Composite Singer Arrays with Hole-Free Coarrays and Enhanced Robustness

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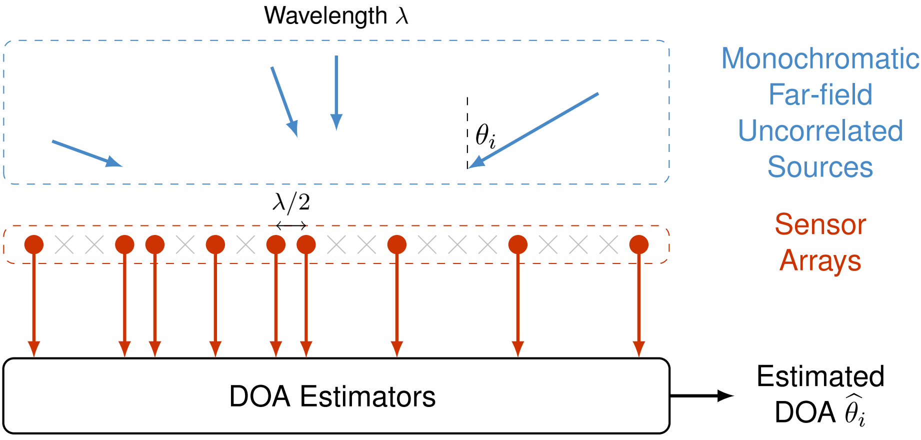
Outline

Outline

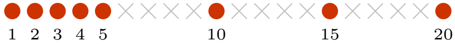
Direction-Of-Arrival (DOA) Estimation



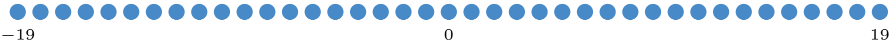
[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

Physical Array and Difference Coarray

Physical array $\mathbb{S} = \{1, 2, 3, 4, 5, 10, 15, 20\}$



Difference coarray $\mathbb{D} = \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$



[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002; Pillai, Bar-Ness, and Haber, *Proc. IEEE*, 1985; Abramovich, Gray, Gorokhov, and Spencer, *IEEE Trans. Signal Process.*, 1998; Pal and Vaidyanathan, *IEEE Trans. Signal Process.*, 2010; Vaidyanathan and Pal *IEEE Trans. Signal Process.*, 2011.

Outline

ULA and Sparse Arrays

ULA (Not Sparse)

- Identify at most $N - 1$ uncorrelated sources[1]

  (N is the number of sensors)
- Can only find fewer sources than sensors
- Robust to sensor failures

Linear Sparse Arrays

1. Minimum redundancy arrays
2. Nested arrays
3. Coprime arrays
4. Super nested arrays

- Identify $\mathcal{O}(N^2)$ uncorrelated sources since the difference coarray has size $\mathcal{O}(N^2)$
- More sources than sensors!

[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968. (MRA)

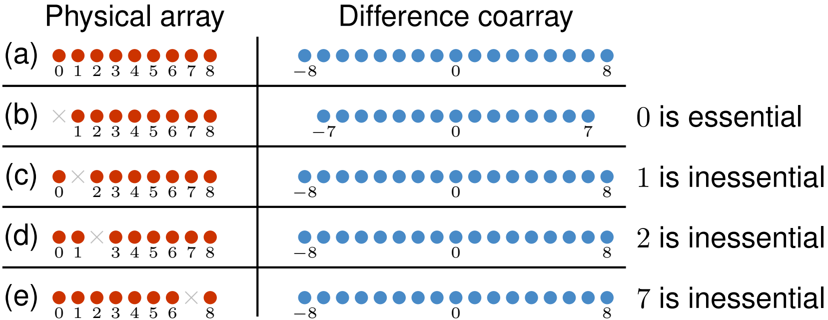
[3] Pal and Vaidyanathan, *IEEE Trans. Signal Process.*, 2010. (Nested arrays)

[4] Vaidyanathan and Pal, *IEEE Trans. Signal Process.*, 2011. (Coprime arrays)

[5] Liu and Vaidyanathan, *IEEE Trans. Signal Process.*, 2016. (Super nested arrays)

The Essentialness Property

The sensor $n \in \mathbb{S}$ is essential with respect to $\mathbb{S}$ if $\overline{\mathbb{D}} \neq \mathbb{D}$.[2]



[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018; *IEEE Trans. Signal Process.*, 2019.

[2] $\mathbb{D}$ is the difference coarray of $\mathbb{S}$ and $\overline{\mathbb{D}}$ is the difference coarray of $\mathbb{S}\backslash\{n\}$.

The Fragility of an Array $\mathbb{S}$

The fragility $F(\mathbb{S}) \triangleq \dfrac{\text{\# of essential sensors}}{N}$.

$\mathbb{S}$: The sensor array; N: The number of sensors

$$\frac{2}{N} \leq F(\mathbb{S}) \leq 1$$ (for $N \geq 4$ and any $\mathbb{S}$)

The Most Robust
Minimum Fragility
e.g. ULA

The Least Robust
Maximally Economic
e.g. Minimum redundancy array
Minimum hole arrays

[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2018; *IEEE Trans. Signal Process.*, 2019.

Sparse Arrays with Minimum Fragility

Main Idea

We would like to study an array configuration $\mathbb{S}$ such that

1. $\mathbb{S}$ owns hole-free difference coarrays of size $\mathcal{O}(N^2)$
2. $\mathbb{S}$ is as robust as ULA with the same number of sensors N
   (In particular, $F(\mathbb{S}) = 2/N$)

Known Solutions: Robust Minimum Redundancy Arrays (RMRA)

- Hole-free difference coarrays of size $\mathcal{O}(N^2)$
- Minimum fragility ($F(\mathbb{S}) = 2/N$)
- No closed-form solutions

[1] Liu and Vaidyanathan, *Proc. of the 52th Asilomar Conference on Signals, Systems, and Computers*, 2018.

The Focus of This Talk

Main Idea

We would like to study an array configuration $\mathbb{S}$ such that

1. $\mathbb{S}$ owns hole-free difference coarrays of size $\mathcal{O}(N^2)$
2. $\mathbb{S}$ is as robust as ULA with the same number of sensors N
   (In particular, $F(\mathbb{S}) = 2/N$)

Solutions to the above problem

whose sensor locations can be found easily

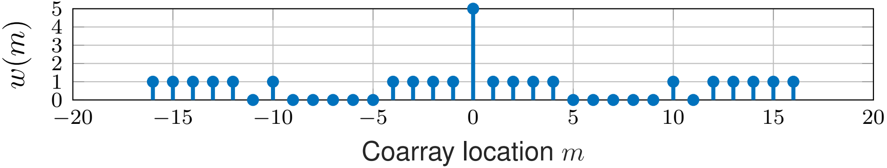
[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2019.

Outline

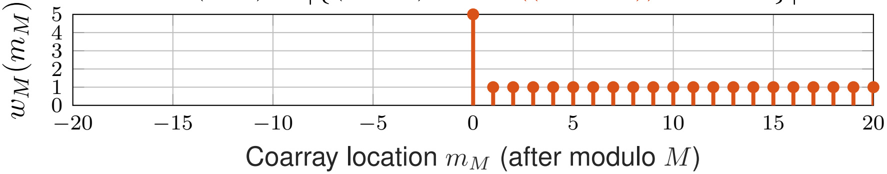
An Example of the Singer Array ($q = 4, M = 21$)

Singer Array $\mathbb{S}$:

$$w(m) \triangleq \left|\left\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\right\}\right|$$



$$w_M(m_M) \triangleq \left|\left\{(n_1, n_2) \in \mathbb{S}^2 : ((n_1 - n_2))_M = m_M\right\}\right|$$



Singer Arrays

Theorem (Rephrase from [1])

Let q be a power of a prime number. Assume that $M = q^2 + q + 1$. Then there exist $q+1$ integers $d_1, d_2, \ldots, d_{q+1}$ such that

1. $0 = d_1 < d_2 < \cdots < d_{q+1} < M$ *and*
2. *The set of pairwise differences between d_i and d_j for $i \neq j$ is equivalent to $\{1, 2, \ldots, M-1\}$, under the modulo-M operation. That is,*

$$\{((d_i - d_j))_M : i \neq j, \quad i, j = 1, 2, \ldots, q+1\} \\ = \{1,\ 2,\ \ldots,\ M-1\},$$

where $((a))_b$ denotes the remainder of a divided by b.

Definition: Singer Arrays

A Singer array with the parameter q (a power of a prime) is defined as the set $\{d_1,\ d_2,\ \ldots,\ d_{q+1}\}$, where $d_1, d_2, \ldots, d_{q+1}$ are given in the above theorem.

[1] J. Singer, *Trans. Amer. Math. Soc.*, 1938.

Composite Arrays

Definition: $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array (Rephrase from [1])

Suppose α is a positive integer. Let $\mathbb{S}_1$ and $\mathbb{S}_2$ denote two sensor arrays with $\min(\mathbb{S}_1) = \min(\mathbb{S}_2) = 0$. Then the $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array $\mathbb{S}_\mathrm{c}$ is defined as

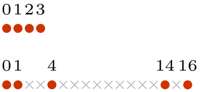
$$\mathbb{S}_\mathrm{c} \triangleq \alpha\mathbb{S}_1 + \mathbb{S}_2 = \{\alpha n_1 + n_2 : n_1 \in \mathbb{S}_1, n_2 \in \mathbb{S}_2\},$$

where $\alpha > \max(\mathbb{S}_2)$.

- This technique was used in the literature to generate arrays with large difference coarrays[1].

[1] Erdös and Gál, *Indagationes Mathematicae*, 1948; Leech, *J. London Math. Soc.*, 1956; Ishiguro, *Radio Science*, 1980; Yang, Haimovich, Yuan, Sun, and Chen, *IEEE Access*, 2018.

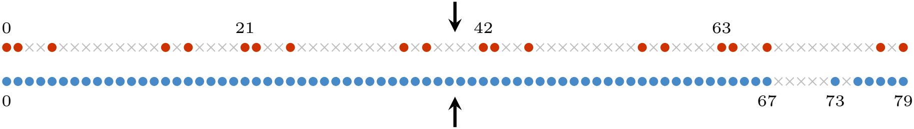
An Example of Singer Arrays and Composite Arrays



$\leftarrow \mathbb{S}_1$: The ULA with 4 sensors

$\leftarrow \mathbb{S}_2$: The Singer array with $q = 4$

$\mathbb{S}_\mathrm{c}$: The $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array with $\alpha = 21$



$\mathbb{D}_\mathrm{c}^+$: The nonnegative part of the difference coarray of $\mathbb{S}_\mathrm{c}$

- $\mathbb{D}_\mathrm{c}$ has a large central ULA segment.
- $\mathbb{S}_\mathrm{c}$ is as robust as $\mathbb{S}_1$ ($F(\mathbb{S}_1) = 1/2$ and $F(\mathbb{S}_\mathrm{c}) = 1/2$).

The Focus of This Talk

Main Idea

We would like to study an array configuration $\mathbb{S}$ such that

1. $\mathbb{S}$ owns hole-free difference coarrays of size $\mathcal{O}(N^2)$
2. $\mathbb{S}$ is as robust as ULA with the same number of sensors N (In particular, $F(\mathbb{S}) = 2/N$)

Solutions to the above problem

whose sensor locations can be found easily

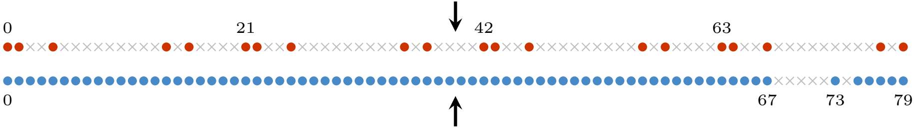
[1] Liu and Vaidyanathan, *IEEE ICASSP*, 2019.

Problems with Composite Arrays plus Singer Arrays

$\leftarrow \mathbb{S}_1$: The ULA with 4 sensors

$\leftarrow \mathbb{S}_2$: The Singer array with $q = 4$

$\mathbb{S}_\mathrm{c}$: The $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-composite array with $\alpha = 21$



$\mathbb{D}_\mathrm{c}^+$: The nonnegative part of the difference coarray of $\mathbb{S}_\mathrm{c}$

- $\mathbb{D}_\mathrm{c}$ has holes
- $\mathbb{S}_\mathrm{c}$ DO NOT have minimum fragility $(F(\mathbb{S}_\mathrm{c}) \neq \frac{2}{N})$

Supplementary Arrays (Proposed)

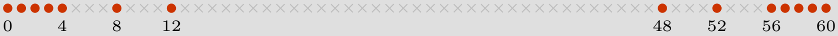
Definition: Supplementary Arrays

Let P and Q be positive integers satisfying $Q > 2P(P-1)$. The supplementary array $\mathbb{S}_{\text{supp}}$ with parameters P and Q is defined as

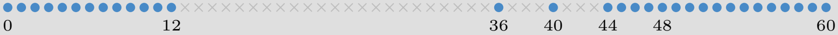
$$\mathbb{S}_{\text{supp}} = \{u,\ uP,\ Q-u,\ Q-uP\ :\ u = 0, 1, \ldots, P-1\}.$$

The Supplementary Array with $P = 4$ and $Q = 60$

$\mathbb{S}_{\text{supp}}$:



$\mathbb{D}^{+}_{\text{supp}}$: (The nonnegative part of the difference coarray of $\mathbb{S}_{\text{supp}}$)



Properties of Supplementary Arrays

Let $\mathbb{S}_{\text{supp}}$ be the supplementary array with parameters P and Q. Let $\mathbb{D}_{\text{supp}}$ be the difference coarray of $\mathbb{S}_{\text{supp}}$. Define the new set

$$\begin{aligned}\mathbb{L} \triangleq \{&1,\ 2,\ 3,\ \ldots,\ P^2 - P,\\ &Q-1,\ Q-2,\ Q-3,\ \ldots,\ Q-(P^2-1)\}.\end{aligned}$$

Then the following properties hold

1. $|\mathbb{S}_{\text{supp}}| = 4P - 2$.
2. $\mathbb{L} \subseteq \mathbb{D}_{\text{supp}}$.
3. Let $n \in \mathbb{S}_{\text{supp}} \backslash \{0, Q\}$. Denote the difference coarray of $\overline{\mathbb{S}}_{\text{supp}} \triangleq \mathbb{S}_{\text{supp}} \backslash \{n\}$ by $\overline{\mathbb{D}}_{\text{supp}}$. Then $\mathbb{L} \subseteq \overline{\mathbb{D}}_{\text{supp}}$.

$\mathbb{S}_{\text{supp}}$ helps to fill the holes and increase the robustness

Composite Singer Arrays $\mathbb{S}_{\mathrm{cs}}$

$$\mathbb{S}_{\mathrm{cs}} = \mathbb{S}_{\mathrm{c}} \cup \mathbb{S}_{\mathrm{supp}}$$

The $(\alpha, \mathbb{S}_1, \mathbb{S}_2)$-Composite Array

- $\mathbb{S}_1$:
  - $F(\mathbb{S}_1) = 2/|\mathbb{S}_1|$
  - Hole-free coarray
- $\mathbb{S}_2$: A Singer array with the parameter q ($\mathbb{S}_2 = \{d_1, d_2, \ldots, d_{q+1}\}$)
- $\alpha = q^2 + q + 1$.

The Supplementary Array

- $P = \left\lceil \sqrt{2d_{q+1} + 1} \right\rceil$.
- $Q = \alpha A_1 + d_{q+1}$, where A_1 is the aperture of $\mathbb{S}_1$.
- $Q > 2P(P-1)$

Outline

Design Parameters

- $\mathbb{S}_1 = \{0, 1, 2, 3, 5, 6\}$: Robust MRA with 6 sensors[1]
  - $F(\mathbb{S}_1) = 2/6$
  - The difference coarray of $\mathbb{S}_1$: $\mathbb{D}_1 = \{0, \pm 1, \pm 2, \ldots, \pm 6\}$
- $\mathbb{S}_2 = \{0, 1, 4, 14, 16\}$: The Singer array with $q = 4$
- $\alpha = q^2 + q + 1 = 21$
- $P = \lceil \sqrt{2 \times 16 + 1} \rceil = 6$
- $Q = 21 \times 6 + 16 = 142$

[1] Liu and Vaidyanathan, ACSSC, 2018.

Array Configurations

 $\leftarrow \mathbb{S}_1 = \{0, 1, 2, 3, 5, 6\}$: RMRA with 6 sensors

 $\leftarrow \mathbb{S}_2 = \{0, 1, 4, 14, 16\}$: The Singer array with $q = 4$

$$\mathbb{S}_{\text{cs}} = \{0, 1, 2, 3, 4, 5, 6, 12, 14, 16, 18, 21, 22, 24, 25, 30, 35, 37, 42, 43, 46, 56, 58, 63, 64, 67, 77, 79, 105, 106, 109, 112, 118, 119, 121, 124, 126, 127, 130, 136, 137, 138, 139, 140, 141, 142\}$$

Essential sensors (red): 0, 142

The composite Singer array (46 sensors)

↓

↑

$\mathbb{D}_{\text{cs}}^{+}$: The nonnegative part of the difference coarray of $\mathbb{S}_c$

$$\mathbb{D}_{\text{cs}}^{+} = \{0, 1, \ldots, 142\}$$

Outline

Concluding Remarks

- Composite Singer arrays
  - Singer arrays, composite arrays, supplementary arrays
- Future work
  - New arrays with hole-free difference coarrays and minimum fragility
  - Two-dimensional arrays
- This work is supported by
  - Office of Naval Research
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Thank you!