Correlation Subspaces: Generalizations and Connection to Difference Coarrays

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Abstract—Direction-of-arrival (DOA) estimation finds applications in many areas of science and engineering. In these applications, sparse arrays like minimum redundancy arrays, nested arrays, and coprime arrays, can be exploited to resolve $O(N^2)$ uncorrelated sources using $N$ physical sensors. Recently, it has been shown that correlation subspaces, which reveal the structure of the covariance matrix, help to improve some existing DOA estimators. However, the bases, the dimension, and other theoretical properties of correlation subspaces remain to be investigated. This paper proposes generalized correlation subspaces in one and multiple dimensions. This leads to new insights into correlation subspaces and DOA estimation with prior knowledge. First, it is shown that the bases and the dimension of correlation subspaces are fundamentally related to difference coarrays, which were previously found to be important in the study of sparse arrays. Furthermore, generalized correlation subspaces can handle certain forms of prior knowledge about source directions. These results allow one to derive a broad class of DOA estimators with improved performance. It is demonstrated through examples that using sparse arrays and generalized correlation subspaces, DOA estimators with source priors exhibit better estimation performance than those without priors, in extreme cases like low SNR and limited snapshots.

Index Terms—DOA estimation, sparse arrays, difference coarrays, generalized correlation subspaces, discrete prolate spheroidal sequences.

I. INTRODUCTION

Direction-of-arrival (DOA) estimation has been a popular research field in array processing for several decades. This problem aims to estimate the source directions from sensor measurements. It arises in many practical scenarios such as radio astronomy, radar, imaging, and communications [1]–[7]. DOA estimators such as MUSIC [8], ESPRIT [9], MODE [10], and SPICE [11]–[13], to name a few [14]–[16] have been developed for this purpose. These estimators are functional for uniform linear arrays (ULAs). It is known that, for an ULA with $N$ physical sensors, these DOA estimators can identify at most $N – 1$ uncorrelated sources [6].

Another family of DOA estimators is applicable to sparse arrays, where the sensors are placed nonuniformly. Some well-known sparse arrays include minimum redundancy arrays [17], nested arrays [18], and coprime arrays [19]–[21]. All these arrays have $O(N^2)$ consecutive elements in the difference coarray, with $N$ being the number of physical sensors. Due to this property, sparse arrays can identify $O(N^2)$ uncorrelated sources using $N$ physical sensors and they lead to better spatial resolution than ULAs. DOA estimators for sparse arrays consist of the augmented covariance matrix method [22], [23], positive definite Toeplitz completion [24], [25], spatial smoothing MUSIC (SS MUSIC) [18], [26], [27], dictionary methods [28]–[31], and gridless approaches [32]–[34]. Note that these estimators are also applicable to ULAs.

Recently, the elegant concept of correlation subspaces was proposed by Rahmani and Atia [16], to improve DOA estimation in a number of ways. For any given array geometry, the correlation subspace is uniquely determined, and imposes some implicit constraints on the structure of the covariance, as we shall see. This subspace can be utilized in denoising the sample covariance matrix. Then the source directions are estimated from the denoised covariance matrix using off-the-shelf DOA estimators, such as the MUSIC algorithm. Note that the correlation subspace depends on the array configurations and prior knowledge about the sources but is independent of the choice of DOA estimators. Hence, a broad class of DOA estimators are applicable to the denoised covariance matrix. However, the explicit expressions for the correlation subspace were not known, so its approximation was computed numerically in [16]. Furthermore, the way in which the correlation subspace is influenced by the array configuration, and by partial knowledge about sources, was not explored.

Inspired by the concept of correlation subspaces introduced in [16], this paper makes a number of new contributions. To analyze the correlation subspace for any array configuration explicitly, we first generalize the definition in [16] to formulate what we call the generalized correlation subspace. This makes it possible to incorporate some types of prior information on source locations, leading to improvements in DOA estimation. Furthermore, we show that the (generalized) correlation subspaces can be uniquely characterized in terms of the difference coarray of the original physical array. In fact we will show how to obtain simple and elegant closed form expressions for the basis vectors of the correlation subspace, in terms of the sensor geometry and the difference coarray geometry. Furthermore, if source directions belong to apriori known intervals [1], [5], [34]–[36], then it is shown that the generalized correlation subspace finds close connections to discrete prolate spheroidal sequences [37] defined on the difference coarrays. Similar results can be developed in multiple dimensions, and are useful in angle-Doppler estimation [38], [39], angle-delay estimation [40], angle-range estimation [41], 2D DOA estimation [6], and harmonic retrieval [42], [43]. These results not only facilitate the implementation of the denoising framework as in [16] but also offer better understanding of several DOA estimators with prior knowledge about sources.

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The original work on correlation subspaces [16] did not emphasize any knowledge about the sources except that they are uncorrelated\(^1\). In our work, we show how to incorporate prior knowledge about source intervals with generalized correlation subspaces. Furthermore, correlation subspaces for 1D and 2D arrays were analyzed numerically in [16] while our work provides closed-form characterizations of generalized correlation subspaces for 1D and multidimensional arrays. Finally, in our work, covariance matrix denoising using the generalized correlation subspace is implemented more efficiently than that in [16].

The outline of this paper is as follows: Section II reviews correlation subspaces. Section III proposes the generalized correlation subspaces while Section IV derives their expressions for unknown and known source intervals. Section V discusses the connections with existing methods while Section VI studies the generalized correlation subspace for multidimensional arrays. Section VII presents several examples and numerical simulations to demonstrate the advantages of the new methods while Section VIII concludes this paper.

**Notations:** Scalars, vectors, matrices, and sets are denoted by lowercase letters (a), lowercase letters in boldface (\(a\)), uppercase letters in boldface (\(A\)), and letters in blackboard boldface (\(\mathbb{A}\)), respectively. An \(R\)-dimensional vector \(n\) is denoted by \((n^{(1)}, n^{(2)}, \ldots, n^{(R)})\) or \([n^{(1)}, n^{(2)}, \ldots, n^{(R)}]^T\), where \(n^{(r)}\) is the \(r\)th coordinate. \(A^T\), \(A^*\), and \(A^H\) are the transpose, complex conjugate, and complex conjugate transpose of \(A\). The Moore-Penrose pseudoinverse of \(A\) is \(A^+\). If \(A\) has full column rank, then \(A^+ = (A^HA)^{-1}A^H\). The Kronecker product between \(A\) and \(B\) is written as \(A \otimes B\). The vectorization operator is defined as \(\text{vec}(a_1, a_2, \ldots, a_N) = [a_1^T, a_2^T, \ldots, a_N^T]^T\), where \(a_1, a_2, \ldots, a_N\) are column vectors. For two Hermitian matrices \(A\) and \(B\), \(A \succeq B\) is equivalent to \(A - B\) being positive semidefinite. \(\text{col}(A)\) stands for the column space of \(A\). The support of a function \(f(x)\) is defined as \(\text{supp}(f) = \{x : f(x) \neq 0\}\), where \(x\) belongs to the domain of \(f\). The indicator function \(1_A(x)\) is one if \(x \in A\) and zero otherwise. \(E[\cdot]\) is the expectation operator. The cardinality of a set \(A\) is denoted by \(|A|\).

The bracket notation [27], [44] is reviewed using the following example. Assume the sensor locations are characterized by an integer set \(S = \{\pm 1, \pm 2, \ldots, \pm K\}\). Assume the sensor measurement on \(S\) is denoted by \(x_S = [10, 11, 12]^T\). The square bracket \([x_S]_n\) represents the \(n\)th entry of \(x_S\) while the triangular bracket \((x_S)_n\) denotes the sample value on the support location \(n\). Hence, we have \([x_S]_1 = 10, [x_S]_2 = 11, [x_S]_3 = 12, (x_S)_{-2} = 10, (x_S)_{-1} = 11, \) and \((x_S)_0 = 12\). Similar notations apply to matrices. For instance, if \(A = xSx_S^T\), then \([A]_{i,j} = [x_S]_i[x_S]_j\) and \((A)_{n_1,n_2} = (x_S)_{n_1}(x_S)_{n_2}\).

### II. REVIEW OF CORRELATION SUBSPACES

Assume that \(D\) monochromatic sources impinge on an one-dimensional sensor array\(^2\). The sensor locations are \(nd\), where \(n\) belongs to an integer set \(S \subseteq \mathbb{Z}\), \(d = \lambda/2\), and \(\lambda\) is the wavelength. Let \(\theta_i \in [-\pi/2, \pi/2]\) be the DOA of the \(i\)th source. The normalized DOA of the \(i\)th source is defined as \(\bar{\theta}_i = (d/\lambda)\sin \theta_i \in [-1/2, 1/2]\). The measurements on the sensor array \(S\) can be modeled as

\[
x_S = \sum_{i=1}^{D} A_i v_S(\bar{\theta}_i) + n_S \quad \in \mathbb{C}^{|S|},
\]

where \(A_i\) is the complex amplitude of the \(i\)th source, \(v_S(\bar{\theta}_i) = [e^{j2\pi\theta_i}], \) \(n_S\) are the steering vectors, and \(n_S\) is the additive noise term. It is assumed that the sources and noise are zero-mean and uncorrelated. Namely, let \(s = [A_1, \ldots, A_D, n_S]^T\). Then \(E[s] = 0\) and \(E[ss^H] = \text{diag}(p_1, \ldots, p_D, p_n, \ldots, p_n)\), where \(p_i\) and \(p_n\) are the power of the \(i\)th sources and the noise, respectively.

The covariance matrix of \(x_S\) can be expressed as

\[
R_S = E[x_Sx_S^H] = \sum_{i=1}^{D} p_i v_S(\bar{\theta}_i)v_S^H(\bar{\theta}_i) + \eta_n I.
\]

Rearranging the elements in (2) leads to

\[
\text{vec}(R_S - \eta_n I) = \sum_{i=1}^{D} p_i c(\bar{\theta}_i),
\]

where the correlation vectors \(c(\bar{\theta}_i)\) are defined as

\[
c(\bar{\theta}_i) \triangleq \text{vec}(v_S(\bar{\theta}_i)v_S^H(\bar{\theta}_i)) \in \mathbb{C}^{|S|^2}.
\]

The relation (3) implies

\[
\text{vec}(R_S - \eta_n I) \in \text{span}\{c(\bar{\theta}_i) : i = 1, 2, \ldots, D\} \subseteq CS \triangleq \text{span}\{c(\bar{\theta}) : -1/2 \leq \bar{\theta} \leq 1/2\},
\]

where the linear span in (6) is defined as the set of all vectors of the form \(\sum_{p=1}^{P} a_p c(\bar{\theta}_p)\) where \(P \in \mathbb{N}, a_p \in \mathbb{C}, -1/2 \leq \bar{\theta}_p \leq 1/2\) [45]. This subspace is called the correlation subspace, denoted by \(CS\). Eq. (6) also indicates that \(\text{vec}(R_S - \eta_n I)\) is constrained in a certain way by \(CS\), and these constraints can be used in designing DOA estimators for improved performance.

It is clear that \(CS\) is a finite-dimensional subspace of \(\mathbb{C}^{|S|^2}\), due to (4). However, the definition of the correlation subspace in (6) is computationally intractable since it involves infinitely many \(c(\bar{\theta})\). The correlation subspace was originally computed by the following definition [16]:

**Definition 1:** The correlation subspace \(CS\) satisfies

\[
CS = \text{col}(S),
\]

where the correlation subspace matrix \(S\) is defined as

\[
S \triangleq \int_{-\pi/2}^{\pi/2} c(\bar{\theta})c(\bar{\theta})^H d\bar{\theta} \in \mathbb{C}^{|S|^2 \times |S|^2}.
\]

In Appendix A, we show that this definition is equivalent to our revised definition given in (6). Note that this integral is carried out over the DOA, \(\bar{\theta} \in [-\pi/2, \pi/2]\) and the relation \(\bar{\theta} = (\sin \theta)/2\) can be utilized to evaluate (8). According to (8), it can be shown that the correlation subspace matrix \(S\) is Hermitian and positive semidefinite.
It was shown in [16] that, the right-hand side of (7) can be simplified further, based on the eigenvectors of $S$ associated with the nonzero eigenvalues. In particular, let the eigen-decomposition of $S$ be

$$S = \begin{bmatrix} Q_{CS} & Q_{CS}^{-1} \\ Q_{CS}^{-1} & Q_{CS} \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} Q_{CS} & Q_{CS}^{-1} \end{bmatrix}^H,$$  \tag{9}

where the diagonal matrix $\Lambda_1$ contains the positive eigenvalues in the descending order and the columns of $Q$ consist of the orthonormal eigenvectors. Then, (7) and (9) lead to $CS = \text{col}(Q_{CS})$. Namely, the correlation subspace $CS$ is the column space of the matrix $Q_{CS}$. Eqs. (7), (8), and (9) indicate that the matrix $S$, its eigenvalues, its eigenvectors, and the correlation subspace depend purely on the array configuration.

For any array geometry, the correlation subspace is uniquely determined, and imposes some implicit constraints on the structure of the covariance matrix, as indicated in (6). This leads to a covariance-matrix denoising approach [16]. To begin with, consider finite snapshot sensor measurements $\tilde{x}_k(k)$ for $k = 1, \ldots, K$. The sample covariance matrix $R_S$ can be estimated by

$$\tilde{R}_S = \frac{1}{K} \sum_{k=1}^K \tilde{x}_k(k) \tilde{x}_k^H(k).$$  \tag{10}

The algorithm in [16] first denotes the sample covariance matrix $R_S$ using the following convex program (P1):

$$(P1): R_{p_1}^* \triangleq \arg \min_{R} \| \tilde{R}_S - p_n I - R \|_2^2$$  \tag{11}

subject to

$$(I - Q_{CS}Q_{CS}^\dagger)\text{vec}(R) = 0,$$  \tag{12}

$R \succeq 0,$  \tag{13}

where the noise power $p_n$ is estimated from the eigenvalues of $R_S$ and $\| \cdot \|_2$ denotes the spectral norm of a matrix (i.e., the largest singular value). The cost function in (11) suggests that the matrix $R_{p_1}^*$ resembles $\sum_{i=1}^D p_i v_\parallel(\hat{\theta}_i) v_\parallel^H(\hat{\theta}_i)$ in (2). The constraint (12) ensures that $\text{vec}(R_{p_1}^*)$ belongs to the correlation subspace while (13) indicates that $R_{p_1}^*$ is positive semidefinite.

The final stage is to perform DOA estimation on $R_{p_1}^*$. It was shown in [16] that, the MUSIC algorithm on $R_{p_1}^*$ can exhibit better estimation performance than that on $R_S$. It should be noted that the DOA estimators are not restricted to the MUSIC algorithm. Other estimators, such as ESPRIT, MODE, and SPICE, can also be exploited. This result shows that the structure of the covariance matrix, as specified by the correlation subspace, helps to improve the estimation performance.

Summarizing, the DOA estimator associated with the correlation subspace is composed of the following three steps [16]:

**Step 1: Correlation subspace.** For a given array geometry, numerically calculate the correlation subspace matrix $S$ and its eigen-decomposition, as in (8) and (9), respectively. Save the matrix $Q_{CS}$ for the correlation subspace.

**Step 2: Denoising.** Given the sensor measurements, evaluate the sample covariance matrix $\tilde{R}_S$, as in (10). Then solve the optimization problem (P1). Let the optimal solution be $R_{p_1}^*$.

**Step 3: DOA estimation.** In the simulations of [16], the MUSIC algorithm is applied to $R_{p_1}^*$. Remarks on Step 1: Note that this step needs to be done only once per array. Once the matrix $Q_{CS}$ is obtained, it can be used repeatedly in Step 2. To calculate $Q_{CS}$, the numerical integration was utilized in [16]. This step is typically done by choosing a dense grid of the parameter $\theta$, which only approximates the integral in (8). Furthermore, the numerical eigen-decomposition in (9) introduces perturbations on zero eigenvalues, making it challenging to determine the correlation subspace precisely. It is desirable to mitigate the negative effects caused by numerical computations. It will be shown in Theorem 1 that, the correlation subspace can be fully characterized by simple, elegant, and closed-form expressions.

Remarks on Step 2: The convex optimization problem (P1) can be solved by numerical solvers. However, it requires several numerical iterations to obtain the optimal solution, which could be an issue if real-time processing is needed. To avoid high computational cost, (P1) is approximated by two sub-problems (P1a) and (P1b) in [16]:

$$(P1a): R_{p_{1a}}^* \triangleq \arg \min_{R} \| \tilde{R}_S - p_n I - R \|_2^2$$  \tag{14}

subject to

$$(I - Q_{CS}Q_{CS}^\dagger)\text{vec}(R) = 0,$$  \tag{15}

$R \succeq 0,$  \tag{16}

In particular, we first compute the solution to (P1a) using the orthogonal projection onto the correlation subspace, $\text{vec}(R_{p_{1a}}^*) = Q_{CS}Q_{CS}^\dagger\text{vec}(\tilde{R}_S - p_n I)$. Then the solution $R_{p_{1b}}^*$ can be obtained explicitly from the eigen-decomposition of $R_{p_{1a}}^*$. It was demonstrated in [16] that this two-step approach can be readily implemented with a moderate degradation in the estimation performance.

It can be seen that we need to estimate the noise power $p_n$ first before solving either (P1) or (P1a). This extra processing is not desirable if the source directions are the only parameters of interest. In Section V, we will present another optimization problem that enjoys good DOA estimation performance without estimating the noise power.

Remarks on Step 3: If $R_{p1}^* \in \mathbb{C}^{[S] \times [S]}$, then the MUSIC algorithm on $R_{p1}^*$ can resolve at most $|S| - 1$ uncorrelated sources, regardless of array configurations [6]. However, sparse arrays can identify more uncorrelated sources than sensors by using augmented covariance matrices [23], positive definite Toeplitz completion [24], [25], and spatial smoothing MUSIC (SS MUSIC) [18], [19], [26], [27]. Furthermore, these approaches usually provide better spatial resolution than the MUSIC algorithm [18], [19], [23]–[27]. Hence, we will use sparse arrays and SS MUSIC in the examples of Section VII.

Furthermore, it was demonstrated numerically in [16] that, prior knowledge about source directions can be embedded into the correlation subspace by changing the intervals of integration in (8). However, the influences of prior knowledge about the correlation subspace and the optimization problem have not been studied in detail. These above points will be addressed by generalized correlation subspaces, as we will present next.
III. GENERALIZED CORRELATION SUBSPACES

The main difficulty in deriving the closed-form expressions for CS using (7), (8), and (9) is as follows. Eq. (8) implies that the entries of $S$ are related to Bessel functions, making it complicated to obtain analytical forms of (9). In order to derive closed-form expressions for CS, we will first propose the generalized correlation subspace $\mathcal{GCS}(\rho)$, which is parameterized by a nonnegative density function $\rho(\bar{\theta})$. Then it will be shown that $\mathcal{GCS}(\rho)$ depends only on the support of $\rho(\bar{\theta})$, and is invariant to its exact shape. Using this property, we can derive simple and closed forms for CS by selecting some density functions $\rho(\bar{\theta})$ such that 1) $CS = \mathcal{GCS}(\rho)$ and 2) it is straightforward to derive closed-form expressions for $\mathcal{GCS}(\rho)$.

As a motivating example, let us consider the definition of $S$ in (8). Since $\bar{\theta} = 0.5 \sin \theta$, we have $d\bar{\theta} = 2(1 - (2\bar{\theta})^2)^{-1/2} d\theta$. Hence, (8) can be rewritten as

$$S = \int_{-1/2}^{1/2} c(\bar{\theta}) c^H(\bar{\theta}) \left(2(1 - (2\bar{\theta})^2)^{-1/2}\right) d\bar{\theta}. \quad (18)$$

Note that (18) can be regarded as a weighted integral with the density function $2(1 - (2\bar{\theta})^2)^{-1/2}$ over $\bar{\theta} \in [-1/2, 1/2]$. Hence, we can generalize the correlation subspace matrix by varying the density function in (18). It is formally defined as

**Definition 2:** Let the correlation vector $c(\bar{\theta})$ be defined as in (4). Let $\rho(\bar{\theta})$ be a nonnegative Lebesgue integrable function over the set $[-1/2, 1/2]$. The generalized correlation subspace matrix associated with $\rho(\bar{\theta})$ is defined as

$$S(\rho) = \int_{-1/2}^{1/2} c(\bar{\theta}) c^H(\bar{\theta}) \rho(\bar{\theta}) d\bar{\theta}. \quad (19)$$

It can be seen that (18) is a special case of Definition 2, with $\rho(\bar{\theta}) = 2(1 - (2\bar{\theta})^2)^{-1/2}$ for $\bar{\theta} \in [-1/2, 1/2]$.

Based on Definition 1, the generalized correlation subspace can be defined as follows:

**Definition 3:** Let $S(\rho)$ be the generalized correlation subspace matrix associated with $\rho(\bar{\theta})$, as in (19). The generalized correlation subspace is defined as $\mathcal{GCS}(\rho) = \text{col}(S(\rho))$.

It can be seen from Definition 2 and 3 that the generalized correlation subspaces are parameterized by the density function $\rho(\bar{\theta})$. For any given support of $\rho(\bar{\theta})$, the generalized correlation subspace is invariant to the exact shape of $\rho(\bar{\theta})$ under that support, as indicated by the following lemma:

**Lemma 1:** Let $\rho_1(\bar{\theta})$ and $\rho_2(\bar{\theta})$ be two nonnegative Lebesgue integrable functions over the set $[-1/2, 1/2]$. If $\text{supp}(\rho_1) = \text{supp}(\rho_2)$, then $\mathcal{GCS}(\rho_1) = \mathcal{GCS}(\rho_2)$.

**Proof:** See Appendix B.

**Corollary 1:** Let the density function in (18) be $\rho_1(\bar{\theta}) = 2(1 - (2\bar{\theta})^2)^{-1/2}$ for $\bar{\theta} \in [-1/2, 1/2]$ and the constant density function be $\rho_2(\bar{\theta}) = 1_{[-1/2, 1/2]}(\bar{\theta})$. Then $CS = \mathcal{GCS}(\rho_1) = \mathcal{GCS}(\rho_2)$.

The density functions $\rho_1(\bar{\theta})$ and $\rho_2(\bar{\theta})$ are illustrated in Fig. 1. It can be observed that these density functions share the same support $[-1/2, 1/2]$. Furthermore, Corollary 1 also enables us to analyze the correlation subspace readily through the generalized correlation subspace $\mathcal{GCS}(\rho_2)$. The details will be developed in Section IV.

IV. PROPERTIES OF GENERALIZED CORRELATION SUBSPACES

In this section, the generalized correlation subspaces for several density functions will be investigated. It will be shown that the correlation subspace and the generalized correlation subspace depend on the difference coarray. Furthermore, we will derive simple, explicit, and computationally tractable representations of the correlation subspace and the generalized correlation subspace in certain cases.

First, it is known from [44] that the correlation vector (4) can be rewritten as:

$$c(\bar{\theta}_i) = Jv_\mathbb{D}(\bar{\theta}_i), \quad (20)$$

where $v_\mathbb{D}(\bar{\theta}_i) = [e^{j2\pi \nu_m \bar{\theta}_i}]_{m \in \mathbb{D}}$ are the steering vectors on the difference coarray. Here the difference coarray $\mathbb{D}$ and the matrix $J$ [44] are defined as:

**Definition 4 (Difference coarray):** The difference coarray $\mathbb{D}$ contains the differences between the elements in $\mathbb{S}$, i.e., $\mathbb{D} = \{n_1 - n_2 : \forall n_1, n_2 \in \mathbb{S}\}$.

**Definition 5 (The matrix J):** The binary matrix $J$ has size $|\mathbb{S}|^2$-by-$|\mathbb{D}|$. The columns of $J$ satisfy $(J)_{i,m} = \text{vec}(I(m))$ for $m \in \mathbb{D}$, where $I(m) \in \{0, 1\}^{|\mathbb{S}| \times |\mathbb{S}|}$ is given by

$$(I(m))_{n_1,n_2} = \begin{cases} 1, & \text{if } n_1 - n_2 = m, \\ 0, & \text{otherwise}. \end{cases} \quad (21)$$

Here the bracket notation $(\cdot)_{n_1,n_2}$ is defined in Section I.

**Example 1:** Assume the sensor locations are characterized by an integer set $\mathbb{S} = \{0, 2\}$. According to Definition 4, the difference coarray becomes $\mathbb{D} = \{-2, 0, 2\}$. Next we will evaluate the matrix $J$, as in Definition 5. First we consider the matrices $I(m)$ for $m \in \mathbb{D}$ as follows:

$$I(0) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad I(2) = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}.$$
where \( n_1 \) and \( n_2 \) are marked in the corresponding rows and columns. Furthermore, due to (21), it can be shown that \( \mathbf{I}(-2) = (\mathbf{I}(2))^T \). Hence, the matrix \( \mathbf{J} \) can be written as

\[
\mathbf{J} = \begin{bmatrix}
\text{vec}(\mathbf{I}(-2)) & \text{vec}(\mathbf{I}(0)) & \text{vec}(\mathbf{I}(2)) \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix},
\]

(22)

where the first, the second, and the third column of \( \mathbf{J} \) correspond to the coarray index \( m = -2, 0, 2 \), respectively. Finally, we will verify (20) in this example. Starting with (4), the correlation vector is given by

\[
\mathbf{c}(\bar{\theta}_i) = \text{vec} \left( \begin{bmatrix}
\frac{1}{e^{j2\pi\bar{\theta}_i}} & 1 & e^{-j2\pi\bar{\theta}_i}
\end{bmatrix} \right) = \begin{bmatrix} 1 & e^{j4\pi\bar{\theta}_i} & e^{-j4\pi\bar{\theta}_i} & 1 \end{bmatrix}.
\]

This result verifies (20).

Using (20) and Definition 2, the generalized correlation subspace matrix can be expressed in terms of the difference coarray as in Lemma 2:

**Lemma 2:** The generalized correlation subspace matrix satisfies \( \mathbf{S}(\rho) = \mathbf{J} \mathbf{S}_D(\rho) \mathbf{J}^H \), where \( \mathbf{J} \) is defined in Definition 5 and \( \mathbf{S}_D(\rho) \) is given by

\[
\mathbf{S}_D(\rho) = \int_{-1/2}^{1/2} \mathbf{v}_D(\bar{\theta}) \mathbf{v}_D^H(\bar{\theta}) \rho(\bar{\theta}) d\bar{\theta}.
\]

(23)

Note that the matrix \( \mathbf{S}_D(\rho) \) depends on the difference coarray, rather than the physical sensor locations. This property suggests that the generalized correlation subspace is fundamentally related to the difference coarray. This is indeed true and we will elaborate this point later. Furthermore, Lemma 2 also allows us to readily analyze the matrix \( \mathbf{S}(\rho) \) of size \( |\mathbb{S}|^2 \)-by-\( |\mathbb{S}|^2 \), by examining a smaller matrix \( \mathbf{S}_D(\rho) \) of size \( |\mathbb{D}| \)-by-\( |\mathbb{D}| \).

Next, we will consider two simple examples of the density function as follows:

- The density function is a constant over \( \bar{\theta} \in [-1/2, 1/2] \). Namely, \( \rho(\bar{\theta}) = 1_{[-1/2, 1/2]}(\bar{\theta}) \).
- The density function is a constant over some known intervals \( \mathbb{I} \). That is, \( \rho(\bar{\theta}) = 1_{\mathbb{I}}(\bar{\theta}) \). This case corresponds to known source intervals.

In both cases, we will present the closed-form expressions of \( \mathbf{S}_D(\rho) \) and \( \mathbf{S}(\rho) \), from which the generalized correlation subspaces can be analyzed systematically.

A. The constant density function

In this case, the entry of \( \mathbf{S}_D(1_{[-1/2,1/2]}) \) associated with coarray locations \( m_1, m_2 \in \mathbb{D} \) becomes

\[
\langle \mathbf{S}_D(1_{[-1/2,1/2]}) \rangle_{m_1,m_2} = \int_{-1/2}^{1/2} e^{j2\pi(m_1-m_2)\bar{\theta}} d\bar{\theta} = \delta_{m_1,m_2},
\]

(24)

since \( m_1 \) and \( m_2 \) are integers. Substituting (24) into Lemma 2 gives

\[
\mathbf{S}(1_{[-1/2,1/2]}) = \mathbf{J} \mathbf{J}^H.
\]

(25)

In order to obtain the eigen-decomposition of \( \mathbf{S}(1_{[-1/2,1/2]}) \), we invoke the definition of the weight function and a lemma regarding the matrix \( \mathbf{J} \):

**Definition 6 (Weight function \( w(m) \)):** The weight function \( w(m) \) of an array \( \mathbb{S} \) is defined as the number of sensor pairs with coarray index \( m \). Namely, \( w(m) = |\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\}| \) for \( m \in \mathbb{D} \).

**Lemma 3:** \( \mathbf{J}^H \mathbf{J} = \mathbf{W} \triangleq \text{diag}(w(m))_{m \in \mathbb{D}} \). Namely, \( \mathbf{J} \) has orthogonal columns and the norm of the column associated with the coarray index \( m \) is \( \sqrt{w(m)} \).

**Proof:** It was proved in [44] that the columns of \( \mathbf{J} \) are orthogonal. It suffices to consider the norms of the individual columns of \( \mathbf{J} \). For the coarray location \( m \in \mathbb{D} \), the norm of the associated column is

\[
||\text{vec}(\mathbf{I}(m))||_2^2 = \sum_{n_1,n_2 \in \mathbb{S}} |(\mathbf{I}(m))_{n_1,n_2}|^2 = \text{the number of ones in} \ \mathbf{I}(m) = w(m),
\]

which proves this lemma.

**Example 2:** Assume the sensor locations are given by \( \mathbb{S} = \{0, 2\} \), as in Example 1. Then the weight functions \( w(m) \) are given by

\[
w(-2) = |\{(0, 2)\}| = 1, \quad w(0) = |\{(0, 0), (2, 2)\}| = 2, \quad w(2) = |\{(2, 0)\}| = 1.
\]

Hence the matrix \( \mathbf{W} \) in Lemma 3 can be written as \( \mathbf{W} = \text{diag}(w(-2), w(0), w(2)) = \text{diag}(1, 2, 1) \). Then, Lemma 3 can be verified using (22) and \( \mathbf{W} \).

Due to Definition 6 and Lemma 3, the generalized correlation subspace matrix \( \mathbf{S}(1_{[-1/2,1/2]}) \) can be expressed as

\[
\mathbf{S}(1_{[-1/2,1/2]}) = (\mathbf{J} \mathbf{W}^{-1/2}) \mathbf{W} (\mathbf{J} \mathbf{W}^{-1/2})^H,
\]

(26)

where the sizes of these terms are \( \mathbf{J} \mathbf{W}^{-1/2} \in \mathbb{C}^{|\mathbb{S}|^2 \times |\mathbb{D}|} \) and \( \mathbf{W} \in \mathbb{C}^{|\mathbb{D}| \times |\mathbb{D}|} \). Eq. (26) also indicates that the matrix \( \mathbf{J} \mathbf{W}^{-1/2} \) corresponds to the orthonormal eigenvectors while the diagonal matrix \( \mathbf{W} \) is associated with the eigenvalues. In particular, the positive eigenvalues and the associated eigenvectors of \( \mathbf{S}(1_{[-1/2,1/2]}) \) are given by

**Positive eigenvalues of** \( \mathbf{S}(1_{[-1/2,1/2]}) = w(m) \),

(27)

**Eigenvectors of** \( \mathbf{S}(1_{[-1/2,1/2]}) = \frac{\text{vec}(\mathbf{I}(m))}{\sqrt{w(m)}} \),

(28)

where \( m \in \mathbb{D} \). Note that (27) and (28) can be calculated readily from the array geometry using Definition 6 and 5, respectively. Namely, the eigen-decomposition of \( \mathbf{S}(1_{[-1/2,1/2]}) \) can be
evaluated without using the numerical integration in Definition 2 and the numerical eigen-decomposition on $S(1,-1/2,1/2)$.

Properties of $S(1,-1/2,1/2)$: The following list some properties regarding the eigen-structure of $S(1,-1/2,1/2)$. Some items are direct consequences of the properties of the weight function $w(m)$, as in [46, Lemma 1].

1) The eigenvalues of $S(1,-1/2,1/2)$ are nonnegative integers, due to (26), (27), and [46, Lemma 1-1].

2) The number of nonzero eigenvalues is $|\mathbb{D}|$, which is the size of the difference coarray. Here repeated eigenvalues are counted separately.

3) The largest eigenvalue of $S(1,-1/2,1/2)$ is $w(0) = |S|$ with algebraic multiplicity 1. This is a direct consequence of [46, Lemma 1-4].

4) For any nonzero eigenvalue that is not the largest, the associated algebraic multiplicity is an even positive integer, due to [46, Lemma 1-2].

5) The eigenvalue zero has algebraic multiplicity $|S|^2 - |\mathbb{D}|$.

6) The orthonormal eigenvectors associated with nonzero eigenvalues are given in (28).

7) For all the eigenvalues of $S(1,-1/2,1/2)$, the geometric multiplicities achieve the algebraic multiplicities.

Finally, based on (26), the generalized correlation subspace $\mathcal{GCS}(1,-1/2,1/2)$ becomes

$$\text{col}(S(1,-1/2,1/2)) = \text{col}(JW^{-1/2}) = \text{col}(J).$$

(29)

The significance of (29) is that, the correlation subspace in (6) can be characterized in closed forms, as in the following theorem:

**Theorem 1:** Let the matrix $J$ be defined as in Definition 5. Then the correlation subspace satisfies

$$\mathcal{CS} = \text{col}(J).$$

(30)

**Proof:** Corollary 1 indicates that $\mathcal{CS} = \mathcal{GCS}(\rho_1) = \mathcal{GCS}(\rho_2)$. The relation $\mathcal{GCS}(\rho_2) = \text{col}(J)$ is due to (29).

This theorem indicates that the correlation subspace is fully characterized by the binary matrix $J$, which can be readily computed from sensor locations and the difference coarray using Definition 5. Namely, to compute the correlation subspace, the numerical integration (8) and the eigen-decomposition (9) can be avoided completely. Due to Theorem 1 and Lemma 3, the dimension of the correlation subspace is given by

**Corollary 2:** The dimension of the correlation subspace is the size of the difference coarray, i.e., $\text{dim}(\mathcal{CS}) = |\mathbb{D}|$.

B. The constant density function with known source intervals

Here we will study the generalized correlation subspace with known source intervals. This scenario arises in practical applications such as stationary radar and the diagnosis of rotating machines in industrial environments [47].

Table I summarizes the procedure for the generalized correlation subspaces with known source intervals. For a given array configuration $\mathbb{S}$ and source intervals $\mathbb{I}$, we can calculate $S_D(\mathbb{I})$ explicitly using either (32), (38), or (42), as we shall explain later. The eigen-decomposition of $S_D(\mathbb{I})$ suggests that $S_D(\mathbb{I})$ can be approximated by $\Psi_L \cdot \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_L) \cdot \Psi_H^H$, where the related quantities are given in Table I. This property leads to an approximation of the generalized correlation subspace $\mathcal{GCS}(\mathbb{I}) \approx \text{col}(J\Psi_L)$. Note that Table I is applicable to a given array $\mathbb{S}$ and given source intervals $\mathbb{I}$ in the following development, we will study the generalized correlation subspaces based on these factors.

1) Hole-free difference coarrays and $\mathbb{I} = [-\alpha/2, \alpha/2]$: In this case, the density function is assumed to be

$$\rho(\bar{\theta}) = 1_{[-\alpha/2, \alpha/2]}(\bar{\theta}) = \begin{cases} 1, & \text{if } -\alpha/2 \leq \bar{\theta} \leq \alpha/2, \\ 0, & \text{otherwise}, \end{cases}$$

(31)

where $0 < \alpha < 1$. Now we can derive the expression for the generalized correlation subspace matrix. According to Lemma 2, the entry of $S_D(\mathbb{I} = [-\alpha/2, \alpha/2])$ associated with coarray locations $m_1, m_2 \in \mathbb{I}$ is given by

$$\langle S_D(\mathbb{I} = [-\alpha/2, \alpha/2]) \rangle_{m_1,m_2} = \int_{-\alpha/2}^{\alpha/2} e^{|\alpha/2\bar{\theta}(m_1-m_2)} d\bar{\theta}$$

$$= \alpha \cdot \text{sinc}(\alpha(m_1 - m_2)), \quad (32)$$

where the normalized sinc function $\text{sinc}(x)$ is 1 for $x = 0$ and $\sin(x)/(\pi x)$ otherwise. The eigen-decomposition of $S_D(\mathbb{I} = [-\alpha/2, \alpha/2])$ is assumed to be

$$S_D(\mathbb{I} = [-\alpha/2, \alpha/2]) = \Psi \Lambda \Psi^H,$$

(33)

where the matrices $\Psi$ and $\Lambda$ are given by $\Psi = [\psi_1, \psi_2, \ldots, \psi_{|\mathbb{D}|}]$ and $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{|\mathbb{D}|})$, respectively. Here the eigenvalues satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{|\mathbb{D}|} \geq 0$ and $\psi_1, \psi_2, \ldots, \psi_{|\mathbb{D}|}$ are the associated orthonormal eigenvectors.

Note that the matrix $S_D(\mathbb{I} = [-\alpha/2, \alpha/2])$ depends purely on the difference coarray $\mathbb{D}$ and the parameter $\alpha$. Next we assume the difference coarray $\mathbb{D}$ is hole-free. Namely, $\mathbb{D}$ is composed of consecutive integers. For instance, Fig. 2 depicts array configurations like (a) ULA [6], (b) nested array [18], (c) coprime array [19], and (d) super nested array [21]. It can be observed from the nonnegative part of the difference coarray that (a), (b), and (d) have hole-free difference coarrays while (c) does not. These array configurations will be elaborated in Section VII later.

3The symbols $\psi_k, \Psi_L$, and $\Psi$ are reserved for the eigenvectors of the matrix $S_D(\mathbb{I})$ while the notations $v_k$ and $V_L$ represent the discrete prolate spheroidal sequences.

<table>
<thead>
<tr>
<th>Input:</th>
<th>Array configuration $\mathbb{S}$, source intervals $\mathbb{I}$, and error tolerance $\delta$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1)</td>
<td>Evaluate the matrix $J$ according to Definition 5.</td>
</tr>
<tr>
<td>2)</td>
<td>Based on $\mathbb{I}$, compute $S_D(\mathbb{I})$ using either (32), (38), or (42).</td>
</tr>
<tr>
<td>3)</td>
<td>Numerical eigen-decomposition: $S_D(\mathbb{I}) = \Psi \Lambda \Psi^H$.</td>
</tr>
<tr>
<td>a)</td>
<td>Eigenvectors: $\Psi = [\psi_1, \psi_2, \ldots, \psi_{</td>
</tr>
<tr>
<td>b)</td>
<td>Eigenvales: $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{</td>
</tr>
<tr>
<td>4)</td>
<td>Determine a positive integer $L$ using (44).</td>
</tr>
<tr>
<td>5)</td>
<td>Construct $\Psi_L = [\psi_1, \psi_2, \ldots, \psi_{</td>
</tr>
<tr>
<td>Output:</td>
<td>$\mathcal{GCS}(\mathbb{I})$ is approximated by $\text{col}(J\Psi_L)$.</td>
</tr>
</tbody>
</table>
ψ

(c) Super nested array:

$$S_D \approx \mathbf{V}_L \cdot \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_L) \cdot \mathbf{V}_L^H,$$  \hspace{1cm} (36)

where $|\alpha| |D| \leq L \leq |D|$ and $\mathbf{V}_L = [v_1, v_2, \ldots, v_N]$ consists of the first $L$ DPSS. Note that the exact value of $L$ depends on the approximation error of (36), which will be elaborated in (44) later. In this example, $|\alpha| |D| = 5$. This means that the first five eigenvalues are close to one, as depicted in Fig. 3(a). The DPSS $v_1, v_2, v_3, \text{and} v_4$ are illustrated in Fig. 3(b). These DPSS can be proved to be orthogonal, real, and unique up to scale [37]. Furthermore, these DPSS satisfy $\langle v_k \rangle_m = (-1)^{k+1} v_k$ for $m \in \mathbb{D}$. Namely, they are either even or odd symmetric.

Substituting (36) into Lemma 2, the generalized correlation subspace matrix can be approximated by

$$S(1_{[-\alpha/2, \alpha/2]}) \approx \mathbf{J} \mathbf{V}_L \cdot \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_L) \cdot (\mathbf{J} \mathbf{V}_L^H),$$  \hspace{1cm} (37)

Since the matrix $\mathbf{J} \mathbf{V}_L$ has full column rank, the generalized correlation subspace can be characterized by the following theorem:

**Theorem 2:** Let the density function be $1_{[-\alpha/2, \alpha/2]}$ for $0 < \alpha < 1$. Assume the difference coarray $\mathbb{D}$ is hole-free. Then we have

$$\mathcal{GCS}(1_{[-\alpha/2, \alpha/2]}) \approx \text{col}(\mathbf{J} \mathbf{V}_L),$$

where the columns of $\mathbf{V}_L$ contain the first $L$ DPSS on $\mathbb{D}$. The parameter $L$ is obtained by (44), for a given error tolerance $\delta$.

The significance of Theorem 2 is that, the interval information $\alpha$ is embedded in the matrix $\mathbf{V}_L$. The approximation $\mathbf{V}_L$ is constructed by selecting the DPSS associated with the largest $L$ eigenvalues of $S_D(1_{[-1/2, 1/2]})$. The details in choosing the parameter $L$ will be developed in (43) further. Note that Theorem 2 can be utilized to denoise the sample covariance matrix, as we shall present in Section 4.

2) **Hole-free difference coarrays and $\mathbb{D} = [\bar{\theta}_{\text{min}}, \bar{\theta}_{\text{max}}]$**: Let us consider another scenario where the known interval is not centered around the origin, such as $[\bar{\theta}_{\text{min}}, \bar{\theta}_{\text{max}}]$ with $-0.5 < \bar{\theta}_{\text{min}} < \bar{\theta}_{\text{max}} < 0.5$. In this case, the density function can be

$$\rho(\bar{\theta}) = 1_{[\bar{\theta}_{\text{min}}, \bar{\theta}_{\text{max}}]}(\bar{\theta}) = \begin{cases} 1, & \text{if } \bar{\theta}_{\text{min}} \leq \bar{\theta} \leq \bar{\theta}_{\text{max}}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, due to Lemma 2, the entry of $S_D(1_{[\bar{\theta}_{\text{min}}, \bar{\theta}_{\text{max}}]})$ associated with the coarray location $\{m_1, m_2\} \in \mathbb{D}$ becomes

$$S_D(1_{[\bar{\theta}_{\text{min}}, \bar{\theta}_{\text{max}}]})_{m_1, m_2} = e^{j2\pi \bar{\theta}_{\text{min}} m_1} \cdot \alpha \cdot \text{sinc}(\alpha (m_1 - m_2)) \cdot e^{-j2\pi \bar{\theta}_{\text{max}} m_2}, \hspace{1cm} (38)$$
where \( \tilde{\theta}_{\text{avg}} = (\theta_{\text{max}} + \theta_{\text{min}})/2 \) and \( \alpha = \theta_{\text{max}} - \tilde{\theta}_{\text{avg}} \) are known. Using similar arguments in Section IV-B1, the matrix \( \mathbf{S}_D(\tilde{\theta}_{\text{min}}, \tilde{\theta}_{\text{max}}) \) owns

\[
\text{Eigenvectors of } \mathbf{S}_D(\tilde{\theta}_{\text{min}}, \tilde{\theta}_{\text{max}}) = \text{diag}(e^{j2\pi \tilde{\theta}_{\text{avg}} m})_{m \in \mathbb{D}} \times \mathbf{v}_k,
\]

where \( \lambda_k \) are defined in (33) and \( \mathbf{v}_k \) are the DPSS. These eigenvectors can be regarded as the modulated version of DPS. Hence, following similar arguments in Example 3, the generalized correlation subspace can be approximated by

\[
\mathcal{GCS}(\tilde{\theta}_{\text{min}}, \tilde{\theta}_{\text{max}}) \approx \text{col} \left( \mathbf{J} \cdot \text{diag}(e^{j2\pi \tilde{\theta}_{\text{avg}} m})_{m \in \mathbb{D}} \cdot \mathbf{V}_L \right),
\]

where the matrix \( \mathbf{J} \) is defined in Definition 5. The matrix \( \mathbf{V}_L = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_L \} \) includes the first \( L \) DPSS, where \( L \) will be chosen as in (44).

3) Hole-free difference coarrays and unions of multiple source intervals: In this case, the interval information \( \mathbb{I} \) is given by \( \cup_{p=1}^P \{ \tilde{\theta}_{\text{min},p}, \tilde{\theta}_{\text{max},p} \} \), where \( \tilde{\theta}_{\text{min},p}, \tilde{\theta}_{\text{max},p} \) for \( p = 1, 2, \ldots, P \) are non-overlapping intervals. The density function is assumed to be \( 1_{\mathbb{I}}(\hat{\theta}) \). Using (38) across all intervals yields

\[
\langle \mathbf{S}_D(1) \rangle_{m_1,m_2} = \sum_{p=1}^P \mathbf{A}_p e^{j2\pi \tilde{\theta}_{\text{avg},p} (m_1 - m_2)} \text{sinc}(\alpha_p (m_1 - m_2)),
\]

where \( \tilde{\theta}_{\text{avg}, p} = (\theta_{\text{max}, p} + \theta_{\text{min}, p})/2 \) is the centroid of the \( p \)th interval and \( \alpha_p = \theta_{\text{max}, p} - \theta_{\text{min}, p} \) is the width of the \( p \)th interval. Here the indices \( m_1, m_2 \in \mathbb{D} \). Although the entries of \( \mathbf{S}_D(1) \) are sums of modulated sinc functions, the eigenvectors of \( \mathbf{S}_D(1) \) cannot be expressed in terms of DS in general. In this case, the generalized correlation subspace \( \mathcal{GCS}(1) \) has to be evaluated numerically using Table 1.

4) Difference coarrays with holes: The results in Section IV-A hold true for 1D arrays, regardless of the difference coarrays. However, it is assumed in Section IV-B1 to IV-B3 that the difference coarrays are hole-free. These arrays with hole-free difference coarrays include ULA [6], minimum redundancy arrays [17], nested arrays [18], and super nested arrays [21]. For such arrays, Eqs. (37) and (41) are applicable.

However, for arrays containing holes in the difference coarrays, such as minimum hole arrays [54], coprime arrays [19], and some generalizations of coprime arrays [20], Eqs. (37) and (41) are not applicable in general. It is because the hole-free property of the difference coarray is used to derive (35) and (40). In this case, \( \mathbf{S}_D(1) \) can still be computed from (32), (38), and (42) accordingly. Then, the generalized correlation subspaces need to be calculated numerically using Table 1.

The choice of the parameter \( L \): Here we will present the details on the parameter \( L \) in Section IV-B. Let \( \mathbf{\Psi}_L \Lambda L \mathbf{\Psi}_L^H \) be the rank-\( \ell \) approximation of \( \mathbf{S}_D(1) \), where the notations are consistent with those in Table 1. Let the matrix \( \Lambda_L \) be \( \text{diag}(\lambda_1, \ldots, \lambda_L) \). Then the relative error \( E(\ell) \) is

\[
E(\ell) \triangleq \frac{\| \mathbf{\Psi}_L \Lambda_L \mathbf{\Psi}_L^H - \mathbf{S}_D(1) \|^2_F}{\| \mathbf{S}_D(1) \|^2_F} = \frac{\sum_{k=L+1}^{[P]} \lambda_k^2}{\sum_{k=1}^{[P]} \lambda_k^2}.
\]

For a given error tolerance \( 0 \leq \delta \ll 1 \), \( L \) is the minimum \( \ell \) such that \( E(\ell) < \delta \), i.e.,

\[
L = \min_{\ell \in \mathbb{Z}, 1 \leq \ell \leq [P]} \ell \text{ subject to } E(\ell) < \delta. \quad (44)
\]

Then this \( L \) is used in computing the generalized correlation subspace.

In particular, the smaller \( \delta \) is, the larger \( L \) is. For instance, Fig. 4 plots the relative error \( E(\ell) \), where the eigenvalues are given in Fig. 3(a). It can be seen that, if the error tolerance \( \delta = 10^{-5} \), then the parameter \( L = 8 \). If \( \delta = 10^{-10} \), then we have \( L = 10 \).

V. Connections with Existing Methods

In this section, we will discuss a covariance matrix denoising framework associated with the (generalized) correlation subspace. This method, denoted by problem (P2), can be regarded as a modified version of the optimization problem (P1). This problem (P2) can be solved by simple, closed-form, and computationally tractable expressions, unlike the problem (P1). We also relate it to redundancy averaging, which is a well-known processing technique in coarray-based DOA estimators, as shown in Example 4. This idea can be extended to the case of known source intervals, as in Example 5.

To develop some feelings for this method, Fig. 5 demonstrates the main idea of (P2), where the sample covariance matrix \( \mathbf{R}_s \) is defined in (10). For a given element \( \text{vec}(\mathbf{R}_s) \), we can calculate its orthogonal projection \( \mathbf{p}_1^* \) onto the subspace \( \mathcal{C} \mathcal{S} + \mathcal{I} \), where \( \mathcal{I} = \text{span}(\text{vec}(\mathbf{I})) \) and the sum of subspaces \( \mathcal{A} + \mathcal{B} \) is defined as \( \mathcal{A} + \mathcal{B} = \{ a + b : a \in \mathcal{A}, b \in \mathcal{B} \} \).

Moreover, if the source interval is known to be \( [-\alpha/2, \alpha/2] \), then the projection can be refined as \( \mathbf{p}_2^* \). These projections \( \mathbf{p}_1^* \) and \( \mathbf{p}_2^* \) can be matricized as indefinite Hermitian matrices, to which some existing DOA estimators can be applied.
The rationale for (P2) is based on the following chain of arguments. According to (6), we have vec($R_{\beta} - p_n I$) $\in CS$. This result implies that vec($R_{\beta}$) can be decomposed as $p_n vec(I)$ plus a vector in CS. Namely, vec($R_{\beta}$) $\in CS + I$, where $I = span(vec(I))$. Hence, in the finite snapshot scenario, we can find the vector $p^*$ in CS + I that minimizes the Euclidean distance to vec($R_{\beta}$). More generally, if the prior knowledge about sources is available, embedded in the generalized correlation subspace $GCS(\rho)$, then we have vec($R_{\beta}$) $\in GCS(\rho) + I$. This idea leads to the following convex program:

(P2): $p^* = \text{arg min}_p \|vec(R_{\beta}) - p\|^2_2$ subject to (45)  

$$(I - Q_{GCS(\rho) + I}Q_{GCS(\rho) + I})p = 0,$$  

(46)

where the columns of $Q_A$ are the bases for the subspace $A$. The solution to (P2) is given by

$$p^* = Q_{GCS(\rho) + I}Q_{GCS(\rho) + I}vec(R_{\beta}).$$  

(47)

Note that (47) can be evaluated directly, given the sample covariance matrix $\tilde{R}_{\beta}$ and the generalized correlation subspace $GCS(\rho)$. The computational complexity of (47) is much less than solving (P2) numerically. It can be observed that (47) shares similar expressions with the solution to (P1a). The main difference is that, estimating the noise power $p_n$ is required in (P1a), but not in (47).

Next, we will demonstrate some instances of (47) using the generalized correlation subspaces in Section IV.

Example 4: First let us consider the correlation subspace. Recall that $CS = GCS(1-1/2/1/2) = col(J)$, as in Theorem 1. The subspace $CS + I$ becomes $CS + I = col(J) + span(vec(I)) = col([J, Je_0]) = col(J)$. Here we use the fact that vec($I$) = $Je_0$ [44]. The column vector $e_0$ $\in \{0, 1\}^|Z|$ satisfies $e_0_m = \delta_{m, 0}$ for $m \in \mathbb{D}$. Next, according to (47), the orthogonal projection $p^*_L$ as shown in Fig. 5, can be written as

$$p^*_L = J\tilde{x}_D \in \mathbb{C}^{2^{|Z|}}.$$  

(48)

Due to (48) and Lemma 3, the sample value of $\tilde{x}_D$ at the coarray location $m \in \mathbb{D}$ is given by

$$\tilde{x}_D = \left(\frac{1}{w(m)}\sum_{n_1-n_2=m} \tilde{R}_{\beta}\right)_{n_1, n_2},$$  

(49)

where $n_1, n_2 \in S$. Eq. (49) was previously known as redundancy averaging [22], [24], [25] and [27, Definition 3]. The vector $\tilde{x}_D$ is known to be the sample autocorrelation vector on the difference coarray, which was used extensively in DOA estimators such as the augmented covariance matrix [22] positive definite Toeplitz completion [24], [25], and SS MUSIC [18], [27]. This example shows that, redundancy averaging is closely related to (47), which uses the concept of the correlation subspace.

Example 5: Eq. (47) can be used to derive a large class of DOA estimators with prior knowledge about source directions. According to Section IV-B, if the difference coarray is hole-free and the known source interval is $[-\alpha/2, \alpha/2]$, then the generalized correlation subspace $GCS(\rho)$ can be approximated by $col(JV_L)$, as in Theorem 2. Hence the subspace $GCS(\rho) + I$ is approximated by

$$col(JV_L) + span(vec(I)) = col([JV_L, Je_0]) = col(JU_L),$$  

where $U_L \triangleq [V_L, e_0]$. Then the associated orthogonal projection $p^*_L$, as illustrated in Fig. 5, is given by

$$p^*_L = (JU_L)(JU_L)^Tvec(R_{\beta}).$$  

(50)

Eq. (50) shows that the orthogonal projection can be evaluated using the matrix $J$ and the DPSS $V_L$. If the matrix $JU_L$ has full column rank, then Eq. (50) can rewritten as:

$$p^*_L = J\tilde{y}_D,$$  

(51)

where

$$\tilde{y}_D = U_L([WU_L]^H U_L)^{-1}(WU_L)^H x_D,$$  

(52)

and the matrix $W$ is defined in Lemma 3. Here $x_D$ is given by (49). Note that (51) shares the same formulation as the first equation of (48). Furthermore, according to (52), the vector $\tilde{y}_D$ can be regarded as the denoised version of $x_D$, as characterized by the oblique projection operator $U_L([WU_L]^H U_L)^{-1}(WU_L)^H$. Eq. (52) can also be interpreted as redundancy averaging with prior knowledge about sources. Most importantly, coarray-based DOA estimators such as the augmented covariance matrix, positive definite Toeplitz completion, and SS MUSIC can work on the denoised sample autocorrelation vector $\tilde{y}_D$, without any modification.

Covariance matrix denoising using the (generalized) correlation subspace is not limited to the optimization problems presented in this paper. The constraints imposed on the covariance matrices by the (generalized) correlation subspace can be readily applied to state-of-the-art covariance matrix denoising methods and DOA estimators such as SPICE [11–13], gridless SPICE [55], and atomic norm denoising [34–36, 56]. This idea could lead to new DOA estimators that enjoy improved performance, which could even approach the Cramér-Rao bounds for sparse arrays [44, 57, 58].

VI. GENERALIZED CORRELATION SUBSPACES IN MULTIPLE DIMENSIONS

The results developed in [16] were not restricted to one dimension. Even though our discussions in this paper were so far restricted to 1D arrays and 1D DOAs, they can also be readily generalized to multiple dimensions as we shall elaborate next. This is useful in many practical scenarios such as angle-Doppler estimation [38, 39], angle-delay estimation [40], angle-range estimation [41], 2D DOA (azimuth and elevation) estimation [6], and harmonic retrieval [42, 43].

IV. GENERALIZED CORRELATION SUBSPACES IN MULTIPLE DIMENSIONS

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Let us consider the data model for the $R$-dimensional $(R-D)$ case. The $R$-D sampling locations $n = (n^{(1)}, n^{(2)}, \ldots, n^{(R)})$ are collected by the set $S \subset \mathbb{Z}^R$ and the harmonic parameters
to be estimated are denoted by \( \bar{\mu} = (\bar{\mu}^{(1)}, \bar{\mu}^{(2)}, \ldots, \bar{\mu}^{(R)}) \in [-1/2, 1/2]^R \). Then we have the following data model:
\[
x_S = \sum_{i=1}^{D} A_i v_S(\bar{\mu}_i) + n_S,  \tag{53}
\]
where the vectors \( x_S, v_S(\bar{\mu}_i), \) and \( n_S \) represent the measurement, the steering vector; and the noise term, respectively. The entry of the steering vector associated with the sample locations \( n \in S \) is given by \( e^{j2\pi n / \lambda} \). The source amplitudes \( A_i \) and the noise term \( n_S \) are assumed to be uncorrelated, as in Section II. Then we can define the R-D difference coarray and the matrix \( J \) as follows:

**Definition 7 (R-D difference coarray):** Let \( S \) be a set of R-D sampling locations. The difference coarray is defined as \( \mathbb{D} = \{ n_1 - n_2 : \forall n_1, n_2 \in S \} \).

**Definition 8 (The matrix \( J \) for R-D):** Let \( S \) be a set of R-D sampling locations and \( \mathbb{D} \) be the R-D difference coarray. The matrix \( J \) is an \( |S|^2 \)-by-\(|\mathbb{D}| \) matrix satisfying \( \langle J \rangle_{m} = \text{vec}(I(m)) \). The matrix \( I(m) \) is given by
\[
\langle I(m) \rangle_{n_1, n_2} = \begin{cases} 1, & \text{if } n_1 - n_2 = m, \\ 0, & \text{otherwise}, \end{cases}
\]
where \( m \in \mathbb{D} \) and \( n_1, n_2 \in S \).

Using Definitions 7 and 8, the correlation vector can be expressed as \( c(\bar{\mu}) \triangleq \text{vec}(v_S(\bar{\mu})v_S^H(\bar{\mu})) = J v_S(\bar{\mu}) \), where \( v_S(\bar{\mu}) \) is the steering vector on the difference coarray. A numerical example is demonstrated in the supplementary document [46, Section II] for clarity. Then, the generalized correlation subspace matrix becomes
\[
S(\rho) \triangleq \int_{[-1/2,1/2]^R} c(\bar{\mu}) e^H(\bar{\mu}) \rho(\bar{\mu}) d\bar{\mu},
\]
\[
= J \left( \int_{[-1/2,1/2]^n} v_S(\bar{\mu}) v_S^H(\bar{\mu}) \rho(\bar{\mu}) d\bar{\mu} \right) J^H.  \tag{55}
\]

Eq. (54) and (55) are analogous to Definition 2 and Lemma 2, respectively. Finally, the generalized correlation subspace \( GCS(\rho) \triangleq \text{col}(S(\rho)) \), as in Definition 3. According to (55), the generalized correlation subspace for R-D depends on the difference coarray, similar to Lemma 2.

Next we will consider some concrete examples in 2D. They are joint angle-Doppler estimation and 2D DOA estimation.

**Example 6 (Joint angle-Doppler estimation):** The joint angle-Doppler estimation [38, 39] corresponds to the data model (53) with \( R = 2 \). It aims to estimate the angle and Doppler frequency from the spatiotemporal measurements \( x_S \). The harmonic parameters \( \bar{\mu} = (\bar{\mu}^{(1)}, \bar{\mu}^{(2)}) \) are related to the DOA and the Doppler frequency as follows:
\[
\bar{\mu}^{(1)} = (d/\lambda) \sin \theta, \quad \bar{\mu}^{(2)} = (T/\lambda)v,
\]
where \( \lambda \) is the wavelength, \( d = \lambda/2 \) is the minimum sensor separation, \( \theta \in [-\pi/2, \pi/2] \) is the DOA, \( T \) is the sampling interval in the temporal domain, and \( v \) is the radial velocity of the source. \( \bar{\mu}^{(1)} \) and \( \bar{\mu}^{(2)} \) are called the normalized angle and normalized Doppler, respectively. If the sampling interval \( T \) is chosen properly, the parameters \( (\bar{\mu}^{(1)}, \bar{\mu}^{(2)}) \) belongs to \([-1/2, 1/2]^2 \). This region is also depicted in Fig. 6(a) for clarity.

Next, suppose we choose the density function \( \rho(\bar{\mu}) \) to be \( 1_{[-1/2,1/2]^2}(\bar{\mu}) \). The generalized correlation subspace matrix becomes
\[
S(1_{[-1/2,1/2]^2}) = JJ^H = (JW^{-1/2}) W(JW^{-1/2})^H,
\]
where \( W = \text{diag}(w(m))_{m \in \mathbb{D}} \). Here \( w(m) = |\{(n_1, n_2) : n_1 - n_2 = m\}| \) denotes 2D weight function. This result is analogous to that in Section IV-A. It can be inferred that the positive eigenvalues of \( S(1_{[-1/2,1/2]^2}) \) are \( w(m) \) and the associated eigenvectors are \( \text{vec}(I(m))/\sqrt{w(m)} \). Note that the results in Example 6 can be trivially extended to the R-D case if the parameters \( \bar{\mu} \in [-1/2, 1/2]^R \).

**Example 7 (2D DOA estimation):** Another example of (53) with \( R = 2 \) is the 2D DOA estimation. The parameters of interest are the azimuth \( \phi \in [0, 2\pi] \) and the elevation \( \theta \in [0, \pi/2] \). The relation between the 2D DOA \((\theta, \phi)\) and the harmonic parameters \( \bar{\mu} = (\bar{\mu}^{(1)}, \bar{\mu}^{(2)}) \) is given by
\[
\bar{\mu}^{(1)} = (d_x/\lambda) \sin \theta \cos \phi, \quad \bar{\mu}^{(2)} = (d_y/\lambda) \sin \theta \sin \phi, \tag{57}
\]
where \( d_x = d_y = \lambda/2 \). According to (57), the visible region of \( \bar{\mu} \) becomes a disc with radius 1/2, i.e.,
\[
\mathbb{K} = \{ \bar{\mu} : \|\bar{\mu}\|_2 \leq 1/2 \}.  \tag{58}
\]

The set \( \mathbb{K} \) is also depicted in Fig. 6(b). Using (55), the generalized correlation subspace matrix with density function \( \rho(\bar{\mu}) = 1_{\mathbb{K}}(\bar{\mu}) \) becomes \( S(1_{\mathbb{K}}) = JS(1_{\mathbb{K}})J^H \). Here the entry of \( S_{\mathbb{K}}(1_{\mathbb{K}}) \) associated with coarray location \( m_1, m_2 \in \mathbb{D} \) can be derived as [46, Section III]
\[
\langle S_{\mathbb{K}}(1_{\mathbb{K}}) \rangle_{m_1, m_2} = \text{jinc}(\|m_1 - m_2\|_2), \tag{59}
\]
where the jinc function \( \text{jinc}(x) = \pi/4 \) for \( x = 0 \) and \( J_0(x)/(2x) \) otherwise [59]. Here \( J_0(x) \) is the first-order Bessel function of the first kind [59]. Eq. (59) can be regarded as an extension of (32) since the jinc function is analogous...
to the sinc function in the 2D polar coordinate [59]. The eigenvectors of $S_{2}(1_{\mathcal{X}})$ are 2D discrete prolate spheroidal sequences (2D DPSS). Details about these sequences can be found in [60] and the references therein. Finally the generalized correlation subspace can be approximated by \text{col}\left(J\Psi_{L}\right)$, where $\Psi_{L}$ contains the first $L$ 2D DPSS and $L$ is defined in (44).

If the prior knowledge about 2D DOA is available, then the visible region could be an annulus or a circular sector. For instance, if we know a priori that the elevation $\theta_{\min} \leq \theta \leq \theta_{\max}$, then the visible region is depicted in Fig. 6(c). On the other hand, if the the prior knowledge is $\phi_{\min} \leq \phi \leq \phi_{\max}$, then the visible region becomes a circular sector, as illustrated in Fig. 6(d).

VII. NUMERICAL EXAMPLES

A. Generalized Correlation Subspaces

In this section, we will consider the following four 1D array configurations: the ULA with 10 sensors, the nested array with $N_{1} = N_{2} = 5$, the coprime array with $M = 3, N = 5$, and the super nested array with $N_{1} = N_{2} = 5, Q = 2$, where the notations are in accordance with [18], [21], [26]. The number of sensors is 10 for each array. The sensor locations and the notations are in accordance with [18], [21], [26]. The number of SNR and the number of snapshots. The equal-power and uncorrelated sources have normalized DOAs $-0.045, 0$, and $0.045$. The number of sources $D$ is 3. The array configuration is the nested array with $N_{1} = N_{2} = 5$ (10 sensors), as depicted in Fig. 2(b). Then the generalized correlation subspace $\hat{GCS}(1[-\alpha/2,\alpha/2])$ can be evaluated according to Section IV. The error tolerance in (44) is $\delta = 10^{-10}$ so the parameter $L$ becomes 10 and 17 for $\alpha = 0.1$ and 0.2, respectively. In each run, the sensor measurements $\mathcal{X}_{k}(k)$ for $k = 1, \ldots, K$ are realized by (1), from which the sample covariance matrix $\mathbf{R}_{s}$ can be determined by (10). Then the covariance matrices are denoised according to 1) Problem (P1) and the cvx package with perfect knowledge about the noise power; 2) Problem (P2) and (47) without knowing the noise power. Finally the source directions are estimated by the SS MUSIC algorithm [18], [27] on the denoised covariance matrices. The estimation performance is measured in terms of root mean-squared errors (RMSE), defined as

$$\text{RMSE} = \sqrt{\frac{1}{D} \sum_{i=1}^{D} (\hat{\theta}_{i} - \tilde{\theta}_{i})^2},$$

where $\tilde{\theta}_{i}$ denotes the estimated normalized DOAs. Each sample point is averaged from 1000 Monte-Carlo runs.

It can be seen from Fig. 8 that if the source interval $[-\alpha /2, \alpha /2]$ is known, the RMSEs decrease except at very low SNRs. In this example, we choose $\alpha = 1, 0.2$, and 0.1 in Fig. 8. The following discuss the performances of these estimators:

1) In most cases, the RMSE for (P1) and (P2) decrease with $\alpha$, for the same SNR and the same number of snapshots. The improvement is significant for low SNR and limited snapshots. This is because smaller source intervals $[-\alpha /2, \alpha /2]$ help to improve the estimation performances.

2) (P1) requires much more time than (P2), provided that the generalized correlation subspace is computed using the results in Section IV. For instance, the computational time for (P1) and $\alpha = 0.1$ in Fig. 8(a) is 9904.5 seconds while that for (P2) and $\alpha = 0.1$ in Fig. 8(a) is 53.7 seconds. The reason is that (P1) is solved numerically using the cvx package but (P2) can be implemented efficiently using (47).

3) As shown in Fig. 8(a), empirically, (P2) has better estimation performance than (P1) for the same $\alpha$ in most cases. For instance, if $\alpha = 0.1$ and SNR $= -10$dB, then the RMSE for (P1) is $4.9 \times 10^{-3}$ while that for (P2) becomes $1.9 \times 10^{-3}$. If $\alpha = 0.1$ and SNR $= 10$dB, then the RMSEs for (P1) and (P2) are about $8.8 \times 10^{-4}$ and $6.9 \times 10^{-4}$, respectively. These phenomena can also be observed in Fig. 8(b).

C. Computational Complexity

Table II investigates the interplay between the computational complexity and the estimation performance by varying the parameter $\alpha$ and the implementation details. These cases are denoted by Cases A to H, respectively. Here Step 1, which computes the (generalized) correlation subspace, can
be realized in two ways. One is the numerical integration (60), followed by numerical eigen-decompositions [16]. The number of grids \( N_{\text{pt}} \) in (60) is \( 2^{14} + 1 \) while the density function \( \rho(\hat{\theta}) \) is \( 2(1 - (2\hat{\theta})^2)^{-1/2}1_{[-1/2,1/2]}(\hat{\theta}) \). The other is to use the proposed analytical expressions, based on Definition 5, Theorem 1, and Table I. Then, in Step 2, either (P1) or (P2) is solved. Finally, Step 3 uses SS MUSIC to estimate the source directions. We assume 100 snapshots and 5dB SNR.

The CPU time and the RMSEs in Table II are averaged from 1000 Monte-Carlo runs, on a Ubuntu 16.04 workstation with an Intel Core i7-2600 3.40GHz processor and 8GB RAM. The remaining parameters are given in Section VII-B.

Some observations can be drawn from Table II. First, the proposed analytical expressions lead to almost the same RMSE but much less CPU time than those with numerical approximations. For instance, the CPU time for Step 1 in Case D is about 0.04% of that for Step 1 in Case B. Second, in these examples, (P2) enjoys smaller RMSE and much less computational time than (P1). As an example, the CPU time of Step 2 in Case B costs only 0.007% of that of Step 2 in Case A, while the RMSE is 0.000991 for Case A and 0.000918 for Case B, respectively. Finally, if \( \alpha = 0.1 \), the proposed analytical approach remains computationally tractable. It can be observed that the CPU time for Step 1 in Case H is approximately 0.2% of that for Step 1 in Case F. These observations show that, for a given \( \alpha \), it is preferable to use the proposed analytical expressions and problem (P2), since they lead to the least total time and the smallest RMSE.

VIII. CONCLUDING REMARKS

In this paper, we presented generalized correlation subspaces with applications to DOA estimation for uncorrelated sources and known source intervals. Generalized correlation subspaces have closed-form characterizations in terms of array profiles such as sensor locations and difference coarrays. Furthermore, our theory provides insights to existing DOA estimators and multidimensional sparse arrays with prior knowledge about sources.

In the future, it is of considerable interest to exploit generalized correlation subspaces in other topics of array
processing, like adaptive beamforming, source detection, and target tracking. Another future topic would be the performance analysis for the proposed approach using generalized correlation subspaces.

**APPENDIX A**

**PROOF OF THE EQUIVALENCE OF (6) AND DEFINITION 1**

To begin with, we need the following lemmas:

**Lemma 4**: Let $M \in \mathbb{C}^{N \times N}$ be a Hermitian, positive semidefinite matrix. Assume $u \in \mathbb{C}^{N}$. Then $u^H M u = 0$ if and only if $u$ belongs to the null space of $M$.

*Proof*: If $u$ belongs to the null space of $M$, then $M u = 0$ and clearly $u^H M u = 0$. Conversely, if $u^H M u = 0$, then $(M^{-1/2} u)^H (M^{-1/2} u) = 0$, where the matrix square root exists due to the positive semidefiniteness of $M$. Hence, $M^{-1/2} u = 0$, so $M u = 0$.

**Lemma 5**: Let $f(x)$ be a real-valued Lebesgue integrable function defined over a measurable set $\mathbb{A}$. Assume that $f(x) \geq 0$ for $x \in \mathbb{A}$ almost everywhere (a.e.). Then $\int_{\mathbb{A}} f(x) dx = 0$ if and only if $f(x) = 0$ a.e.

*Proof*: See [61, Chapter V] for details.

Next it will be first shown that $CS^\perp = \text{null}(S^H)$, where $CS^\perp$ denotes the orthogonal complement of $CS$ and $\text{null}(A)$ represents the null space of $A$.

First consider any $v \in \text{null}(S^H)$. Since $S$ is a Hermitian positive semidefinite matrix, Lemma 4 implies that $v \in \text{null}(S^H)$ is equivalent to $v^H S^H v = 0$. Substituting the definition of $S$, (8), into $v^H S^H v = 0$ yields

$$\int_{-\pi/2}^{\pi/2} |v^H c(\bar{\theta})|^2 \ d\bar{\theta} = 0.\quad (61)$$

Since the nonnegative function $|v^H c(\bar{\theta})|^2$ is continuous in $\theta$, Lemma 5 indicates that (61) is equivalent to $|v^H c(\bar{\theta})|^2 = 0$ for all $\bar{\theta} \in [-\pi/2, \pi/2]$. Namely, $v \in CS^\perp$. Note that all the above arguments are both necessary and sufficient so we have $CS^\perp = \text{null}(S^H)$.

Finally, the correlation subspace can be obtained by $CS = CS^\perp = \text{null}(S^H)^\perp = \text{col}(S)$, which proves this observation.

**APPENDIX B**

**PROOF OF LEMMA 1**

Let $\rho_1(\bar{\theta})$ and $\rho_2(\bar{\theta})$ be two nonnegative Lebesgue integrable functions with the same support. The support of $\rho_1(\bar{\theta})$ and $\rho_2(\bar{\theta})$ satisfies $\text{supp}(\rho_1) = \text{supp}(\rho_2) = \mathbb{A} \subseteq [-1/2, 1/2]$. The corresponding generalized correlation subspace matrices are denoted by $S(\rho_1)$ and $S(\rho_2)$, respectively.

Consider a nonzero and finite-valued vector $u \in \mathbb{C}^{[\mathbb{A}]^2}$. It will be shown that $u^H S(\rho_1) u = 0$ if and only if $u^H S(\rho_2) u = 0$. Based on Definition 2, the condition $u^H S(\rho_1) u = 0$ is equivalent to the following:

$$u^H \left( \int_{\mathbb{A}} c(\bar{\theta}) c^H(\bar{\theta}) \rho_1(\bar{\theta}) \ d\bar{\theta} \right) u = 0$$

if and only if $\int_{\mathbb{A}} |c(\bar{\theta}) u|^2 \rho_1(\bar{\theta}) d\bar{\theta} = 0.\quad (62)$

Due to Lemma 5, Eq. (62) is equivalent to the statement that $|c(\bar{\theta}) u|^2 \rho_1(\bar{\theta}) = 0$ for $\bar{\theta} \in \mathbb{A}$ a.e. Since $\bar{\theta}$ belongs
to the support $\mathcal{A}$, multiplying both sides with $\rho_2(\hat{\theta})/\rho_1(\hat{\theta})$ yields $|\mathbf{r}(\hat{\theta})\mathbf{u}|^2 \rho_2(\hat{\theta}) = 0$ for $\hat{\theta} \in \mathcal{A}$ a.e. Invoking Lemma 5 again gives $\mathbf{u}^H \mathbf{S}(\rho_2) \mathbf{u} = 0$. Combining the above arguments with Lemma 4 leads to $\text{null}(\mathbf{S}(\rho_1)) = \text{null}(\mathbf{S}(\rho_2))$, where $\text{null}(\mathbf{S})$ denotes the null space of $\mathbf{S}$. Taking the orthogonal complement on the null spaces gives $\text{col}(\mathbf{S}(\rho_1)) = \text{col}(\mathbf{S}(\rho_2))$, which proves this theorem.

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