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ROBUSTNESS OF COARRAYS OF SPARSE ARRAYS TO SENSOR FAILURES

Chun-Lin Liu[1] *and P. P. Vaidyanathan*[2]

Dept. of Electrical Engineering, MC 136-93
California Institute of Technology, Pasadena, CA 91125, USA
E-mail: cl.liu@caltech.edu[1] and ppvnath@systems.caltech.edu[2]

ABSTRACT

Sparse arrays can identify $\mathcal{O}(N^2)$ uncorrelated sources using N physical sensors. This property is because the difference coarray, defined as the differences between sensor locations, has uniform linear array (ULA) segments of length $\mathcal{O}(N^2)$. It is empirically known that, for sparse arrays like minimum redundancy arrays, nested arrays, and coprime arrays, this $\mathcal{O}(N^2)$ segment is susceptible to sensor failure, which is an important issue in practical systems. This paper presents the (k-)essentialness property, which characterizes the combinations of the failing sensors that shrink the difference coarray. Based on this, the notion of *fragility* is proposed to quantify the reliability of sparse arrays with faulty sensors, along with comprehensive studies of their properties. It is demonstrated through examples that there do exist sparse arrays that are as robust as ULA and at the same time, they enjoy $\mathcal{O}(N^2)$ consecutive elements in the difference coarray.

***Index Terms*—** Sparse arrays, robustness, essentialness, fragility.

1. INTRODUCTION

Sparse arrays, which have nonuniform sensor spacing, have recently attracted considerable attention in array signal processing [1–4]. Unlike uniform linear arrays (ULA), which resolve $\mathcal{O}(N)$ uncorrelated sources, some sparse arrays are capable of identifying $\mathcal{O}(N^2)$ uncorrelated sources using N physical sensors. These arrays include minimum redundancy arrays (MRA) [2], nested arrays [3], coprime arrays [4], and their generalizations [5, 6]. This $\mathcal{O}(N^2)$ property is because the difference coarray, defined as the differences between the sensor locations, possesses an $\mathcal{O}(N^2)$-long central ULA segment. By analyzing the samples on the difference coarray, quite a few direction-of-arrival (DOA) estimators have been shown to resolve more uncorrelated sources than sensors [3, 7, 8].

In practice, *sensor failure* is an important issue for the reliability of the overall system. It is *empirically* known that, for most sparse arrays, faulty sensors could shrink the $\mathcal{O}(N^2)$-long ULA segment in the difference coarray significantly. Furthermore, small ULA segments in the difference coarray typically lead to performance degradation [3, 7–9]. Due to these observations, in the past, sparse arrays were considered to be not robust to sensor failure. However, the impact of damaged sensors on sparse arrays remains to be analyzed, since these observations assume specific array configurations.

The issue of sensor failure was addressed in the literature in two aspects, including 1) developing new algorithms that are functional in the presence of sensor failure and 2) analyzing the robustness of array geometries. In the first part, various approaches have been developed, including DOA estimators based on minimal resource allocation network [10], impaired array beamforming and DOA estimation [11], and so on [12, 13]. However, the interplay between the array configuration and the exact condition under which these algorithms are applicable, remains to be investigated. The second aspect assesses the robustness of array configurations with faulty elements [14, 15]. For instance, Alexiou and Manikas [14] proposed various measures to quantify the robustness of arrays while Carlin *et al.* [15] performed a statistical study on the beampattern with a given element failure rate. Even so, the impact of damaged elements on the difference coarray has not yet been analyzed in a deterministic fashion, which is crucial for sparse arrays.

In this paper, we propose the k-essentialness property and the *fragility* to investigate the influence of faulty sensors on the difference coarray. The main focus of this paper is not to develop new algorithms, but to analyze the robustness of arrays. An element is said to be *essential* if its deletion changes the difference coarray. Note that the essentialness property, introduced to study the economy of sensors [16], depends purely on the array geometry, instead of the source parameters and the estimators. It will be shown in this paper that the essentialness property can be used to assess the robustness of the array geometry, in the sense of preserving the difference coarray. Furthermore, if there are multiple damaged elements, the concept of *k-essentialness* enables us to study the interplay between failure patterns and the difference coarray. By enumerating the k-essential subarrays, the robustness is quantified by the *fragility*, which ranges from 0 to 1. An array is more robust or less fragile if the fragility is closer to 0. It will be shown that, some sparse arrays are as robust as the ULA even though they possess a $\mathcal{O}(N^2)$-long ULA segment in the difference coarray.

Paper outline: Section 2 reviews the theory of sparse arrays. Section 3 proposes the essentialness, the k-essentialness, the fragility, and the related properties. Section 4 compares the essentialness property and the fragility among several known array configurations while Section 5 concludes this paper.

2. REVIEW OF SPARSE ARRAYS

Assume that D monochromatic and far-field sources with wavelength λ impinge on a sensor array, where the sensor locations are $n\lambda/2$. Here n belongs to an integer set $\mathbb{S}$. Let $\theta_i \in [-\pi/2, \pi/2]$ and $A_i \in \mathbb{C}$ be the DOA and the complex amplitude of the ith source, respectively. The array output of the sensor array $\mathbb{S}$, denoted by $\mathbf{x}_\mathbb{S}$, is modeled as

$$\mathbf{x}_\mathbb{S} = \sum_{i=1}^{D} A_i \mathbf{v}_\mathbb{S}(\bar{\theta}_i) + \mathbf{n}_\mathbb{S} \quad \in \mathbb{C}^{|\mathbb{S}|}, \tag{1}$$

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where $\mathbf{v}_{\mathbb{S}}(\bar{\theta}_i) \triangleq [e^{j2\pi\bar{\theta}_i n}]_{n\in\mathbb{S}}$ is the steering vector and $\mathbf{n}_{\mathbb{S}}$ is the noise term. The normalized DOA is defined as $\bar{\theta}_i \triangleq (\sin\theta_i)/2 \in [-1/2, 1/2]$. It is assumed that the sources and the noise are zero-mean and uncorrelated. Namely, if $\mathbf{s} \triangleq [A_1, \ldots, A_D, \mathbf{n}_{\mathbb{S}}^T]^T$, then we have $\mathbb{E}[\mathbf{s}] = \mathbf{0}$ and $\mathbb{E}[\mathbf{s}\mathbf{s}^H] = \mathrm{diag}(p_1, \ldots, p_D, p_n\mathbf{I})$, where p_i and p_n are the powers of the ith source and the noise, respectively. Under these assumptions, the covariance matrix of $\mathbf{x}_{\mathbb{S}}$ becomes [3]:

$$\mathbf{R}_{\mathbb{S}} = \mathbb{E}[\mathbf{x}_{\mathbb{S}}\mathbf{x}_{\mathbb{S}}^H] = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{S}}(\bar{\theta}_i)\mathbf{v}_{\mathbb{S}}^H(\bar{\theta}_i) + p_n\mathbf{I}. \tag{2}$$

Vectorizing (2) and removing duplicated entries lead to the autocorrelation vector $\mathbf{x}_{\mathbb{D}}$ on the difference coarray:

$$\mathbf{x}_{\mathbb{D}} = \sum_{i=1}^{D} p_i \mathbf{v}_{\mathbb{D}}(\bar{\theta}_i) + p_n \mathbf{e}_0 \quad \in \mathbb{C}^{|\mathbb{D}|}, \tag{3}$$

where $\mathbf{e}_0$ is a column vector with 1 in the middle (the $(|\mathbb{D}|+1)/2$-th element) and 0 elsewhere. The difference coarray $\mathbb{D}$ is defined as

Definition 1. The difference coarray of the sensor array $\mathbb{S}$ is defined as $\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$.

Note that (3) can be regarded as the output defined on the difference coarray, instead of that on the physical array (1). If sensor locations are designed properly, the size of the difference coarray can be much larger than the size of the physical array. In particular, $|\mathbb{D}| = \mathcal{O}(|\mathbb{S}|^2)$. This property makes it possible to develop coarray-based DOA estimators that resolve more uncorrelated sources than sensors and achieve better spatial resolution [3, 4, 7, 8].

Next we will define some useful quantities regarding the difference coarray. The central ULA segment of $\mathbb{D}$ is defined as $\mathbb{U} \triangleq \{m : \{0, \pm1, \ldots, \pm|m|\} \subseteq \mathbb{D}\}$. The smallest ULA containing $\mathbb{D}$ is denoted by $\mathbb{V} \triangleq \{m \in \mathbb{Z} : \min(\mathbb{D}) \leq m \leq \max(\mathbb{D})\}$. An integer h is said to be a hole in the difference coarray if $h \in \mathbb{V}$ but $h \notin \mathbb{D}$. A difference coarray is *hole-free* if $\mathbb{D} = \mathbb{U} = \mathbb{V}$.

Definition 2. The weight function $w(m)$ of an array $\mathbb{S}$ is defined as the number of sensor pairs with coarray index m. That is, $w(m) = \left|\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\}\right|$.

Example 1. Consider the array $\mathbb{S} = \{0, 1, 4, 5\}$. According to Definition 1, the difference coarray is $\mathbb{D} = \{0, \pm1, \pm3, \pm4, \pm5\}$. It can be shown that $\mathbb{U} = \{0, \pm1\}$ and $\mathbb{V} = \{0, \pm1, \ldots, \pm5\}$. Therefore, ±2 are holes in the difference coarray. The weight function $w(1)$, for instance, is 2 since there are two sensor pairs $(1, 0)$ and $(5, 4)$ with difference 1.

It is known that the difference coarray plays a significant role in DOA estimation based on (3). For instance, the performance of coarray MUSIC relies on $\mathbb{U}$ [3, 8, 9, 17]. In addition, the performance of any unbiased DOA estimator using sparse arrays is known to be limited by the difference coarray [9, 18, 19].

Now let us review some existing array geometries and their difference coarrays. First, the ULA with N elements [1] is denoted by the set $\mathbb{S}_{\text{ULA}} \triangleq \{0, 1, \ldots, N-1\}$. The difference coarray for ULA is $\mathbb{D}_{\text{ULA}} = \{\pm0, \pm1, \ldots, \pm(N-1)\}$. It can be shown that $|\mathbb{D}_{\text{ULA}}| = 2N - 1 = \mathcal{O}(N)$. Next, the nested array [3] is defined as $\mathbb{S}_{\text{nested}} \triangleq \{1, 2, \ldots, N_1, (N_1+1), 2(N_1+1), \ldots, N_2(N_1+1)\}$, where N_1 and N_2 are positive integers. The difference coarray of the nested array is $\mathbb{D}_{\text{nested}} = \{0, \pm1, \ldots, \pm(N_2(N_1+1) - 1)\}$. Given N elements, if N_1 and N_2 are approximately $N/2$, the size of the difference coarray can be shown to be $|\mathbb{D}| = \mathcal{O}(N^2)$ [3]. Finally, the coprime array is parameterized by a pair of integers (M, N) whose greatest common divisor is 1. The sensors for the coprime array are located at $\mathbb{S}_{\text{coprime}} \triangleq \{0, M, \ldots, (N-1)M, N, 2N, \ldots, (2M-1)N\}$. It can be shown that the difference coarray for the coprime array has holes [4] and the central ULA segment is $\mathbb{U}_{\text{coprime}} = \{0, \pm1, \ldots, \pm(MN + M - 1)\}$ [5]. Namely, $|\mathbb{U}_{\text{coprime}}| = 2MN + 2M - 1 = \mathcal{O}(MN)$, and there are $|\mathbb{S}_{\text{coprime}}| = N + 2M - 1 = \mathcal{O}(M + N)$ physical sensors.

3. ESSENTIALNESS AND FRAGILITY

It was known in [8] that coarray MUSIC is applicable to the autocorrelation vector on $\mathbb{U}$ as long as $|\mathbb{U}| > 1$. However, it will be demonstrated in Example 2 that $\mathbb{U}$ is susceptible to sensor failure. For certain array geometries, even one damaged physical element could alter $\mathbb{U}$ significantly and coarray MUSIC may fail.

Example 2. Consider the array geometry $\mathbb{S} = \{0, 1, 2, 4, 6\}$ which has difference coarray $\mathbb{D} = \{0, \pm1, \ldots, \pm6\} = \mathbb{U}$. In this case, the coarray MUSIC algorithm can be used, since $|\mathbb{U}| = 13 > 1$. Now suppose the sensor located at 1 fails. The new array configuration and the associated difference coarray becomes $\mathbb{S}_1 = \{0, 2, 4, 6\}$ and $\mathbb{D}_1 = \{0, \pm2, \pm4, \pm6\}$, respectively. So $|\mathbb{U}_1| = 1$ and the coarray MUSIC algorithm is not applicable. On the other hand, if the element at 2 fails, we have $\mathbb{S}_2 = \{0, 1, 4, 6\}$ and $\mathbb{D}_2 = \{0, \pm1, \ldots, \pm6\}$. Since $|\mathbb{U}_2| = 13 > 1$, the coarray MUSIC algorithm can still be implemented.

Example 2 shows that, the location of the faulty sensors could modify the difference coarray, which influences the applicability of coarray MUSIC. Note that, even if the difference coarray changes, there might exist other DOA estimators that work on the new difference coarray. However, this scenario will be left for future work.

The concept that difference coarray changes after the removal of sensors from the array configuration is described by the essentialness property [16]:

Definition 3. The sensor located at $n \in \mathbb{S}$ is said to be essential with respect to $\mathbb{S}$ if the difference coarray changes when sensor at n is deleted from the array. That is, if $\widetilde{\mathbb{S}} = \mathbb{S}\backslash\{n\}$, then $\widetilde{\mathbb{D}} \neq \mathbb{D}$. Here $\mathbb{D}$ and $\widetilde{\mathbb{D}}$ are the difference coarrays for $\mathbb{S}$ and $\widetilde{\mathbb{S}}$, respectively.

The sensor at $n \in \mathbb{S}$ is said to be inessential if it is not essential. For instance, in Example 2, the element at 1 is essential but the one at 2 is inessential. In addition, it can be shown that, for any $\mathbb{S}$, the sensors at $\max(\mathbb{S})$ and $\min(\mathbb{S})$ are both essential. A sensor array $\mathbb{S}$ is said to be *maximally economic* if all the sensors in $\mathbb{S}$ are essential [16]. It can be shown [16] that some well-known array configurations, such as MRA [2], minimum hole arrays [20], nested arrays with $N_2 \geq 2$ [3], and Cantor arrays [21], are maximally economic.

The essentialness property was originally introduced in [16] to study symmetric arrays and Cantor arrays. The main focus in [16] was the economy of sensors. However, the emphasis in this paper is the observation that the essentialness property also characterizes the importance of each element in preserving the difference coarray, as demonstrated in Example 2. This interpretation leads to the robustness analysis of array configurations, as we will develop later. In addition, the essentialness property enables us to deploy sensors of different quality, since the essential sensors typically require devices with low failure rate, as opposed to inessential sensors.

If two sensors are inessential, it means that either one of them can be removed without changing the coarray. But if both sensors are removed, the coarray may change. The k-essentialness property is useful to handle multiple sensor failures:

Definition 4. A subarray $\mathbb{A} \subseteq \mathbb{S}$ is said to be k-essential if 1) $|\mathbb{A}| = k$, and 2) the difference coarray changes when $\mathbb{A}$ is removed from $\mathbb{S}$. Namely, $\widetilde{\mathbb{D}} \neq \mathbb{D}$ if $\widetilde{\mathbb{S}} = \mathbb{S}\backslash\mathbb{A}$. Here $\mathbb{D}$ and $\widetilde{\mathbb{D}}$ are the difference coarrays of $\mathbb{S}$ and $\widetilde{\mathbb{S}}$, respectively.

Note that essentialness, as defined in Definition 3, is equivalent to 1-essentialness ($k = 1$ in Definition 4). Namely, $n \in \mathbb{S}$ is essential if and only if $\{n\} \subseteq \mathbb{S}$ is 1-essential. For brevity, we will use these terms interchangeably.

Example 3. Assume the array configuration is the ULA with 6 elements. We have $\mathbb{S} = \{0, 1, \ldots, 5\}$ and $\mathbb{D} = \{0, \pm 1, \ldots, \pm 5\}$. It can be shown that 0 and 5 are both essential, but the remaining elements are inessential. Using Definition 4, it can be further shown that $\{1, 4\}$ is 2-essential. This is since the difference coarray for $\mathbb{S}\backslash\{1, 4\}$ is $\{0, \pm 1, \pm 2, \pm 3, \pm 5\}$. This example shows that two elements, such as 1 and 4, could be individually inessential, but together the subarray, $\{1, 4\}$, could be 2-essential.

Definition 4 enables us to define the family of sets that contains all the k-essential subarrays, called the k-essential family:

Definition 5. The k-essential family $\mathcal{E}_k$ with respect to a sensor array $\mathbb{S}$ is defined as

$$\mathcal{E}_k \triangleq \{\mathbb{A} : \mathbb{A} \text{ is } k\text{-essential with respect to } \mathbb{S}\}, \tag{4}$$

where $k = 1, 2, \ldots, |\mathbb{S}|$.

Given an array configuration $\mathbb{S}$, the k-essential family is uniquely determined, by examining all possible subarrays. From the computational perspective, enumerating all these subarrays becomes intractable as the number of sensors increases. Even so, it is still possible to determine or bound the cardinality of the k-essential family, as in Proposition 1. This is proved at the end of this Section.

Proposition 1. Let $\mathcal{E}_k$ be the k-essential family with respect to a nonempty integer set $\mathbb{S}$. Then the following properties hold true:

1. $(|\mathbb{S}| - k)|\mathcal{E}_k| \leq (k + 1)|\mathcal{E}_{k+1}|$ for all $1 \leq k \leq |\mathbb{S}| - 1$. The equality holds if and only if $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$.
2. $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$ for all $|\mathbb{S}| - |\mathcal{E}_1| + 1 \leq k \leq |\mathbb{S}|$.
3. If $\mathbb{S}$ is maximally economic, then $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$ for all $1 \leq k \leq |\mathbb{S}|$.
4. Let $M_p = |\{m \in \mathbb{D} : w(m) = p\}|$ be the number of elements in the difference coarray such that the associated weight function is p. If $|\mathbb{S}| \geq 2$, then
$$\left\lceil \frac{1 + \sqrt{1 + 4M_1}}{2} \right\rceil \leq |\mathcal{E}_1| \leq \min\left\{ M_1 + \left\lfloor \frac{M_2}{2} \right\rfloor, |\mathbb{S}| \right\}, \tag{5}$$
where $\lceil\cdot\rceil$ and $\lfloor\cdot\rfloor$ are the ceiling function and the floor function, respectively.

These propositions show that the size of the k-essential family is closely related to the number of sensors $|\mathbb{S}|$ and the weight function $w(m)$, which can be readily computed from the array geometry.

If $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$ for some k, then $\mathbb{S}$ is not necessarily maximally economic. For instance, for the array configuration in Example 2, it can be shown that $\mathcal{E}_5 = \{\{0, 1, 2, 4, 6\}\}$ and $|\mathcal{E}_5| = 1 = \binom{5}{5}$. However, as shown in Example 2, $\mathbb{S}$ is not maximally economic.

Proposition 1.4 shows an elegant relation between the number of essential sensors and the weight function. This result is analogous to Cheeger inequalities in graph theory [22], where the Cheeger constant is bounded by the expressions regarding the eigenvalues of the adjacency matrix of certain graphs. Here in (5), the number of essential sensors is analogous to the Cheeger constant. The bounds in (5) depend on the weight functions, which are known to be the eigenvalues of the matrix $\mathbf{J}^H\mathbf{J}$, or the singular values squared of the matrix $\mathbf{J}$ [23]. It was shown in [23] that the matrix $\mathbf{J}$ indicates the topology of the physical array and the difference coarray, which is analogous to the adjacency matrix of a graph.

Example 4. Let us verify Proposition 1.4 using an example. The MRA with 6 sensors has $\mathbb{S} = \{0, 1, 6, 9, 11, 13\}$. According to the Definition 2, we obtain $M_1 = 22$ and $M_2 = 4$. The lower bound in (5) is given by $\lceil(1 + \sqrt{89})/2\rceil = \lceil 5.217\rceil = 6$ while the upper bound becomes $\min\{24, 6\} = 6$. On the other hand, using Definition 5, we have $|\mathcal{E}_1| = 6$, which verifies (5).

The size of the k-essential family also indicates the likelihood that the difference coarray changes after the removal of k elements. For instance, if $|\mathcal{E}_k| = \binom{|\mathbb{S}|}{k}$, it means any k-sensor failure out of $|\mathbb{S}|$ sensors changes the difference coarray. The notion of *fragility* is useful to capture this idea.

Definition 6. The fragility or k-fragility of a sensor array $\mathbb{S}$ is defined as

$$F_k \triangleq \frac{|\mathcal{E}_k|}{\binom{|\mathbb{S}|}{k}}, \tag{6}$$

where $k = 1, 2, \ldots, |\mathbb{S}|$.

The physical interpretation of fragility is as follows. Suppose there are k faulty sensors in the array. ***The fragility is the probability that the difference coarray changes,*** given all failure patterns occur with equal probability. ***Larger F_k indicates that this array configuration is less robust, or more fragile to sensor failure, in the sense of changing the difference coarray.***

The following present some properties of the k-fragility F_k:

Proposition 2. Let $\mathbb{S}$ be an integer set denoting the sensor locations. The k-fragility F_k with respect to $\mathbb{S}$ has these properties:

1. $0 \leq F_k \leq 1$ for all $1 \leq k \leq |\mathbb{S}|$.
2. $F_k \leq F_{k+1}$ for all $1 \leq k \leq |\mathbb{S}| - 1$. The equality holds if and only if $F_k = 1$.
3. $F_k = 1$ for all $|\mathbb{S}| - |\mathcal{E}_1| + 1 \leq k \leq |\mathbb{S}|$.
4. If $\mathbb{S}$ is maximally economic, then $F_k = 1$ for all $1 \leq k \leq |\mathbb{S}|$.

Proof. These results are direct consequences of Proposition 1 and Definition 6. □

The k-fragility F_k is a notion of probability due to Proposition 2.1. Proposition 2.3 indicates that the difference coarray definitely changes after the deletion of more than $|\mathbb{S}| - |\mathcal{E}_1|$ elements. Proposition 2.4 implies that maximally economic sparse arrays are the most fragile or the least robust arrays.

Proof of Proposition 1: Before proving Proposition 1, the following lemmas will be invoked:

Lemma 1. Let $\mathbb{D}$ and $\widetilde{\mathbb{D}}$ be the difference coarrays of $\mathbb{S}$ and $\widetilde{\mathbb{S}}$, respectively. If $\widetilde{\mathbb{S}} \subseteq \mathbb{S}$, then $\widetilde{\mathbb{D}} \subseteq \mathbb{D}$.

Proof. Let $m \in \widetilde{\mathbb{D}}$. By definition, there exist $n_1, n_2 \in \widetilde{\mathbb{S}}$ such that $n_1 - n_2 = m$. Since $\widetilde{\mathbb{S}} \subseteq \mathbb{S}$, we have $n_1, n_2 \in \mathbb{S}$, implying $m \in \mathbb{D}$. This completes the proof. □

Lemma 2. Assume that $\mathbb{A}$ and $\mathbb{B}$ are sets such that $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{S}$. If $\mathbb{A} \in \mathcal{E}_{|\mathbb{A}|}$, then $\mathbb{B} \in \mathcal{E}_{|\mathbb{B}|}$.

Proof. Assume that $\mathbb{S}_1 \triangleq \mathbb{S}\backslash\mathbb{A}$ and $\mathbb{S}_2 \triangleq \mathbb{S}\backslash\mathbb{B}$. The difference coarrays of $\mathbb{S}$, $\mathbb{S}_1$, and $\mathbb{S}_2$ are denoted by $\mathbb{D}$, $\mathbb{D}_1$, and $\mathbb{D}_2$, respectively. The notation $\mathbb{X} \subset \mathbb{Y}$ denotes that $\mathbb{X}$ is a subset of $\mathbb{Y}$ but $\mathbb{X} \neq \mathbb{Y}$. We will show that $\mathbb{D}_2 \subseteq \mathbb{D}_1 \subset \mathbb{D}$. First, since $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{S}$, we have $\mathbb{S}_2 \subseteq \mathbb{S}_1$, implying $\mathbb{D}_2 \subseteq \mathbb{D}_1$ due to Lemma 1. Second, due to the definition of the k-essential family, $\mathbb{A} \in \mathcal{E}_{|\mathbb{A}|}$ is equivalent to $\mathbb{D}_1 \subset \mathbb{D}$. Hence $\mathbb{D}_2 \subset \mathbb{D}$, which means $\mathbb{B} \in \mathcal{E}_{|\mathbb{B}|}$. □

Lemma 3. [16]. If $n_1, n_2 \in \mathbb{S}$ and $w(n_1 - n_2) = 1$, then n_1 and n_2 are both essential.

Now let us move on to the proof of Proposition 1:

1): This proof technique can be found in [24]. Let us count the number of pairs $(\mathbb{A}, \mathbb{B})$ for all $\mathbb{A} \in \mathcal{E}_k$ and all $\mathbb{B} \in \mathcal{E}_{k+1}$ such that $\mathbb{A} \subset \mathbb{B}$. Let P be the number of such pairs. For every $n_1 \in \mathbb{S}$ but $n_1 \notin \mathbb{A}$, it can be shown that $\mathbb{A} \subseteq \mathbb{A} \cup \{n_1\} \subseteq \mathbb{S}$ and therefore

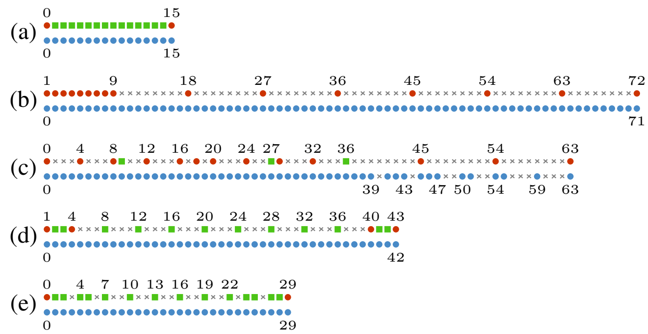

Fig. 1. The physical array (red dots and green squares) and the nonnegative parts of the difference coarray (blue dots) for (a) ULA, (b) the nested array with $N_1 = N_2 = 8$, (c) the coprime array with $M = 4, N = 9$, (d) the concatenated nested array $N_1 = 3, N_2 = 10$, and (e) the symmetric MRA. The number of physical sensors is 16 for all arrays. The red dots and green squares represent the essential sensors and the inessential sensors, respectively. The gray crosses denote empty space.

$\mathbb{A} \cup \{n_1\} \in \mathcal{E}_{k+1}$, due to Lemma 2. Since $(\mathbb{A}, \mathbb{A} \cup \{n_1\})$ has $|\mathbb{S}\backslash\mathbb{A}||\mathcal{E}_k|$ choices, we have

$$P = (|\mathbb{S}| - k)|\mathcal{E}_k|. \tag{7}$$

Similarly, it can be shown that $\mathbb{B}\backslash\{n_2\} \subset \mathbb{B} \subseteq \mathbb{S}$, for all $\mathbb{B} \in \mathcal{E}_{k+1}$ and $n_2 \in \mathbb{B}$. However, the statement that $\mathbb{B}\backslash\{n_2\} \in \mathcal{E}_k$, (the converse of Lemma 2), is not necessarily true. Therefore, by counting the number of n_2 and $\mathbb{B}$, we have

$$P \leq (k+1)|\mathcal{E}_{k+1}|, \tag{8}$$

with equality if and only if $\mathbb{B}\backslash\{n_2\} \in \mathcal{E}_k$ for all $\mathbb{B} \in \mathcal{E}_{k+1}$ and all $n_2 \in \mathbb{B}$. Combining (7) and (8) proves the inequality. The equality holds if and only if $(\mathbb{A} \cup \{n_1\})\backslash\{n_2\} \in \mathcal{E}_k$. Therefore $|\mathcal{E}_k| = 0$ or $\binom{|\mathbb{S}|}{k}$. Since $\min(\mathbb{S})$ and $\max(\mathbb{S})$ are essential, $\mathcal{E}_k$ is not empty. This proves the condition for equality.

2): Let us consider any subarray $\mathbb{A} \subseteq \mathbb{S}$ such that $|\mathbb{A}| = k \geq |\mathbb{S}| - |\mathcal{E}_1| + 1$. The cardinality of $\mathbb{S}\backslash\mathbb{A}$ becomes $|\mathbb{S}| - k \leq |\mathcal{E}_1| - 1 < |\mathcal{E}_1|$, implying that there is at least one essential element in $\mathbb{A}$. Due to Lemma 2, $\mathbb{A}$ is k-essential.

3): Let $\mathbb{A}$ be any subarray of $\mathbb{S}$ with size k. Let n be any element in $\mathbb{A}$. Since $\mathbb{S}$ is maximally economic, n is essential. Furthermore, since $\{n\} \subseteq \mathbb{A} \subseteq \mathbb{S}$, we have $\mathbb{A} \in \mathcal{E}_k$, due to Lemma 2. The above arguments prove this proposition.

4): The proof is sketched as follows. Let $\mathbb{S}_p = \{n_1, n_2 : w(n_1 - n_2) = p\} \subseteq \mathbb{S}$ be the sensors such that the associated weight function is p. The set $\mathbb{G}_p$ is defined as

$$\mathbb{G}_p = \{n : \{n\} \in \mathcal{E}_1,\ n \in \mathbb{S}_p,\ n \notin \mathbb{S}_{p-1},\ \ldots,\ n \notin \mathbb{S}_1\}. \tag{9}$$

It can be shown that $\{\mathbb{G}_1, \mathbb{G}_2\}$ is a partition of all the essential elements. Due to Lemma 3, the size of $\mathbb{G}_1$ satisfies $|\mathbb{G}_1| \leq M_1 \leq 2\binom{|\mathbb{G}_1|}{2}$, implying $(1 + \sqrt{1 + 4M_1})/2 \leq |\mathbb{G}_1| \leq M_1$. For the size of $\mathbb{G}_2$, it can be shown that $0 \leq |\mathbb{G}_2| \leq M_2/2$. The second inequality is due to the case that $\mathbb{S} = \{0, s, 2s\}$, where s is a positive integer. We have $w(s) = w(-s) = 2, M_2 = 2$, but $s \in \mathbb{G}_2$. This means two instances of $w(m) = 2$ could lead to one essential element. Combining the inequalities for $\mathbb{G}_1$ and $\mathbb{G}_2$ proves this proposition.

4. NUMERICAL EXAMPLES

Let us consider the following arrays: ULA [1], the nested array [3], the coprime array [4], the concatenated nested array [6], and the symmetric MRA [16, Fig. 1(c)]. The number of sensors is assumed to be 16 for all arrays and the corresponding parameters are listed in the caption of Fig. 1.

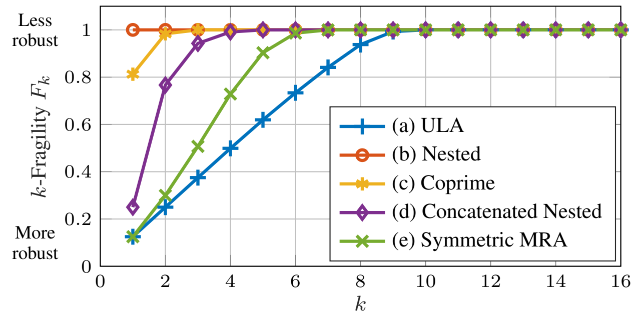

Fig. 2. The k-fragility F_k for the array configurations in Fig. 1.

Fig. 1 depicts the array geometries and the difference coarrays. Note that only the nonnegative part of the difference coarray is shown in Fig. 1, due to its symmetry [23]. It can be deduced that the sizes of the difference coarrays are 31 for ULA, 143 for the nested array, 103 for the coprime array, 85 for the concatenated nested array, and 59 for the symmetric MRA. Most of the arrays have hole-free difference coarrays, except the coprime array. Furthermore, these arrays have $\mathcal{O}(N^2)$ elements in the difference coarray, except the ULA ($\mathcal{O}(N)$) [1, 3, 4, 6, 16].

The essentialness property of these arrays is also shown in Fig. 1, where the essential sensors and the inessential sensors are illustrated in red dots and green squares, respectively. The least number of essential sensors is exhibited by ULA and symmetric MRA (2), followed by the concatenated nested array (4), the coprime array (13), and finally the nested array (16), which is maximally economic.

Fig. 2 plots the k-fragility F_k over all these array configurations. Here these curves are computed by enumerating all possible subsets in Definition 5. These results verify Proposition 2. For instance, as in Propositions 2.1 and 2.2, the k-fragility F_k is an increasing function ranging from 0 to 1. Proposition 2.4 can be verified through the nested array example, where this maximally economic sparse array has the k-fragility $F_k = 1$, as shown in Fig. 1(b) and Fig. 2, respectively. By comparing the k-fragility F_k for all k, it can be deduced that the ULA is the most robust array configuration, followed by the symmetric MRA, the concatenated nested array, the coprime array, and finally the nested array. In particular, as far as F_1 is concerned, the symmetric MRA is as robust as ULA, but the symmetric MRA possesses $\mathcal{O}(N^2)$ elements in the hole-free difference coarray. Note that the symmetric nested array, defined as the union of the nested array and its reversed version [16], can be shown to have $F_1 = 2/N$ for $N \geq 4$, which is as robust as ULA, and at the same time enjoys hole-free difference coarrays of size $\mathcal{O}(N^2)$ [25].

5. CONCLUDING REMARKS

This paper presented the concept of essentialness, k-essentialness, and fragility for arbitrary array configurations. These novel measures quantify the robustness of arrays in the presence of sensor failure. It was shown that maximally economic sparse arrays are the least robust arrays while the ULA, empirically, is the most robust one. There also exist sparse arrays with $\mathcal{O}(N^2)$ elements in the hole-free difference coarray that are as robust as ULA.

In the future, it is of considerable interest to analyze the k-essential family for several array configurations, such as ULA and coprime arrays. Another direction is to analyze the probability that the difference coarray changes under random sensor failures.

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