R$ are not disjoint, then there exists $1 \leq v \leq M-1$ such that $\{vN, (M-v)N\} \subseteq \mathbb{A}$. As a result, $vN \notin \mathbb{P}_2 \backslash \mathbb{A}$ and $(M-v)N \notin \mathbb{P}_2 \backslash \mathbb{A}$. Due to Lemma 7, for some $1 \leq u \leq N-1$, the difference $uM - vN \in \mathbb{D}_3$ is related to the pair $(uM, vN)$ or $((M-v)N, (N-u)M)$. These pairs cannot be found in the cross difference between $\mathbb{P}_1$ and $\mathbb{P}_2 \backslash \mathbb{A}$. Hence $\overline{\mathbb{D}}_3 \neq \mathbb{D}_3$.

Second, let us consider the case of $MN \in \mathbb{A}$. The set $\mathbb{B}$ and $\mathbb{B}_R$ are defined as $\mathbb{B} \triangleq \mathbb{A} \backslash \{MN\} \subseteq \mathbb{P}_2$ and $\mathbb{B}_R \triangleq \{MN - b : b \in \mathbb{B}\} \subseteq \mathbb{P}_2$, respectively. Due to the first part of the proof, we have $\overline{\mathbb{D}}_3 = \mathbb{D}_3$ if and only if $\mathbb{B}$ and $\mathbb{B}_R$ are disjoint. Since $0 \notin \mathbb{B}$ and $MN \notin \mathbb{B}_R$, $\mathbb{B}$ and $\mathbb{B}_R$ being disjoint is equivalent to $\mathbb{A}$ and $\mathbb{A}_R$ being disjoint, which completes the proof. ■

*Lemma 12:* $\overline{\mathbb{D}} = \mathbb{D}$ if and only if $\overline{\mathbb{D}}_1 = \mathbb{D}_1$ and $\overline{\mathbb{D}}_3 = \mathbb{D}_3$.

*Proof:* The sufficiency part of Lemma 12 is trivial using (66). The following shows the necessity part.

Let $m \in \mathbb{D}_1$ but $m \notin \overline{\mathbb{D}}_1$. We denote $m = rN$ for $-(2M-1) \leq r \leq 2M-1$. We will show that the union of $\mathrm{SD}(\{0\} \cup \mathbb{P}_1)$, $\overline{\mathbb{D}}_3$, and $\mathrm{CD}(\mathbb{P}_1, \mathbb{P}_3)$, does not contain $m$, implying that $\overline{\mathbb{D}} \neq \mathbb{D}$. If $m \in \mathrm{SD}(\{0\} \cup \mathbb{P}_1)$, then there exists $-(N-1) \leq s \leq N-1$ such that $rN = sM$. Due to the coprimeness of $M$ and $N$, the parameter $s$ is an integer multiple of $N$, implying $s = 0$ and $m = 0$. But $0 \in \overline{\mathbb{D}}_1$, which contradicts $m \notin \overline{\mathbb{D}}_1$. If $m \in \overline{\mathbb{D}}_3$, then there exists a sensor pair in $\{0\} \cup \mathbb{P}_2 \cup \{MN\} \cup \mathbb{P}_3$ (because $m \in \mathbb{D}_1$) whose difference is $uM - vN$ for $1 \leq u \leq N-1$ and $1 \leq v \leq M-1$ (since $m \in \overline{\mathbb{D}}_3 \subseteq \mathbb{D}_3$). This result contradicts Lemma 7. If $m \in \mathrm{CD}(\mathbb{P}_1, \mathbb{P}_3)$, then there exist $1 \leq p_1 \leq N-1$ and $M+1 \leq p_3 \leq 2M-1$ such that $rN = p_1 M - p_3 N$, implying $(r + p_3)N = p_1 M$. Since $M$ and $N$ are coprime, we have that $p_1$ is divisible by $N$, which violates $1 \leq p_1 \leq N-1$.

If $m \in \mathbb{D}_3$ but $m \notin \overline{\mathbb{D}}_3$, then $m$ can be expressed as $uM - vN$ for $1 \leq u \leq N-1$ and $1 \leq v \leq M-1$. Lemma 7 indicates that, such difference can only be found in the cross difference between $\mathbb{P}_1$ and $\mathbb{P}_2$. Therefore, $m$ does not belong to the union of $\mathrm{SD}(\{0\} \cup \mathbb{P}_1)$, $\overline{\mathbb{D}}_1$, and $\mathrm{CD}(\mathbb{P}_1, \mathbb{P}_3)$. These arguments complete the proof. ■

Now let us consider how the subarray $\mathbb{A} \subseteq \mathbb{P}_2 \cup \{MN\}$ influences the difference coarray $\overline{\mathbb{D}}$. Based on the parameter $M$, we have the following cases:

1) $M = 2$: In this case, we have $\mathbb{A} \subseteq \mathbb{P}_2 \cup \{MN\} = \{N, 2N\}$. Due to Theorem 2, Lemma 11, and Lemma 12, it can be shown that $\overline{\mathbb{D}} \neq \mathbb{D}$ for $\mathbb{A} = \{N\}$ and $\overline{\mathbb{D}} = \mathbb{D}$ for $\mathbb{A} = \{2N\}$. Therefore, $N$ is essential but $2N$ is inessential. If $\mathbb{A} = \{N, 2N\}$, then $\mathbb{A}$ contains the essential element $N$, implying that $\mathbb{A} \notin \mathcal{E}'_2$. These arguments prove (47) to (49) for $M = 2$.
2) $M = 3$: This case leads to $\mathbb{A} \subseteq \mathbb{P}_2 \cup \{MN\} = \{N, 2N, 3N\}$. If $\mathbb{A} = \{N\}$, $\{2N\}$, or $\{3N\}$, then it can be shown that $\overline{\mathbb{D}} = \mathbb{D}$, due to Lemmas 10 to 12. Hence these elements are inessential. If $\mathbb{A} = \{N, 2N\}$, then $\mathbb{A} = \mathbb{A}_R$, so $\overline{\mathbb{D}} \neq \mathbb{D}$, due to Lemmas 11 and 12. Similarly, it can be shown that $\{2N, 3N\}$ is 2-essential, due to $\overline{\mathbb{D}}_1 \neq \mathbb{D}_1$, while $\{N, 3N\}$ is not 2-essential. If $\mathbb{A} = \{N, 2N, 3N\}$, then it is a superset of $\{N, 2N\}$, which is 2-essential. Therefore $\mathbb{A} \notin \mathcal{E}'_3$. As a result, we prove (47) to (49) for $M = 3$.
3) $M \geq 4$: According to the value of $k$, we have the following cases:
   a) $k = 1$: Due to Lemma 10, we have $\overline{\mathbb{D}}_1 = \mathbb{D}_1$. Therefore, based on Lemmas 11 and 12, we have $\overline{\mathbb{D}} \neq \mathbb{D}$ if and only if $\mathbb{A}$ and $\mathbb{A}_R$ are not disjoint. For the essential sensors, since $|\mathbb{A}| = 1$, we have $\mathbb{A} = \mathbb{A}_R$, implying that $n = MN - n$ for some $n \in \mathbb{P}_2$. If $M$ is an odd number, then $n$ is not an integer and $n \notin \mathbb{P}_2$. If $M$ is an even number, then this essential sensor becomes $n = MN/2$, which proves (47).
   b) $k = 2$: Similar to the case of $M \geq 4$ and $k = 1$, we have $\overline{\mathbb{D}} \neq \mathbb{D}$ if and only if $\mathbb{A}$ and $\mathbb{A}_R$ are not disjoint, due to Lemmas 10 to 12. This result means that, all the subarrays of the form $\{n, MN - n\}$ for $n \in \mathbb{P}_2$ belongs to $\mathcal{E}'_2$, which proves (48).
   c) $3 \leq k \leq \lceil M/2 \rceil$: In this case, we have $k \leq \lceil M/2 \rceil \leq M - 2$, which implies $\overline{\mathbb{D}}_1 = \mathbb{D}_1$ due to Lemma 10. Next, according to the set $\mathbb{A}$, we have two cases. If $\mathbb{A}$ and $\mathbb{A}_R$ are disjoint, then $\overline{\mathbb{D}}_3 = \mathbb{D}_3$, due to Lemma 11. Therefore, $\overline{\mathbb{D}} = \mathbb{D}$ and $\mathbb{A} \notin \mathcal{E}'_k$. On the other hand, if $\mathbb{A}$ and $\mathbb{A}_R$ are not disjoint, then there exists $\{n, MN - n\} \subseteq \mathbb{A}$ for some $n \in \mathbb{P}_2$. Since $\{n, MN - n\} \in \mathcal{E}'_2$, we have $\mathbb{A} \notin \mathcal{E}'_k$. These arguments show that $\mathcal{E}'_k$ is empty.
   d) $k \geq \lceil M/2 \rceil + 1$: For any choice of $\mathbb{A}$, it can be shown that there exists $n \in \mathbb{P}_2$ such that $\{n, MN - n\}$ is a subset of $\mathbb{A}$. Hence $\mathbb{A} \in \mathcal{E}_k$ but $\mathbb{A} \notin \mathcal{E}'_k$, implying that $\mathcal{E}'_k$ is empty. All these arguments proves Theorem 3. ■

## VI. Numerical Examples

In this section, we will study the DOA estimation performance of arrays in the presence of random sensor failure, through several numerical examples.

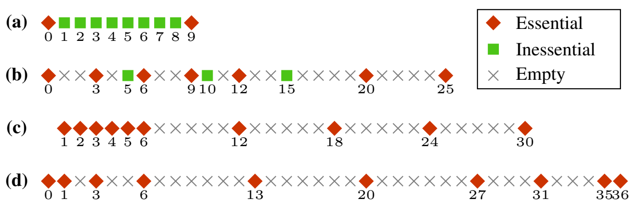


Fig. 6. The array configurations for (a) ULA with 10 elements, (b) the coprime array with $M = 3, N = 5$, (c) the nested array with $N_1 = N_2 = 5$, and (d) the MRA with 10 elements.

TABLE I
THE ARRAY PROFILES IN SECTION VI-A

| Array | Description | $\lvert\mathbb{S}\rvert$ | $\lvert\mathbb{D}\rvert$ | $\lvert\mathbb{U}\rvert$ | $F_1$ |
|---|---|---|---|---|---|
| **(a)** | ULA | 10 | 19 | 19 | 0.2 |
| **(b)** | Coprime array $M = 3, N = 5$ | 10 | 43 | 35 | 0.7 |
| **(c)** | Nested array $N_1 = 5, N_2 = 5$ | 10 | 59 | 59 | 1 |
| **(d)** | MRA | 10 | 73 | 73 | 1 |

### *A. Comparison of ULA, MRA, Nested Arrays, and Coprime Arrays*

Fig. 6 depicts (a) the ULA, (b) the coprime array with $M = 3, N = 5$, (c) the nested array with $N_1 = N_2 = 5$, and (d) the MRA. All these arrays have 10 physical sensors. Here the essential sensors and the inessential sensors are denoted by diamonds and rectangles, respectively. It can be shown that the difference coarrays are $\{0, \pm 1, \ldots, \pm 9\}$ for the ULA, $\{0, \pm 1, \ldots, \pm 17, \pm 19, \pm 20, \pm 22, \pm 25\}$ for the coprime array, $\{0, \pm 1, \ldots, \pm 29\}$ for the nested array, and $\{0, \pm 1, \ldots, \pm 36\}$ for the MRA. Details such as the size of the difference coarray $|\mathbb{D}|$, the size of the central ULA segment $|\mathbb{U}|$, and the fragility $F_1$ are summarized in Table I.

In Fig. 7, the DOA estimation is done by the coarray MUSIC algorithm. The reason why the MHA and the Cantor array with 10 sensors are not included is two-fold. First, coarray MUSIC is usually not deployed for MHAs, since they do not necessarily own a large central ULA segment in the difference coarray. Second, the Cantor array is defined only for

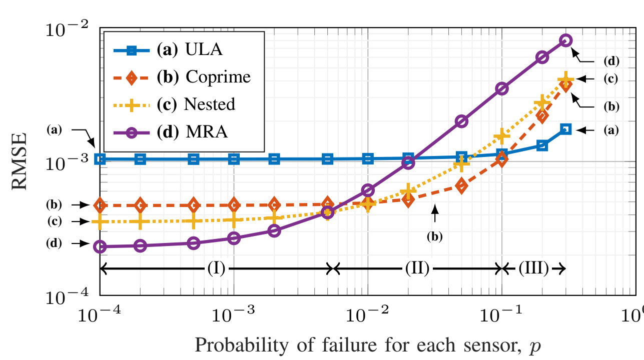


Fig. 7. The dependence of RMSE on the element failure probability $p$ for the array configurations in Fig. 6. There are 10 sensors. The number of snapshots is 100 and the SNR is 0dB. There is one source ($D = 1$) with $\bar{\theta}_1 = 0.25$. Each data point is averaged over $10^6$ independent Monte-Carlo runs.

$|\mathbb{S}| = 2^r$ sensors, where $r$ is a nonnegative integer. Therefore we cannot obtain the Cantor array for 10 sensors.

*1) One Source:* Fig. 7 plots the DOA estimation performance of these arrays as a function of the sensor failure probability $p$, in the range from $10^{-4}$ to $0.3$.[1] Here the number of snapshots is 100 and the signal-to-noise ratio (SNR) is 0dB. There is one source ($D = 1$) at $\bar{\theta}_1 = 0.25$. In each run, each sensor fails independently with probability $p$ and the array output is generated based on [9, (1)], from which coarray MUSIC [29] computes the estimated source direction $\widehat{\bar{\theta}}_i$. For all $10^6$ Monte-Carlo runs, we only collect the instances where coarray MUSIC works, from which the root-mean-square error (RMSE $= \left(\sum_{i=1}^{D} (\bar{\theta}_i - \widehat{\bar{\theta}}_i)/D\right)^{1/2}$) is calculated and averaged.

In this example, coarray MUSIC works for almost all Monte Carlo runs, when $p$ is sufficiently small. In particular, if $p = 0.1$, then the coarray MUSIC is operational in $99.994\%$ of the instances for ULA, in $96.386\%$ of those for the coprime array, in $99.635\%$ of those for the nested array, and in $96.391\%$ of those for the MRA.

Fig. 7 can be divided into three regions:

- Region (I): The MRA owns the smallest RMSE, which is mainly governed by the size of the difference coarray.
- Region (II): Neither the MRA nor the ULA has the smallest RMSE.
- Region (III): The ULA has the least RMSE, which is primarily controlled by the robustness of the array.

In Region (I), the best performance is enjoyed by the MRA, followed by the nested array, then the coprime array, and finally the ULA. This is because for sufficiently small $p$, all the sensors tend to be operational and the performance of coarray MUSIC is dominated by the size of the difference coarray [18]. Note that, as $p$ goes to zero, the RMSE does not approach zero due to finite snapshots and nonzero noise (0dB SNR) [18].

In Region (III), it can be deduced that the RMSE is in accordance with the robustness of these arrays. This is since for large $p$, it is very likely to have multiple faulty elements and the ULA has the least probability that the difference coarray changes. Another observation is that, empirically, for large $p$, the nested array has smaller RMSE than the MRA, even though they are both maximally economic. The is because the $k$-essentialness property only characterizes the integrity of the difference coarray, instead of the central ULA segment of the difference coarray. It is known that the latter has significant influence on the applicability of coarray MUSIC [18], [29].

Another remark is on Region (II). It is observed in Table I that the coprime array does not have the largest difference coarray, nor does it have the smallest fragility $F_1$, but it has the least RMSE in most of Region (II) in Fig. 7. This result shows the existence of sparse arrays that strike a balance between the size and the robustness of the difference coarray. Future research can be directed towards designing such array geometries, which work the best in Region (II).

[1] Based on the exponential distribution, the sensor failure probability can be modeled as $p = 1 - e^{-\lambda t}$, where $\lambda$ is the failure rate and $t$ is time duration [6, Section 2.6.3]. For instance, if $\lambda = 100$ failures per million hours (0.876 failures per year) and $t = 5$ hours, then $p \approx 5 \times 10^{-4}$. Interested readers are referred to [6], [7] and the references therein.

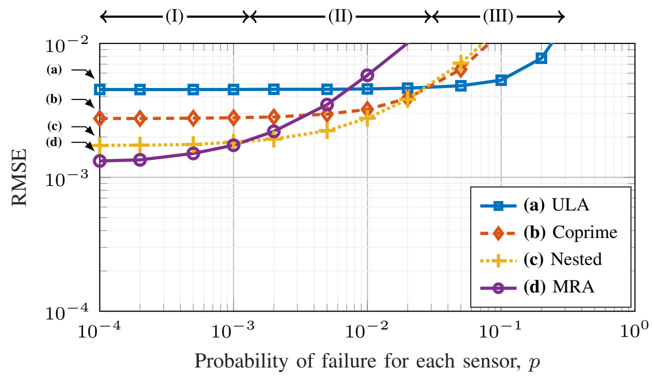


Fig. 8. The dependence of RMSE on the element failure probability $p$ for the array configurations in Fig. 6. The number of snapshots is 100 and the SNR is $-10$dB. There is one source ($D = 1$) with $\bar{\theta}_1 = 0.25$. Each data point is average over $10^6$ independent Monte-Carlo runs.

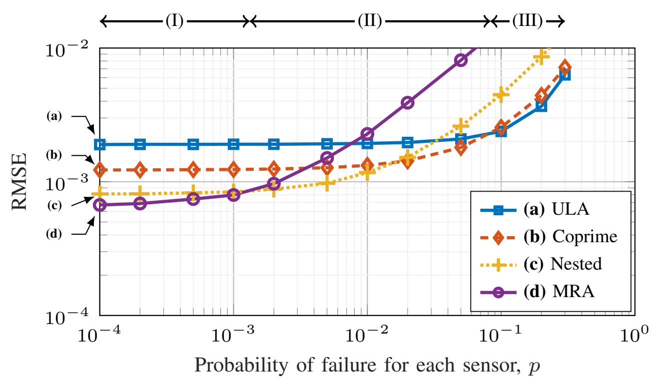


Fig. 9. The dependence of RMSE on the element failure probability $p$ for the array configurations in Fig. 6. The number of snapshots is 100 and the SNR is 0dB. There are five sources ($D = 5$) with $\bar{\theta}_i = -0.25 + 0.125(i-1)$ for $i = 1, 2, \ldots, 5$. Each data point is average over $10^6$ independent Monte-Carlo runs.

Next we will investigate an example with low SNR, namely $-10$dB. Fig. 8 shows the estimation performance of the arrays in Table I under sensor failure. The number of snapshots is 100 and the only source has $\bar{\theta}_1 = 0.25$. Several observations can be drawn from Figs. 7 and 8. First, since Fig. 8 has lower SNR, the RMSEs in Fig. 8 are larger than those in Fig. 7. Second, the ranges of the three regions in Fig. 8 are different from those in Fig. 7. Region (I) is now approximately $10^{-4} < p < 1.2 \times 10^{-3}$, Region (II) has around $1.2 \times 10^{-3} < p < 2.8 \times 10^{-2}$, and Region (III) corresponds to $p > 2.8 \times 10^{-2}$. Furthermore, in Fig. 8, the nested array has smaller RMSE than the coprime array in Region (II).

*2) Multiple Sources:* Fig. 9 demonstrates the RMSE of the array geometries in Fig. 6. We consider $D = 5$ sources with $\bar{\theta}_i = -0.25 + 0.125(i-1)$ for $i = 1, 2, \ldots, 5$. The SNR is 0dB and the number of snapshots is 100. It can be deduced that the RMSEs increase when there are multiple sources, compared with the case of one source in Fig. 7. Furthermore, in Fig. 8, Region (I) becomes $10^{-4} < p < 1.1 \times 10^{-3}$, Region (II) corresponds to $1.1 \times 10^{-3} < p < 7.8 \times 10^{-2}$, and Region (III) is $p > 7.8 \times 10^{-2}$. Among all the array configurations, the nested array has the least RMSE around $p = 5 \times 10^{-3}$ while the coprime array owns the smallest RMSE near $p = 5 \times 10^{-2}$.

TABLE II
THE ARRAY PROFILES IN SECTION VI-B

| Array | Description | $\lvert\mathbb{S}\rvert$ | $\lvert\mathbb{D}\rvert$ | $\lvert\mathbb{U}\rvert$ | $F_1$ |
|---|---|---|---|---|---|
| **(N1)** | Nested array $N_1 = 8, N_2 = 8$ | 16 | 143 | 143 | 1 |
| **(N2)** | Nested array $N_1 = 9, N_2 = 7$ | 16 | 139 | 139 | 1 |
| **(N3)** | Nested array $N_1 = 10, N_2 = 6$ | 16 | 131 | 131 | 1 |
| **(C1)** | Coprime array $M = 4, N = 9$ | 16 | 103 | 79 | 0.8125 |
| **(C2)** | Coprime array $M = 5, N = 7$ | 16 | 103 | 79 | 0.6875 |
| **(C3)** | Coprime array $M = 6, N = 5$ | 16 | 91 | 71 | 0.6875 |

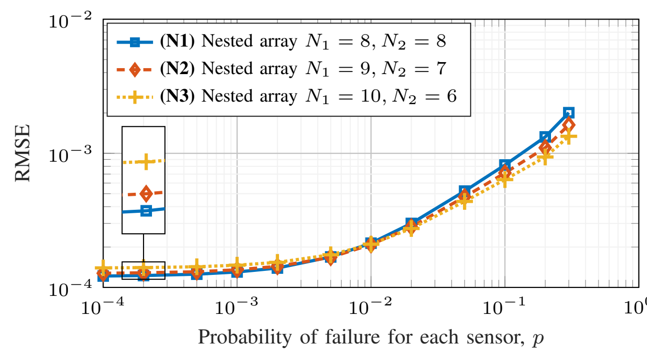


Fig. 10. The dependence of RMSE on the element failure probability $p$ for the nested arrays in Table II. The number of snapshots is 100 and the SNR is 0dB. There is one source with $\bar{\theta}_1 = 0.25$. Each data point is average over $10^6$ independent Monte-Carlo runs.

## B. Comparison of Different Possible Configurations of Nested and Coprime Arrays

For a fixed number of sensors, there are several ways to configure a nested array, and similarly for a coprime array. In this section, we compare the performance of these different configurations under sensor failure. We assume the number of sensors is 16. We select three possible nested arrays, denoted by **(N1)**, **(N2)**, and **(N3)**, as well as three coprime arrays, denoted by **(C1)**, **(C2)**, and **(C3)**, according to Definitions 12 and 15. Table II lists the size of the difference coarray $|\mathbb{D}|$, the size of the central ULA segment of the difference coarray $|\mathbb{U}|$, and the 1-fragility $F_1$ for these array configurations. In this example, for these nested arrays, $|\mathbb{D}|$ decreases monotonically as the parameter $N_1$ increases, but $F_1$ remains unity. For the coprime arrays, all of $|\mathbb{D}|$, $|\mathbb{U}|$, and $F_1$ are monotonically decreasing when the parameter $M$ grows.

Fig. 10 shows the dependence of RMSE on the element failure probability $p$ for the nested arrays. We consider 100 snapshots, 0dB SNR, and $10^6$ Monte-Carlo runs. There is one source at $\bar{\theta}_1 = 0.25$. For $p \approx 2 \times 10^{-4}$, the smallest RMSE is exhibited by **(N1)**, followed by **(N2)**, and finally **(N3)**. This is because the size of the difference coarray is ordered by **(N1)** (largest), **(N2)**, and **(N3)** (smallest). For $p \approx 10^{-1}$, the smallest RMSE is given by **(N3)**, **(N2)**, and finally **(N1)**. This result cannot be explained by examining $F_1$, since $F_1 = 1$

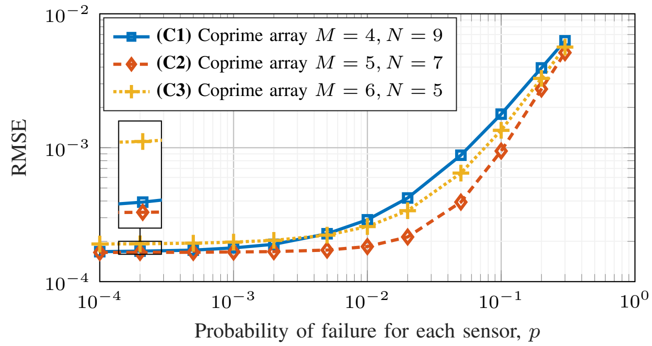


Fig. 11. The dependence of RMSE on the element failure probability $p$ for the coprime arrays in Table II. The number of snapshots is 100 and the SNR is 0dB. There is one source with $\bar{\theta}_1 = 0.25$. Each data point is average over $10^6$ independent Monte-Carlo runs.

for all these nested arrays. It is possible that this phenomenon can be explained by the robustness of the ULA segment in the difference coarray, on which the coarray MUSIC relies. This requires further thought.

Next let us move on to coprime arrays, whose profiles are listed in Table II and the DOA estimation performance is plotted in Fig. 11. For $p \approx 2 \times 10^{-4}$, the smallest RMSE is exhibited by **(C2)**, then **(C1)**, and finally **(C3)**. This result can be explained as follows.

1) Since $p \approx 2 \times 10^{-4}$ is relatively small, the DOA estimation performance is primarily governed by the size of $\mathbb{U}$. Table II shows that, **(C1)** and **(C2)** have the same $|\mathbb{U}|$ while **(C3)** owns the smallest $|\mathbb{U}|$. This is roughly consistent with the RMSE for $p \approx 2 \times 10^{-4}$ in Fig. 11.
2) It is also observed that the RMSE of **(C2)** is slightly smaller than that of **(C1)** for $p \approx 2 \times 10^{-4}$. The reason is that (a) **(C1)** and **(C2)** have the same $|\mathbb{U}|$, and (b) $F_1$ of **(C2)** is smaller than that of **(C1)**, implying that **(C2)** is more robust than **(C1)**.

For the coprime arrays with $p > 10^{-2}$, **(C2)** has the smallest RMSE, followed by **(C3)**, and finally **(C1)**. This phenomenon is due to the following.

1) Since $p$ is relatively large, we can first compare the robustness of these arrays. Table II indicates that $F_1$ of **(C1)** is the largest and the remaining are identical. As a result, the RMSE of **(C1)** is the largest in this region.
2) Since **(C2)** and **(C3)** have the same fragility, the sizes of $\mathbb{U}$ should be compared to get more insight. We have that $|\mathbb{U}|$ of **(C2)** is larger than that of **(C3)**. Therefore the RMSE of **(C3)** is smaller than that of **(C2)**.

## VII. Concluding Remarks

In this paper, we studied the robustness of the difference coarrays for MRA, MHA, nested arrays, Cantor arrays, ULA, and coprime arrays, with respect to sensor failures, through the theory presented in the companion paper [9]. The proposed closed-form expressions for the $k$-essential Sperner family not only indicate the importance of elements in these arrays, but also provide many insights into the reliability and the DOA estimation performance based on these arrays.

Future research will be directed towards designing novel sparse array geometries that strike a balance between performance and robustness [30], [31]. For instance, it could be possible to robustify a given array geometry by adding or redistributing the elements in the array. Another future direction is to focus on the robustness of the central ULA segment in the difference coarray, which has an impact on the applicability of coarray MUSIC.

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