Optimizing Minimum Redundancy Arrays for Robustness

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Outline

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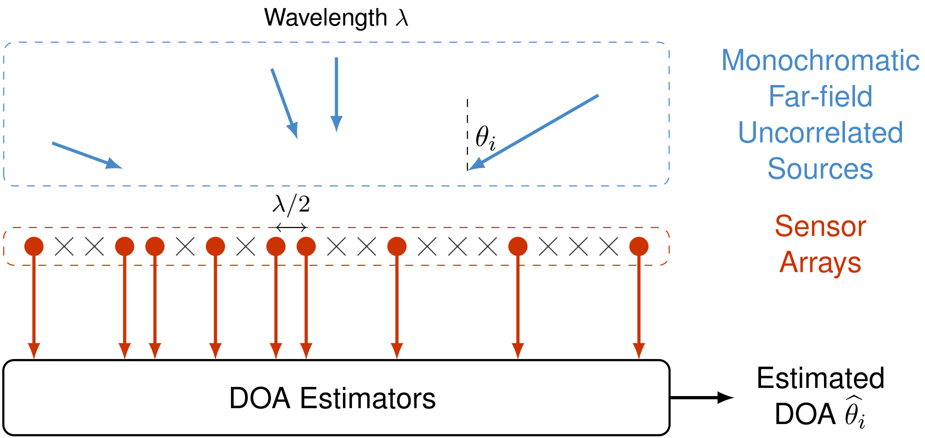
Direction-Of-Arrival (DOA) Estimation



[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

Physical Array and Difference Coarray

Physical array $\mathbb{S}$

Difference coarray $\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$

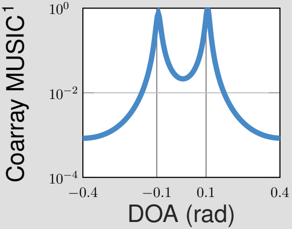
[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

Sensor Failures

Array #1

5 elements

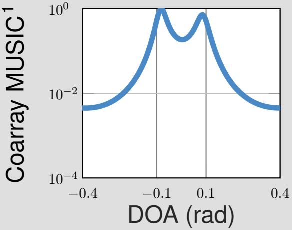
RMSE = 0.00617



Array #2 (2 fails)

4 elements

RMSE = 0.014367



Array #3 (1 fails)

4 elements

Coarray MUSIC is not applicable here!

[1] Liu and Vaidyanathan, *IEEE Signal Process. Letters*, 2015.

[2] 100 snapshots, 0dB SNR, $D = 2$ sources, $\theta_1 = -0.1$, $\theta_2 = 0.1$, equal-power, uncorrelated sources.

Outline

ULA and Sparse Arrays

ULA (not sparse)

- Identify at most $N-1$ uncorrelated sources.[1]
  (N is the number of sensors)
- Can only find fewer sources than sensors.

Linear sparse arrays

1. Minimum redundancy arrays[2]
2. Nested arrays[3]
3. Coprime arrays[4]
4. Super nested arrays[5]

- Identify $\mathcal{O}(N^2)$ uncorrelated sources.
- More sources than sensors!

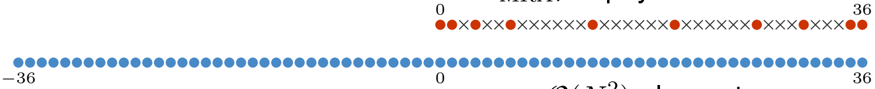
[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.
[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968.
[3] Pal and Vaidyanathan, *IEEE Trans. Signal Process.*, 2010.
[4] Vaidyanathan and Pal, *IEEE Trans. Signal Process.*, 2011.
[5] Liu and Vaidyanathan, *IEEE Trans. Signal Process.*, 2016.

Minimum Redundancy Arrays

$$\mathbb{S}_{\mathrm{MRA}} \triangleq \arg\max_{\mathbb{S}} |\mathbb{D}| \quad \text{subject to} \quad |\mathbb{S}| = N, \quad \mathbb{D} = \mathbb{U}.$$

- N physical sensors
- Hole-free $\mathbb{D}$
- No closed-form expressions for $\mathbb{S}_{\mathrm{MRA}}$

$\mathbb{S}_{\mathrm{MRA}}$, N physical sensors



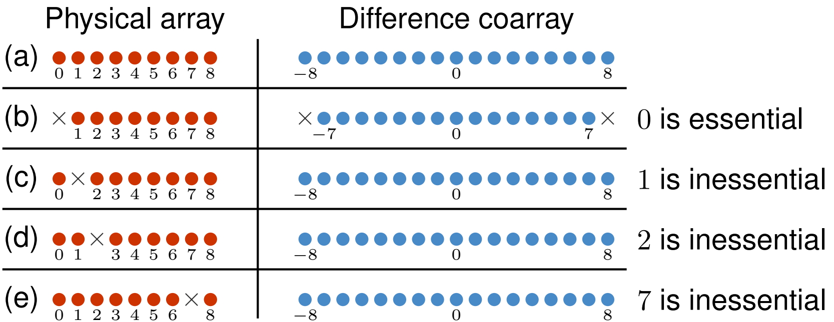
$\mathbb{D}_{\mathrm{MRA}} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}_{\mathrm{MRA}}\}$

$\mathcal{O}(N^2)$ elements

[1] Moffet, *IEEE Trans. Antennas Propag.*, 1968.

The Essentialness Property

The sensor $n \in \mathbb{S}$ is essential with respect to $\mathbb{S}$ if $\overline{\mathbb{D}} \neq \mathbb{D}$.

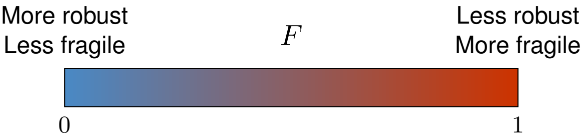


[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018; $\mathbb{D}$ is the difference coarray of $\mathbb{S}$ and $\overline{\mathbb{D}}$ is the difference coarray of $\mathbb{S}\backslash\{n\}$.

The Fragility F: Definition

$$\text{The fragility } F \triangleq \frac{\text{\# of essential sensors}}{\text{\# of sensors}}.$$

$$\frac{2}{N} \leq F \leq 1, \qquad \text{for all } N \geq 4$$



[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018.

The Fragility F and the Normalized Size of $\mathbb{D}$

$$F \triangleq \frac{\text{\# of essential sensors}}{\text{\# of all sensors } (N)}$$

$$\mathfrak{D} \triangleq \frac{|\mathbb{D}|}{N^2 - N + 1}$$

- More robust
- Less fragile

- More sources than sensors
- Higher resolution

[1] Liu and Vaidyanathan, *IEEE SAM*, 2018.

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RMRA: Definition

$$(\text{P1}): \qquad \mathbb{S}_{\text{RMRA}} \triangleq \arg\max_{\mathbb{S}} |\mathbb{D}| \qquad \text{subject to} \tag{1}$$

$$|\mathbb{S}| = N, \tag{2}$$

$$\mathbb{D} = \mathbb{U}, \tag{3}$$

$$F = \frac{2}{N}, \qquad N \geq 4. \tag{4}$$

- (2): N physical sensors
- (3): Hole-free $\mathbb{D}$
- (4): The minimum fragility (the most robust)

[1] Liu and Vaidyanathan, *ACSSC*, 2018.

RMRA: Array Geometries Found by Direct Searching

◆ Essential ■ Inessential

N	Array Configuration
4	#1: ◆ ■ ■ ◆ (0, 3)
5	#1: ◆ ■ ■ ■ ◆ (0, 4)
6	#1: ◆ ■ ■ ■ × ■ ◆ #2: ◆ ■ ■ × ■ ■ ◆ (0, 2, 4, 6)
7	#1: ◆ ■ ■ × ■ × ■ × ■ ◆ #2: ◆ ■ ■ × × ■ ■ × ■ ◆ (0, 2, 5, 9)
8	#1: ◆ ■ ■ ■ × ■ × × ■ × × ■ ◆ #2: ◆ ■ ■ ■ × × × ■ ■ × × ■ ◆ (0, 3, 5, 8, 12)
9	#1: ◆ ■ ■ ■ ■ × × × × ■ ■ × × × ■ ◆ (0, 4, 9, 15)
10	#1: ◆ ■ ■ × × × ■ ■ × × × ■ × × × ■ ■ × ■ ◆ #2: ◆ ■ ■ × × × ■ ■ ■ × × × × × × ■ ■ × ■ ◆ (0, 2, 6, 11, 15, 19)

[1] Liu and Vaidyanathan, *ACSSC*, 2018.

RMRA: Properties

$$(\mathrm{P1}): \qquad \mathbb{S}_{\mathrm{RMRA}} \triangleq \arg\max_{\mathbb{S}} |\mathbb{D}| \qquad \text{subject to}$$
$$|\mathbb{S}| = N,$$
$$\mathbb{D} = \mathbb{U},$$
$$F = \frac{2}{N}, \qquad N \geq 4.$$

- The solution to $(\mathrm{P1})$ exists.
- The solution to $(\mathrm{P1})$ is not unique.
- To the best of our knowledge, closed-form expressions for $\mathbb{S}_{\mathrm{RMRA}}$ is not known.
- The difference coarray of $\mathbb{S}_{\mathrm{RMRA}}$ has size $\mathcal{O}(N^2)$.

[1] Liu and Vaidyanathan, *ACSSC*, 2018.

Theorem 1: The Size of the Difference Coarray

Theorem

Let $\mathbb{S}_{\mathrm{RMRA}}$ *be a solution to* $(\mathrm{P1})$ *with* $N \geq 4$ *physical sensors. The aperture of* $\mathbb{S}_{\mathrm{RMRA}}$ *is denoted by* A_{RMRA}. *Then*

$$L_{\mathrm{RMRA}} \leq \frac{N^2}{A_{\mathrm{RMRA}}} < U_{\mathrm{RMRA}}, \tag{5}$$

where

$$L_{\mathrm{RMRA}} \triangleq 4 + \frac{4\sqrt{2}}{3\pi}, \qquad U_{\mathrm{RMRA}} \triangleq 16. \tag{6}$$

The smaller N^2/A is,
the better the array is (in the sense of large hole-free $\mathbb{D}$).

[1] Liu and Vaidyanathan, *ACSSC*, 2018.

Comparison between MRA and RMRA

MRA [ErdősGál1948, RédeiRényi1948, Leech1956]

For MRA with N sensors,

$$L_{\mathrm{MRA}} \leq \lim_{N\to\infty} \frac{N^2}{A_{\mathrm{MRA}}} \leq U_{\mathrm{MRA}}, \tag{7}$$

where

$$L_{\mathrm{MRA}} \triangleq 2.434\ldots, \qquad U_{\mathrm{MRA}} \triangleq 3.348\ldots. \tag{8}$$

For sufficiently large N, we have

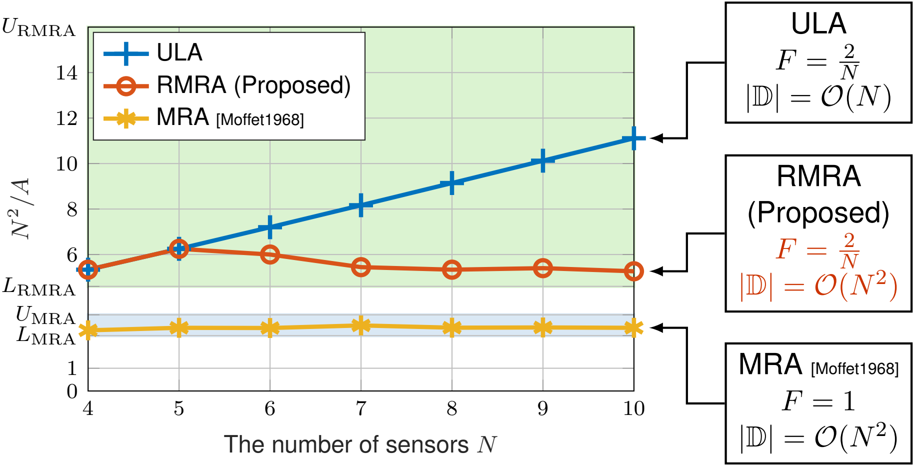
$$0.2987N^2 \leq A_{\mathrm{MRA}} \leq 0.4108N^2,$$ Due to the above

$$0.0625N^2 < A_{\mathrm{RMRA}} \leq 0.2174N^2.$$ This talk

[1] Liu and Vaidyanathan, *ACSSC*, 2018.

[2] Erdős and Gál, Indagationes Mathematicae, 1948; Rédei and Rényi, Recueil Mathématique, 1948; Leech, J. London Math. Soc.,1956, Moffet, *IEEE Trans. Antennas Propag.*, 1968.

A Numerical Example of Theorem 1



$$L_{\mathrm{RMRA}} = 4 + \frac{4\sqrt{2}}{3\pi} \approx 4.6002, \quad U_{\mathrm{RMRA}} = 16, \quad L_{\mathrm{MRA}} = 2.434\ldots, \quad U_{\mathrm{MRA}} = 3.348\ldots.$$

Outline

Concluding Remarks

- Robust array geometries with large difference coarray
- Robust minimum redundancy array (RMRA) with N sensors
  - Minimum fragility ($F = 2/N$)
  - Hole-free difference coarray
  - $|\mathbb{D}_{\mathrm{RMRA}}| = \mathcal{O}(N^2)$
- Future work
  - Suboptimal solutions to RMRA with large N

Thank you!