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ONE-BIT NORMALIZED SCATTER MATRIX ESTIMATION FOR COMPLEX ELLIPTICALLY SYMMETRIC DISTRIBUTIONS

Chun-Lin Liu* *and P. P. Vaidyanathan*†

*Dept. of Electrical Engineering
*Graduate Institute of Communication Engineering
National Taiwan University, Taipei, Taiwan 10617
chunlinliu@ntu.edu.tw

†Dept. of Electrical Engineering, MC 136-93
California Institute of Technology
Pasadena, CA 91125, USA
ppvnath@systems.caltech.edu

ABSTRACT

One-bit quantization has attracted attention in massive MIMO, radar, and array processing, due to its simplicity, low cost, and capability of parameter estimation. Specifically, the shape of the covariance of the unquantized data can be estimated from the arcsine law and one-bit data, if the unquantized data is Gaussian. However, in practice, the Gaussian assumption is not satisfied due to outliers. It is known from the literature that outliers can be modeled by complex elliptically symmetric (CES) distributions with heavy tails. This paper shows that the arcsine law remains applicable to CES distributions. Therefore, the normalized scatter matrix of the unquantized data can be readily estimated from one-bit samples derived from CES distributions. The proposed estimator is not only computationally fast but also robust to CES distributions with heavy tails. These attributes will be demonstrated through numerical examples, in terms of computational time and the estimation error. An application in DOA estimation with MUSIC spectrum is also presented.

Index Terms— One-bit quantization, complex elliptically symmetric distributions, arcsine law, robust statistics, scatter matrices.

1. INTRODUCTION

Signal processing with one-bit quantization has drawn attention in massive MIMO [1–5], array processing [6–9], and radar [10]. From the hardware perspective, one-bit analog-to-digital converters (ADCs) feature low cost, low power consumption, and simple hardware designs, compared to the high-resolution ones [11, 12]. Furthermore, one-bit information can be utilized to estimate information of interest such as the channel information in massive MIMO [3, 5], the direction-of-arrival in sensor arrays [6–9], and the target parameters in radar systems [10].

The advances in this field are partly founded on the relation between second-order statistics of the unquantized data and the quantized data, such as the arcsine law [13] and the Bussgang theorem [14]. However, these assume *the Gaussianity of the unquantized data*. This assumption has been shown to be a poor fit for models like radar clutters [15–17] and noise in mobile communications [18, 19]. This phenomenon is due to the presence of *outliers*, which are samples significantly deviating from other samples.

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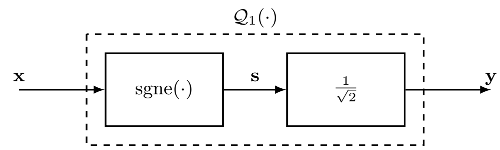

Fig. 1. A one-bit quantization system, where $\mathbf{x} \in \mathbb{C}^N$ is the input vector, $\mathbf{s} \in \{\pm 1 \pm j\}^N$ denotes the sign vector, and $\mathbf{y} \in \{(\pm 1 \pm j)/\sqrt{2}\}^N$ represents the one-bit quantized vector.

In the literature, the complex elliptically symmetric (CES) distributions [20, 21] have been shown to be suitable for modeling outliers with *heavy-tailed distributions*. A CES distribution may be parametrized by the symmetric center (analogous to the mean vector), the scatter matrix (analogous to the covariance matrix), and the density generator, which determines the shape of the probability density function. Based on these, CES distributions unify complex Gaussian distributions [22], complex t-distributions [20], and complex generalized Gaussian distributions [23], to name a few [21]. Furthermore, the theory of CES distributions is related to robust estimators for the scatter matrix [24–26]. Interested readers are referred to the overview article [21] and the references therein.

A notable application of CES distributions is in *robust direction-of-arrival (DOA) estimation* in array processing [21, 26]. For CES distributions with finite second-order moments, the scatter matrix $\mathbf{\Sigma}$ and the covariance matrix $\mathbf{R}$ are related by $\mathbf{R} = c\mathbf{\Sigma}$ for some constant $c > 0$. This property implies that the scatter matrix and the covariance matrix share the same set of eigenvectors. Therefore, robust DOA estimation can be done by replacing the covariance matrix with the scatter matrix in subspace-based DOA estimators [27, 28].

In this paper, we will show that *the arcsine law remains applicable to CES distributions*. In particular, the normalized scatter matrix of the unquantized CES samples and the covariance matrix of one-bit quantized data are related by the arcsine function. This relation has been known for complex Gaussian distributions [6, 29] and real elliptically symmetric distributions [30, 31]. The main contribution of this paper resides in the arcsine law for CES distributions and a new proof arising from the *complex angular central Gaussian distributions*. Owing to the arcsine law, we study the estimation of the normalized scatter matrix of the input, based on one-bit measurements, and this method will be referred to as CES-COBASL (CES distributions with complex one-bit arcsine law) in this paper. We will see that *CES-COBASL is more robust to heavy-tailed CES distribution than the estimator based on sample covariance matrices (SCM), for the problem of estimating the normalized scatter matrix*. Furthermore, the computational time of CES-COBASL is comparable to that of the method based on SCM.

The outline of this paper is as follows. Section 2 reviews one-bit quantization and CES distributions. In Section 3, we will prove that the arcsine law remains applicable to CES distributions, and this property leads to a low-complexity and robust estimator of the normalized scatter matrix. Section 4 demonstrates the estimation performance of CES-COBASL while Section 5 concludes this paper.

2. PRELIMINARIES

2.1. One-Bit Quantization

Let $\mathbf{x} \triangleq [x_1, x_2, \ldots, x_N]^T$ be a complex vector of length N. The sign vector $\mathbf{s}$ of $\mathbf{x}$ is denoted by

$$\mathbf{s} \triangleq \mathrm{sgne}(\mathbf{x}) = \begin{bmatrix} \mathrm{sgn}\{\mathrm{Re}(x_1)\} \\ \vdots \\ \mathrm{sgn}\{\mathrm{Re}(x_N)\} \end{bmatrix} + j \begin{bmatrix} \mathrm{sgn}\{\mathrm{Im}(x_1)\} \\ \vdots \\ \mathrm{sgn}\{\mathrm{Im}(x_N)\} \end{bmatrix}, \quad (1)$$

where the sign function $\mathrm{sgn}(t)$ is 1 if $t \geq 0$ and -1 if $t < 0$, and the notations $\mathrm{Re}(\cdot)$ and $\mathrm{Im}(\cdot)$ stand for the real and imaginary parts of a complex number, respectively. The right-hand side of (1) is denoted by $\mathrm{sgne}(\mathbf{x})$, indicating the entrywise sign function on the real and imaginary parts. Based on (1), the one-bit quantized vector $\mathbf{y}$ is defined as $\mathbf{y} \triangleq \mathcal{Q}_1(\mathbf{x}) = \mathbf{s}/\sqrt{2}$. The scaling factor $1/\sqrt{2}$ aims to unify the arcsine law for the real and the complex case [6,29]. For clarity, the relations among the vectors $\mathbf{x}$, $\mathbf{s}$, and $\mathbf{y}$ are illustrated in Fig. 1.

In what follows, we assume that $\mathbf{x}$, $\mathbf{s}$, and $\mathbf{y}$ are random vectors and the relation among the second-order statistics of $\mathbf{x}$, $\mathbf{s}$, and $\mathbf{y}$ will be reviewed. The traditional ***arcsine law*** relates the covariance matrix of $\mathbf{x}$ to that of $\mathbf{y}$, assuming $\mathbf{x}$ is Gaussian-distributed [6, 13, 29]. To begin with, we define the entrywise sine function $\mathrm{sine}(\mathbf{A})$ and the entrywise arcsine function $\mathrm{sine}^{-1}(\mathbf{A})$ for a complex matrix $\mathbf{A}$. The real and imaginary parts of the (p, q)th entry of $\mathbf{A}$ are denoted by $a^R_{p,q}$ and $a^I_{p,q}$, respectively. Then the (p, q)th entries of $\mathrm{sine}(\mathbf{A})$ and $\mathrm{sine}^{-1}(\mathbf{A})$ are

$$[\mathrm{sine}(\mathbf{A})]_{p,q} = \sin(a^R_{p,q}) + j \sin(a^I_{p,q}), \quad (2)$$

$$\left[\mathrm{sine}^{-1}(\mathbf{A})\right]_{p,q} = \sin^{-1}(a^R_{p,q}) + j \sin^{-1}(a^I_{p,q}), \quad (3)$$

where $\sin^{-1}(\cdot)$ is the arcsine function. Furthermore, for a square matrix $\mathbf{M}$ with positive diagonal entries, the normalized version[1] of $\mathbf{M}$ is defined as $\overline{\mathbf{M}} \triangleq \mathbf{Q}^{-\frac{1}{2}}\mathbf{M}\mathbf{Q}^{-\frac{1}{2}}$, where $\mathbf{Q}$ is diagonal with $[\mathbf{Q}]_{p,p} = [\mathbf{M}]_{p,p}$. Note that $[\overline{\mathbf{M}}]_{p,p} = 1$. With these notations, the arcsine law states that [6, 13, 29]

Lemma 1. If $\mathbf{x}$ is a circularly-symmetric complex Gaussian vector with zero mean and covariance $\mathbf{R}_\mathbf{x}$ (i.e., $\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{R}_\mathbf{x})$) and $\mathbf{y} \triangleq \mathcal{Q}_1(\mathbf{x})$, then the covariance matrix $\mathbf{R}_\mathbf{y}$ and the normalized covariance matrix $\overline{\mathbf{R}}_\mathbf{x}$ of $\mathbf{x}$ satisfy

$$\mathbf{R}_\mathbf{y} = (2/\pi)\mathrm{sine}^{-1}\left(\overline{\mathbf{R}}_\mathbf{x}\right). \quad (4)$$

Here are some remarks on Lemma 1. First, the normalized covariance $\overline{\mathbf{R}}_\mathbf{x}$ of the unquantized samples $\mathbf{x}$ can be inferred from one-bit samples $\mathbf{y}$. Second, the computational complexity based on one-bit data is low since the one-bit vector $\mathbf{y}$ can be obtained from two-level comparators. This attribute makes it favorable for high-speed and low-complexity systems.

[1]In the literature, the normalization of a matrix has various definitions. For example, a matrix may be normalized subject to a fixed trace [21].

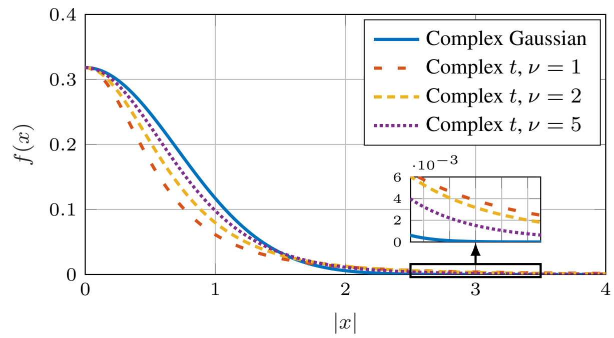

Fig. 2. The pdf of CES distributions as a function of $|x|$.

Historically, the arcsine law was first developed for real, scalar, stationary Gaussian input processes [13]. This relation has been extended to complex, scalar, stationary Gaussian processes and other scenarios [6, 29, 32]. Furthermore, for Gaussian input passing through zero-memory nonlinear devices (rather than one-bit quantization), Price's theorem [33] studied the statistics of the input and the output, which can be treated as a generalization of the arcsine law.

The arcsine law in Lemma 1 is founded on *the Gaussianity of the input vector* $\mathbf{x}$. However, the *outliers* in real data typically invalidate this assumption. In what follows, we will review the CES distribution, which is a popular model for distributions with outliers.

2.2. Complex Elliptically Symmetric Distributions

Assume that $\mathbf{x} \in \mathbb{C}^N$ is CES distributed where the probability density function (pdf) of $\mathbf{x}$ exists. Then this pdf can be expressed as [21]

$$f(\mathbf{x}) = C_{N,g} \det(\mathbf{\Sigma}_\mathbf{x})^{-1} g((\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})^H \mathbf{\Sigma}_\mathbf{x}^{-1}(\mathbf{x} - \boldsymbol{\mu}_\mathbf{x})), \quad (5)$$

where $g : [0, \infty) \to (0, \infty)$ is the density generator and $\boldsymbol{\mu}_\mathbf{x} \in \mathbb{C}^N$ is the symmetric center. The scatter matrix $\mathbf{\Sigma}_\mathbf{x}$ is positive definite. The normalization factor $C_{N,g} = \Gamma(N)/(\pi^N \delta_{N,g})$, where $\Gamma(\cdot)$ denotes the gamma function and $\delta_{N,g} \triangleq \int_0^\infty t^{N-1} g(t) \mathrm{d}t < \infty$. We write $\mathbf{x} \sim \mathcal{CE}(\boldsymbol{\mu}_\mathbf{x}, \mathbf{\Sigma}_\mathbf{x}, g)$ for Equation (5).

The pdf in (5) is a generalization of several well-known distributions. For instance, if $g(t) = e^{-t}$, then we have $C_{N,g} = \pi^{-N}$ so $f(\mathbf{x})$ becomes *the circularly-symmetric complex Gaussian distribution* with mean $\boldsymbol{\mu}_\mathbf{x}$ and covariance $\mathbf{\Sigma}_\mathbf{x}$. If $g(t) = (1 + 2t/\nu)^{-(2N+\nu)/2}$ for $\nu \in (0, \infty)$, then (5) simplifies to *the complex t-distribution*. Another example is *the complex generalized Gaussian distribution*, which corresponds to $g(t) = e^{-t^s/b}$ for $s > 0$ and $b > 0$.

For example, consider the CES distribution $\mathcal{CE}(0, 1, g)$ so that the complex random vector $\mathbf{x}$ reduces to a complex random variable x. In this case, the complex Gaussian distribution corresponds to a pdf $f(x) = \frac{1}{\pi} e^{-|x|^2}$ while the complex t-distribution has $f(x) = \frac{2\Gamma(1+\nu/2)}{\pi\nu\Gamma(\nu/2)} \left(1 + 2|x|^2/\nu\right)^{-(1+\nu/2)}$. Fig. 2 illustrates the pdfs $f(x)$ of the complex Gaussian distribution and the complex t-distribution for $\nu = 1, 2, 5$, with zero mean and unit variance. It can be observed that the tails of the complex t-distribution for $\nu = 1, 2, 5$ are *heavier* (i.e., the tails decay slower) than the complex Gaussian distribution. Therefore, the complex t-distributions for $\nu = 1, 2, 5$ belong to the family of *heavy-tailed distributions*, in which the outliers occur more frequently than the complex Gaussian distribution. Note that the complex t-distribution for $\nu = 1$ is also called the *complex Cauchy distribution*.

The scatter matrix $\mathbf{\Sigma_x}$ is a parameter in a CES distribution, which is not necessarily the covariance matrix. If the second-order moment of a CES distribution is finite, then $\mathbf{\Sigma_x}$ is proportional to the covariance matrix $\mathbf{R_x}$ [21]. However, for complex Cauchy distributions, the scatter matrix $\mathbf{\Sigma_x}$ exists but its covariance matrix is undefined.

Next we will review the *complex angular central Gaussian distribution*, defined as [21, 34]:

Definition 1. Assume that the random vector $\mathbf{x} \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma_x})$. Let the random vector $\mathbf{x}_a$ be $\mathbf{x}/\|\mathbf{x}\|$. Then $\mathbf{x}_a$ is said to have the complex angular central Gaussian (ACG) distribution, denoted by $\mathbf{x}_a \sim \mathcal{CAG}(\mathbf{0}, \mathbf{\Sigma_x})$.

In Definition 1, the case that $\mathbf{x} = \mathbf{0}$ is omitted since $\Pr[\mathbf{x} = \mathbf{0}] = 0$. By definition, $\mathbf{x}_a$ is confined to a unit sphere ($\|\mathbf{x}_a\| = 1$), which is different from the random vector $\mathbf{x} \in \mathbb{C}^N$ in a CES distribution. As a result, the complex ACG distribution does not belong to the family of CES distributions. However, according to Definition 1, the pdf of $\mathbf{x}_a$ is shown to be $f(\mathbf{x}_a) = \frac{\Gamma(N)}{2\pi^N \det(\mathbf{\Sigma_x})} \left(\mathbf{x}_a^H \mathbf{\Sigma_x}^{-1} \mathbf{x}_a\right)^{-N}$ [21, 34]. Based on this, it can be shown that the pdfs of $\mathcal{CAG}(\mathbf{0}, \mathbf{\Sigma_x})$ and $\mathcal{CAG}(\mathbf{0}, c\mathbf{\Sigma_x})$ are identical for a positive constant c.

The CES distribution is associated with the complex ACG distribution as follows [21]:

Lemma 2. If $\mathbf{x} \sim \mathcal{CE}(\mathbf{0}, \mathbf{\Sigma_x}, g)$, then $\mathbf{x}/\|\mathbf{x}\| \sim \mathcal{CAG}(\mathbf{0}, \mathbf{\Sigma_x})$.

If the density function $g(t) = e^{-t}$, then Lemma 2 is consistent with Definition 1. However, for CES distributions such as complex t-distributions and complex generalized Gaussian distributions, the random vector $\mathbf{x}/\|\mathbf{x}\|$ still follows the complex ACG distribution.

3. NORMALIZED SCATTER MATRIX ESTIMATION FROM ONE-BIT DATA

In this section, we will present the arcsine law for CES distributions, which has similar forms as Lemma 1 for Gaussian distributions. This property makes it possible to estimate the normalized scatter matrix from one-bit samples derived from CES distributions. To begin with, the arcsine law for CES distributions is given by

Theorem 1. Let $\mathbf{x} \sim \mathcal{CE}(\mathbf{0}, \mathbf{\Sigma_x}, g)$ and $\mathbf{y} \triangleq \mathcal{Q}_1(\mathbf{x})$. Then the covariance matrix of $\mathbf{y}$ satisfies

$$\mathbf{R_y} = (2/\pi)\mathrm{sine}^{-1}\left(\overline{\mathbf{\Sigma}}_\mathbf{x}\right), \tag{6}$$

where $\overline{\mathbf{\Sigma}}_\mathbf{x}$ is the normalized version of the scatter matrix $\mathbf{\Sigma_x}$ (see the definition following Eq. (3)).

Proof. Since $\mathbf{x} \sim \mathcal{CE}(\mathbf{0}, \mathbf{\Sigma_x}, g)$, we have $\Pr[\mathbf{x} = \mathbf{0}] = 0$. Then with probability 1, the vector $\mathbf{y}$ can be expressed as

$$\mathbf{y} = \mathcal{Q}_1\left(\|\mathbf{x}\| \cdot \mathbf{x}/\|\mathbf{x}\|\right) = \mathcal{Q}_1\left(\mathbf{x}/\|\mathbf{x}\|\right), \tag{7}$$

where the last equality results from the identity that $\mathcal{Q}_1(\alpha\mathbf{v}) = \mathcal{Q}_1(\mathbf{v})$ for $\mathbf{v} \in \mathbb{C}^N$ and $\alpha > 0$. Due to Lemma 2, $\mathbf{x}/\|\mathbf{x}\|$ has the distribution $\mathcal{CAG}(\mathbf{0}, \mathbf{\Sigma_x})$. This distribution can be represented by $\mathbf{z}/\|\mathbf{z}\|$ where $\mathbf{z} \sim \mathcal{CN}(\mathbf{0}, \mathbf{\Sigma_x})$, due to Definition 1. Based on these arguments, Equation (7) becomes

$$\mathbf{y} = \mathcal{Q}_1\left(\mathbf{z}/\|\mathbf{z}\|\right) = \mathcal{Q}_1\left(\mathbf{z}\right). \tag{8}$$

Namely, the vector $\mathbf{y}$ can be derived from a multivariate complex Gaussian vector $\mathbf{z}$. Therefore, due to Lemma 1, the arcsine law between $\mathbf{z}$ and $\mathbf{y}$ holds, which proves (6). □

Theorem 1 indicates that ***the arcsine law remains valid if the input vector follows CES distributions***. As a result, for heavy-tailed CES distributions such as the complex Cauchy distribution (examples shown in Fig. 2), the arcsine law offers a simple and low complexity approach to the estimation of the normalized scatter matrix.

Algorithm 1 Normalized scatter matrix estimation from CES distributions and the complex one-bit arcsine law (CES-COBASL)

Require: Input vectors $\widetilde{\mathbf{x}}(k)$ for $k = 1, 2, \dots, K$ drawn from a CES distribution.

1: Compute the sign vectors $\widetilde{\mathbf{s}}(k) = \mathrm{sgne}(\widetilde{\mathbf{x}}(k))$.

2: Estimate the covariance of the sign vectors by $\widehat{\mathbf{R}}_\mathbf{s} = \frac{1}{K}\sum_{k=1}^{K} \widetilde{\mathbf{s}}(k)\widetilde{\mathbf{s}}^H(k)$.

3: Estimate the normalized scatter matrix of the CES distribution according to the arcsine law $\widehat{\overline{\mathbf{\Sigma}}}_\mathbf{x} = \mathrm{sine}\left(\frac{\pi}{4}\widehat{\mathbf{R}}_\mathbf{s}\right)$, where the entrywise sine function $\mathrm{sine}(\cdot)$ is defined in (2).

In the literature, the arcsine law for elliptically symmetric distributions has been reported [30, 31] based on different assumptions. McGraw and Wagner studied the analog of Price's theorem for bivariate real elliptically symmetric distributions [30]. Similar results were proved for Kendall's tau [35] and multivariate real elliptically symmetric distribution [31]. The contribution of Theorem 1 is as follows. First, the arcsine law holds true for *multivariate complex elliptically symmetric distributions*, and second, the proof is based on complex ACG distributions, which is simpler than the analysis of orthant probabilities [30, 31].

Due to Theorem 1, the normalized scatter matrix $\overline{\mathbf{\Sigma}}_\mathbf{x}$ can be estimated by using Algorithm 1. In particular, after generating the sign vectors $\widetilde{\mathbf{s}}(k)$ from the input vectors $\widetilde{\mathbf{x}}(k)$ for $k = 1, 2, \dots, K$, the covariance of the sign vectors is estimated by Step 2. Then utilizing the relation $\mathbf{R_y} = \mathbf{R_s}/2$ and Theorem 1, the normalized scatter matrix can be estimated by Step 3. In the following development, this method will be referred to as CES-COBASL.

CES-COBASL has low computational complexity, owing to the following reasons. First, the sign vectors $\widetilde{\mathbf{s}}(k)$ can be obtained from the sign of the entries of $\widetilde{\mathbf{x}}(k)$. Next, since $\widetilde{\mathbf{s}}(k) \in \{\pm 1 \pm j\}^N$, the entries of $\widehat{\mathbf{R}}_\mathbf{s}$ are of the form $(a + jb)/K$, where $a, b \in \{0, \pm 1, \dots, \pm 2K\}$. This property makes it possible to represent $\widehat{\mathbf{R}}_\mathbf{s}$ by *fixed-point numbers*. Furthermore, the entrywise sine function in Step 3 of Algorithm 1 can be readily realized through *table lookup* in hardware. Therefore, CES-COBASL can be implemented with no multiplication.

CES-COBASL is robust to heavy-tailed CES distributions, since the arcsine law is invariant to the density generator $g(t)$ in CES distributions (Theorem 1). As a result, for heavy-tailed CES distributions like complex Cauchy distributions, CES-COBASL could outperform methods to estimate the normalized scatter matrix from the sample covariance matrix (SCM), which are known to be not robust to heavy-tailed distributions.

There are some works related to CES-COBASL such as complex polarity coincidence correlator [29, 32], and the one-bit DOA estimator for sparse arrays [9]. However, these works are built on the Gaussianity of the unquantized data. In this paper, CES-COBASL has been shown to be functional for CES distributions. Furthermore, CES-COBASL utilizes the sign vectors to reduce the computational complexity, which was not considered in [9].

One-bit robust DOA estimation can be achieved through CES-COBASL. Assume that in array processing, the array outputs follow from a CES distribution with a finite covariance matrix and the source amplitudes are zero-mean, uncorrelated, and equal-powered [9]. It can be shown that the covariance matrix, the normalized covariance matrix, and the normalized scatter matrix all share the same

Algorithm 2 Normalized scatter matrix estimation from CES distributions and the sample covariance matrix (CES-SCM)

Require: Input vectors $\widetilde{\mathbf{x}}(k)$ for $k = 1, 2, \ldots, K$ drawn from a CES distribution.

1: Estimate the covariance of the input vectors by $\widehat{\mathbf{R}}_{\mathbf{x}} = \frac{1}{K}\sum_{k=1}^{K} \widetilde{\mathbf{x}}(k)\widetilde{\mathbf{x}}^H(k)$.

2: Construct a diagonal matrix $\widehat{\mathbf{Q}}$ satisfying $[\widehat{\mathbf{Q}}]_{i,i} = [\widehat{\mathbf{R}}_{\mathbf{x}}]_{i,i}$ for $i = 1, 2, \ldots, N$.

3: Infer the normalized scatter matrix by $\widehat{\overline{\boldsymbol{\Sigma}}}_{\mathbf{x}} = \widehat{\mathbf{Q}}^{-\frac{1}{2}}\widehat{\mathbf{R}}_{\mathbf{x}}\widehat{\mathbf{Q}}^{-\frac{1}{2}}$.

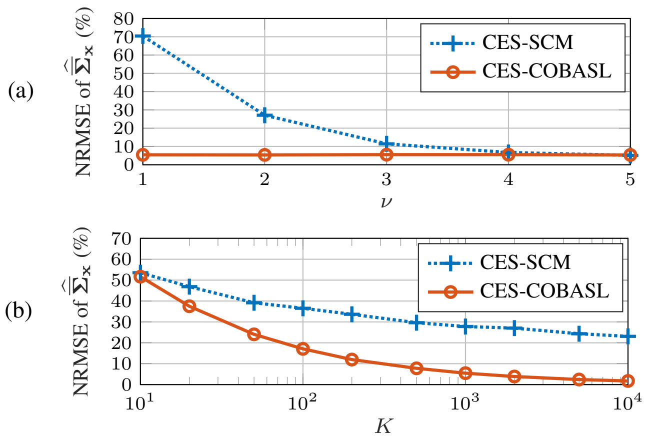

Fig. 3. The dependence of the NRMSE in percentage on (a) the parameter ν in the complex t-distribution and (b) the number of samples K. It is assumed that (a) $K = 1000$ and (b) $\nu = 2$. Each data point is averaged from 1000 Monte-Carlo trials.

eigenvectors. Therefore DOAs can be estimated by the MUSIC algorithm [27] and the normalized scatter matrix in CES-COBASL.

4. NUMERICAL EXAMPLES

In this section, we will compare two approaches to the estimation of the normalized scatter matrix. One is based on the arcsine law (Algorithm 1) while the other is built on the sample covariance matrix, as summarized in Algorithm 2. Algorithms 1 and 2 will be denoted by CES-COBASL and CES-SCM in the discussion. CES-COBASL relies on one-bit data while the data in CES-SCM is unquantized.

Let $\widetilde{\mathbf{x}}(k) \in \mathbb{C}^N$ for $k = 1, 2, \ldots, K$ be i.i.d. samples drawn from the complex t-distribution with $N = 3$ and $\boldsymbol{\mu} = \mathbf{0}$. The true scatter matrix is assumed to be

$$\boldsymbol{\Sigma}_{\mathbf{x}} = \begin{bmatrix} 1 & \rho_{2,1}^* & \rho_{3,1}^* \\ \rho_{2,1} & 1 & \rho_{3,2}^* \\ \rho_{3,1} & \rho_{3,2} & 1 \end{bmatrix}, \qquad (9)$$

where the correlation coefficients are $\rho_{2,1} = 0.5e^{j\pi/4}$, $\rho_{3,1} = 0.2e^{-j\pi/6}$, and $\rho_{3,2} = 0.4e^{j\pi/5}$. The normalized root-mean-square error is defined as $\mathrm{NRMSE} \triangleq \|\widehat{\overline{\boldsymbol{\Sigma}}}_{\mathbf{x}} - \overline{\boldsymbol{\Sigma}}_{\mathbf{x}}\|_F / \|\overline{\boldsymbol{\Sigma}}_{\mathbf{x}}\|_F$, where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix.

Fig. 3(a) compares the NRMSE in percentage of CES-SCM and CES-COBASL with respect to the parameter ν in complex t-distributions, where there are $K = 1000$ snapshots and 1000 Monte-Carlo trials for each data point. In this example, it can be observed that the CES-SCM is not robust to heavy-tailed distributions (e.g. NRMSE is approximately 70% for $\nu = 1$), because the estimator $\widehat{\mathbf{R}}_{\mathbf{x}}$ in Step 1 of Algorithm 2 is sensitive to outliers. On the other hand, CES-COBASL exhibits similar NRMSE (around 5%) across various values of ν, so that it is more robust.

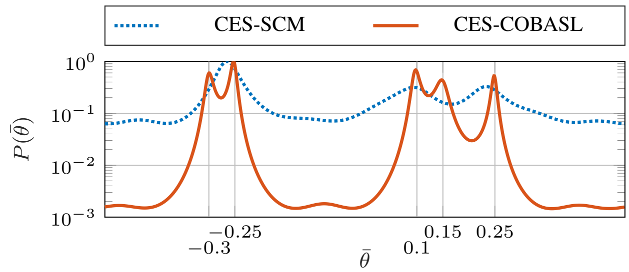

Fig. 4. Normalized MUSIC spectra $P(\bar{\theta})$ from CES-SCM and CES-COBASL. The true normalized DOAs are shown by vertical lines.

The NRMSEs of CES-SCM and CES-COBASL as a function of K are shown in Fig. 3(b), assuming $\nu = 2$ and 1000 Monte-Carlo trials. We see that the NRMSE of CES-COBASL decreases faster than that of CES-SCM as K increases. The reason is that the larger K is, the more outliers tend to occur. These outliers degrade the estimation performance of Step 1 in Algorithm 2, causing a slow decay in the NRMSE of CES-SCM. On the other hand, in CES-COBASL, the sign information is less sensitive to outliers. As a result, CES-COBASL tends to exhibit smaller NRMSE than CES-SCM for large K.

Next the computational time of CES-SCM and CES-COBASL is measured on a workstation with Intel Core i7-8700 CPU 3.20GHz, 32GB RAM, Ubuntu 16.04.6 LTS, and MATLAB R2018b. For $K = 100$ in Fig. 3(b), the computational time for one Monte-Carlo trial is 2.03×10^{-5} second for CES-SCM and 1.869×10^{-5} second for CES-COBASL, averaged from 1000 Monte-Carlo trials. Therefore, these two methods demonstrate similar computational complexities. Note that we have not taken advantage of fixed-point representations and table lookup in this implementation.

Finally, Fig. 4 demonstrates the DOA estimation performance based on CES-SCM and CES-COBASL. Consider a uniform linear array with $N = 10$ sensors. There are $D = 5$ sources and $K = 1000$ snapshots. The array output is modeled as $\mathbf{x} = \mathbf{A}\mathbf{s} + \mathbf{n}$. The source vector $\mathbf{s}$ and the noise vector $\mathbf{n}$ follow $[\mathbf{s}^T, \mathbf{n}^T]^T \sim \mathcal{CE}(\mathbf{0}, \mathbf{I}, g)$ with g being the complex t-distribution and $\nu = 2$. The array manifold matrix is given by $\mathbf{A} = [e^{j2\pi n\bar{\theta}_i}]_{n,i} \in \mathbb{C}^{N\times D}$, where $n = 0, 1, \ldots, N-1$, $i = 1, 2, \ldots, D$, and the true normalized DOAs $\bar{\theta}_i \in [-1/2, 1/2]$ are marked by the vertical lines in Fig. 4. We use the MUSIC algorithm [27] to estimate the normalized DOAs. We see that the normalized MUSIC spectrum $P(\bar{\theta})$ of CES-SCM does not have five peaks matching the true $\bar{\theta}_i$'s. On the other hand, the MUSIC spectrum of CES-COBASL exhibits five distinguishable peaks consistent with the ground truth. The reason is that CES-COBASL is more robust to CES distributions than CES-SCM.

5. CONCLUDING REMARKS

This paper shows that the arcsine law remains valid for CES distributions. The proof was based on the angular Gaussian distributions. This property has led to an estimator for the normalized scatter matrix based on one-bit data and CES-COBASL. CES-COBASL not only enjoys low computational complexity but also is robust to heavy-tailed CES distributions.

In the future, it is of interest to analyze the estimation performance of CES-COBASL for CES distributions. Another possible direction is to estimate the scatter matrix, by using CES-COBASL and robust estimators of the diagonals of the scatter matrix.

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