One-Bit Normalized Scatter Matrix Estimation for Complex Elliptically Symmetric Distributions

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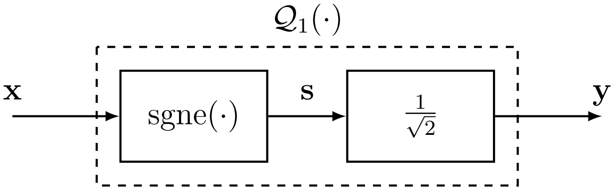
One-Bit Quantized Measurements



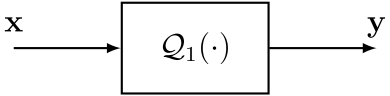
$$\mathbf{x} \triangleq \begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^T \in \mathbb{C}^N.$$

$$\mathbf{s} \triangleq \mathrm{sgne}(\mathbf{x}) = \begin{bmatrix} \mathrm{sgn}\left\{\mathrm{Re}(x_1)\right\} \\ \vdots \\ \mathrm{sgn}\left\{\mathrm{Re}(x_N)\right\} \end{bmatrix} + j \begin{bmatrix} \mathrm{sgn}\left\{\mathrm{Im}(x_1)\right\} \\ \vdots \\ \mathrm{sgn}\left\{\mathrm{Im}(x_N)\right\} \end{bmatrix}.$$

- Low cost.
- Low complexity.
- Reduced data rate.
- Moderate performance loss.
- Applications[1]
  - Massive MIMO.
  - Array processing.
  - Radar.

[1] Lu, Li, Swindlehurst, Ashikhmin, and Zhang, *IEEE J. Sel. Topics Signal Process.*, 2014; Larsson, Edfors, Tufvesson, and Marzetta, *IEEE Commun. Mag.*, 2014; Risi, Persson, and Larsson, 2014; Björnson, Larsson, and Marzetta, *IEEE Commun. Mag.*, 2016; Li, Tao, Seco-Granados, Mezghani, Swindlehurst, and Liu, *IEEE Trans. Signal Process.*, 2017; Bar-Shalom and Weiss, *IEEE Trans. Aerosp. Electron. Syst.*, 2002; Stöckle, Munir, Mezghani, and Nossek, *IEEE SPAWC*, 2015; Stein, Barbe, and Nossek, *WSA*, 2016; Liu and Vaidyanathan, *IEEE ICASSP*, 2017. Ameri, Bose, Li, and Soltanalian, *IEEE Trans. Signal Process.*, 2019.

Second-Order Statistics of One-Bit Data



Arcsine Law[1]

If $\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_\mathbf{x})$ and $\mathbf{y} \triangleq \mathcal{Q}_1(\mathbf{x})$, then

$$\mathbf{R}_\mathbf{y} = \left(\frac{2}{\pi}\right) \operatorname{sine}^{-1}\left(\overline{\mathbf{R}}_\mathbf{x}\right).$$

- $\mathbf{R}_\mathbf{y} \triangleq \mathbb{E}[\mathbf{y}\mathbf{y}^H]$.
- Normalized covariance matrix of $\mathbf{x}$.

$$\overline{\mathbf{R}}_\mathbf{x} \triangleq \mathbf{Q}^{-1/2}\mathbf{R}_\mathbf{x}\mathbf{Q}^{-1/2},$$
$$\mathbf{Q} \triangleq \operatorname{diag}([\mathbf{R}_\mathbf{x}]_{1,1}, [\mathbf{R}_\mathbf{x}]_{2,2}, \ldots, [\mathbf{R}_\mathbf{x}]_{N,N}).$$

- $\operatorname{sine}^{-1}(\cdot)$: Entrywise arcsine function[2].

What if $\mathbf{x}$ is not Gaussian distributed?

[1] Van Vleck and Middleton, *Proc. IEEE*,1966; Jacovitti and Neri, *IEEE Trans. Inf. Theory*,1994; Bar-Shalom and Weiss, *IEEE Trans. Aerosp. Electron. Syst.*,2002.

[2] $[\operatorname{sine}^{-1}(\mathbf{A})]_{p,q} = \sin^{-1}(\operatorname{Re}([\mathbf{A}]_{p,q})) + j\sin^{-1}(\operatorname{Im}([\mathbf{A}]_{p,q}))$.

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Complex Elliptically Symmetric (CES) Distributions

A CES distribution $\mathcal{CE}(\boldsymbol{\mu}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\mathbf{x}}, g)$ has the probability density function (if exists)[1]

$$f(\mathbf{x}) = \frac{\Gamma(N)}{\pi^N \det(\boldsymbol{\Sigma}_{\mathbf{x}}) \delta_{N,g}} g\left((\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}})^H \boldsymbol{\Sigma}_{\mathbf{x}}^{-1} (\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}}) \right),$$

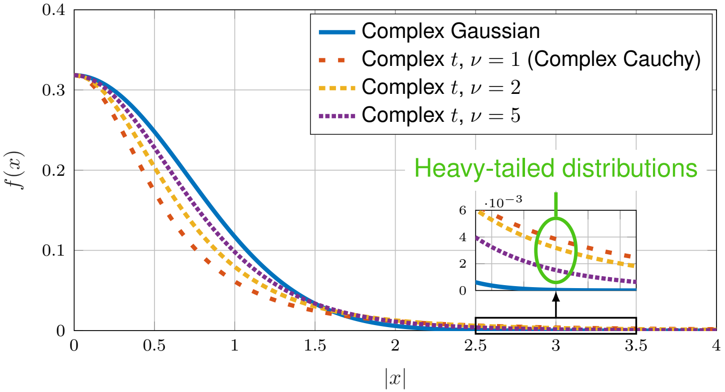
- The symmetric center: $\boldsymbol{\mu}_{\mathbf{x}} \in \mathbb{C}^N$. *Analogous* to the mean vector.
- The scatter matrix: $\boldsymbol{\Sigma}_{\mathbf{x}} \in \mathbb{C}^{N \times N}$. *Analogous* to the covariance matrix.
- The density generator: $g(\cdot) : [0, \infty) \to (0, \infty)$.
  - Complex Gaussian, complex t, complex generalized Gaussian, etc[1].

[1] Ollila, Tyler, Koivunen, and Poor, *IEEE Trans. Signal Process.*, 2012; Wooding, *Biometrika*, 1956; Novey, Adali, and Roy, *IEEE Trans. Signal Process.*, 2010; Huber, *Robust Statistics (Wiley Series in Probability and Statistics)*, Wiley-Interscience, 2003; Tyler, *Ann. Statist.*, 1987; Ollila and Koivunen, *PIMRC*, 2003.

[2] $\Gamma(\cdot)$ denotes the gamma function and $\delta_{N,g} \triangleq \int_0^\infty t^{N-1} g(t) \mathrm{d}t < \infty$.

Examples of CES Distributions

- For $\boldsymbol{\mu}_{\mathbf{x}} = 0$, $\Sigma_{\mathbf{x}} = 1$, and a complex Gaussian, we have $f(x) = \frac{1}{\pi} e^{-|x|^2}$.
- For $\boldsymbol{\mu}_{\mathbf{x}} = 0$, $\Sigma_{\mathbf{x}} = 1$, and a complex t, we have $f(x) = \frac{2\Gamma(1+\nu/2)}{\pi\nu\Gamma(\nu/2)} \left(1 + \frac{2|x|^2}{\nu}\right)^{-\left(1+\frac{\nu}{2}\right)}$.



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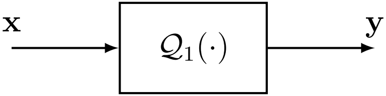
Arcsine Law for CES Distributions



Arcsine Law for CES Distributions

If $\mathbf{x} \sim \mathcal{CE}(\mathbf{0}, \mathbf{\Sigma}_{\mathbf{x}}, g)$ and $\mathbf{y} \triangleq \mathcal{Q}_1(\mathbf{x})$, then

$$\mathbf{R}_{\mathbf{y}} = \left(\frac{2}{\pi}\right) \operatorname{sine}^{-1}\left(\overline{\mathbf{\Sigma}}_{\mathbf{x}}\right), \tag{1}$$

where $\mathbf{R}_{\mathbf{y}} = \mathbb{E}[\mathbf{y}\mathbf{y}^H]$,

$$\overline{\mathbf{\Sigma}}_{\mathbf{x}} \triangleq \mathbf{Q}^{-1/2}\mathbf{\Sigma}_{\mathbf{x}}\mathbf{Q}^{-1/2},$$

$$\mathbf{Q} = \operatorname{diag}([\mathbf{\Sigma}_{\mathbf{x}}]_{1,1}, [\mathbf{\Sigma}_{\mathbf{x}}]_{2,2}, \ldots, [\mathbf{\Sigma}_{\mathbf{x}}]_{N,N}).$$

- Eq. (1) has the same form as in the Gaussian case

$$\mathbf{R}_{\mathbf{y}} = \left(\frac{2}{\pi}\right) \operatorname{sine}^{-1}\left(\overline{\mathbf{R}}_{\mathbf{x}}\right).$$

- $\mathbf{R}_{\mathbf{x}}$ and $\mathbf{\Sigma}_{\mathbf{x}}$.
- Related work[1].
- Our contributions:
  - CES distributions.
  - A simple proof.

[1] McGraw and Wagner, *IEEE Trans. Inf. Theory*, 1968; Lindskog, McNeil, and Schmock, in *Credit Risk*, Heidelberg, 2003.

Sketch of the Proof

Complex Angular central Gaussian[1]

If $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R})$, then

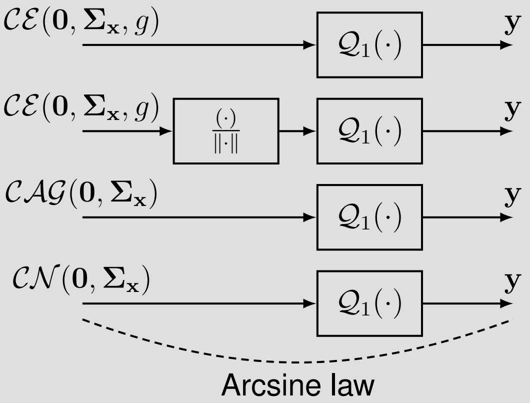
$$\frac{\mathbf{z}}{\|\mathbf{z}\|} \sim \mathcal{CAG}(\mathbf{0}, \mathbf{R}).$$

Lemma

If $\mathbf{x} \sim \mathcal{CE}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{x}}, g)$, *then*

$$\frac{\mathbf{x}}{\|\mathbf{x}\|} \sim \mathcal{CAG}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{x}}).$$

Contribution: Sketch of the Proof



[1] Ollila, Tyler, Koivunen, and Poor, *IEEE Trans. Signal Process.*, 2012, and the references therein.

Normalized Scatter Matrix Estimation

- Require: i.i.d. vectors $\widetilde{\mathbf{x}}(k)$ for $k = 1, 2, \ldots, K$ drawn from $\mathcal{CE}(\mathbf{0}, \boldsymbol{\Sigma}_{\mathbf{x}}, g)$.
- Goal: Estimate the normalized scatter matrix $\overline{\boldsymbol{\Sigma}}_{\mathbf{x}}$.

Sample Covariance Matrix (SCM)

1. Estimate the covariance of $\mathbf{x}$.
$$\widehat{\mathbf{R}}_{\mathbf{x}} = \frac{1}{K}\sum_{k=1}^{K} \widetilde{\mathbf{x}}(k)\widetilde{\mathbf{x}}^H(k).$$
2. Estimate the diagonal matrix $\mathbf{Q}$.
$$[\widehat{\mathbf{Q}}]_{i,i} = [\widehat{\mathbf{R}}_{\mathbf{x}}]_{i,i},$$
for $i = 1, 2, \ldots, N$.
3. Estimate $\overline{\boldsymbol{\Sigma}}_{\mathbf{x}}$.
$$\widehat{\overline{\boldsymbol{\Sigma}}}_{\mathbf{x}} = \widehat{\mathbf{Q}}^{-\frac{1}{2}}\widehat{\mathbf{R}}_{\mathbf{x}}\widehat{\mathbf{Q}}^{-\frac{1}{2}}.$$

Complex One-Bit Arcsine Law (COBASL)

1. Compute sign vectors.
$$\widetilde{\mathbf{s}}(k) = \operatorname{sgne}(\widetilde{\mathbf{x}}(k)).$$
2. Estimate the covariance of $\mathbf{s}$.
$$\widehat{\mathbf{R}}_{\mathbf{s}} = \frac{1}{K}\sum_{k=1}^{K} \widetilde{\mathbf{s}}(k)\widetilde{\mathbf{s}}^H(k).$$
3. Estimate $\overline{\boldsymbol{\Sigma}}_{\mathbf{x}}$.
$$\widehat{\overline{\boldsymbol{\Sigma}}}_{\mathbf{x}} = \operatorname{sine}\left(\frac{\pi}{4}\widehat{\mathbf{R}}_{\mathbf{s}}\right).$$

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Estimation of the Normalized Scatter Matrix (1/2)

- $\widetilde{\mathbf{x}}(k) \in \mathbb{C}^N$ for $k = 1, 2, \ldots, K$.
- $N = 3$ and $K = 1000$.
- Complex t with ν, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}_{\mathbf{x}}$

$$\boldsymbol{\mu} = \mathbf{0},$$

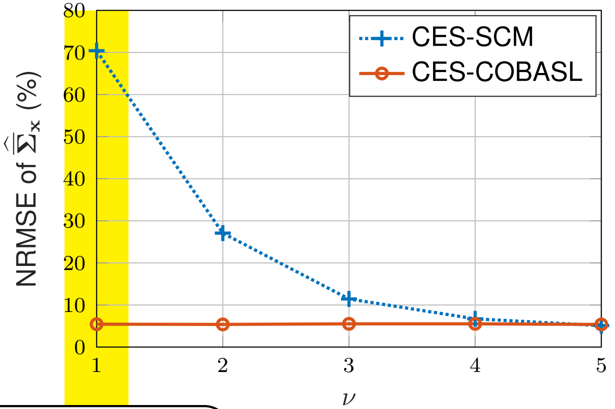
$$\boldsymbol{\Sigma}_{\mathbf{x}} = \begin{bmatrix} 1 & \rho_{2,1}^* & \rho_{3,1}^* \\ \rho_{2,1} & 1 & \rho_{3,2}^* \\ \rho_{3,1} & \rho_{3,2} & 1 \end{bmatrix},$$

$$\rho_{2,1} = 0.5e^{j\pi/4},$$

$$\rho_{3,1} = 0.2e^{-j\pi/6},$$

$$\rho_{3,2} = 0.4e^{j\pi/5}.$$

- $\text{NRMSE} \triangleq \frac{\|\widehat{\overline{\boldsymbol{\Sigma}}}_{\mathbf{x}} - \overline{\boldsymbol{\Sigma}}_{\mathbf{x}}\|_F}{\|\overline{\boldsymbol{\Sigma}}_{\mathbf{x}}\|_F} \times 100\%.$
- 1000 Monte-Carlo trials.



Complex Cauchy (with the heaviest tail)

Estimation of the Normalized Scatter Matrix (2/2)

- $\widetilde{\mathbf{x}}(k) \in \mathbb{C}^N$ for $k = 1, 2, \ldots, K$.
- $N = 3$.
- Complex t with ν, $\boldsymbol{\mu}$, and $\boldsymbol{\Sigma}_{\mathbf{x}}$

$$\nu = 2, \quad \boldsymbol{\mu} = \mathbf{0},$$

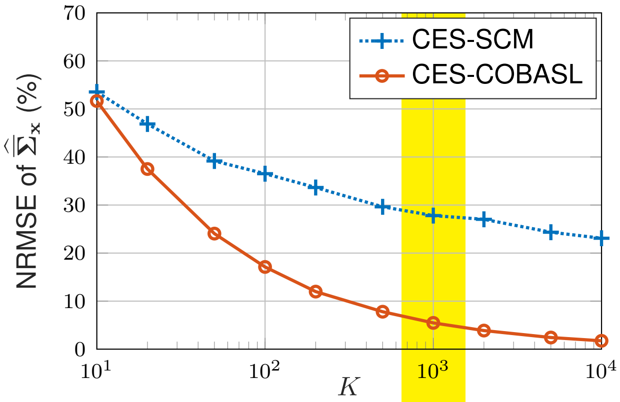
$$\boldsymbol{\Sigma}_{\mathbf{x}} = \begin{bmatrix} 1 & \rho_{2,1}^* & \rho_{3,1}^* \\ \rho_{2,1} & 1 & \rho_{3,2}^* \\ \rho_{3,1} & \rho_{3,2} & 1 \end{bmatrix},$$

$$\rho_{2,1} = 0.5e^{j\pi/4},$$

$$\rho_{3,1} = 0.2e^{-j\pi/6},$$

$$\rho_{3,2} = 0.4e^{j\pi/5}.$$

- $\mathrm{NRMSE} \triangleq \frac{\|\widehat{\overline{\boldsymbol{\Sigma}}}_{\mathbf{x}} - \overline{\boldsymbol{\Sigma}}_{\mathbf{x}}\|_F}{\|\overline{\boldsymbol{\Sigma}}_{\mathbf{x}}\|_F} \times 100\%$.
- 1000 Monte-Carlo trials.

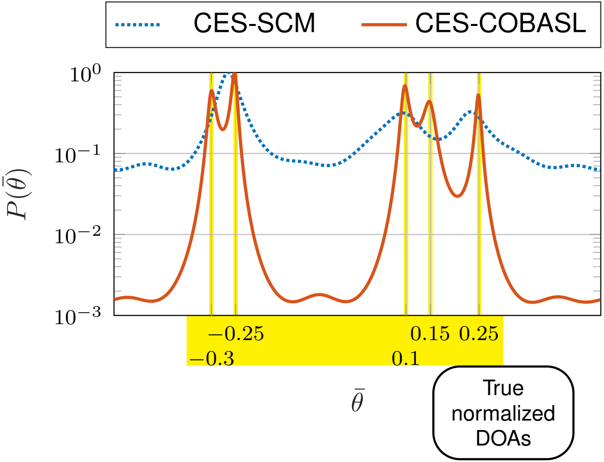


CES-COBASL is more robust to outliers than CES-SCM.

Direction-of-Arrival (DOA) Estimation

$$\underbrace{\mathbf{x}}_{\text{Array output}} = \underbrace{\mathbf{A}}_{\text{Array manifold matrix}} \underbrace{\mathbf{s}}_{\text{Source}} + \underbrace{\mathbf{n}}_{\text{Noise}} \in \mathbb{C}^N,$$

- $D = 5$ sources
- A uniform linear array with $N = 10$ sensors
- $[\mathbf{s}^T, \mathbf{n}^T]^T$ follows a complex t distribution with $\nu = 2$, $\boldsymbol{\mu} = \mathbf{0}$, and $\boldsymbol{\Sigma} = \mathbf{I}$.
- $K = 1000$ snapshots.
- $P(\bar{\theta})$: MUSIC spectrum based on $\widehat{\widehat{\Sigma}}_{\mathbf{x}}$.



[1] $[\mathbf{A}]_{n,i} = e^{j2\pi n\bar{\theta}_i}$ for $0 \le n \le N-1$ and $1 \le i \le D$. $\bar{\theta}_i = (1/2)\sin\theta_i \in [-1/2, 1/2]$ are the normalized DOAs.

Outline

Concluding Remarks

- This presentation
  - Arcsine law for CES.
  - CES-COBASL.
  - Normalized scatter matrix estimation.
  - DOA estimation.
- Future work
  - Performance analysis
  - Scatter matrix estimation

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Thank you!