The Generalized Fractional Fourier Transform

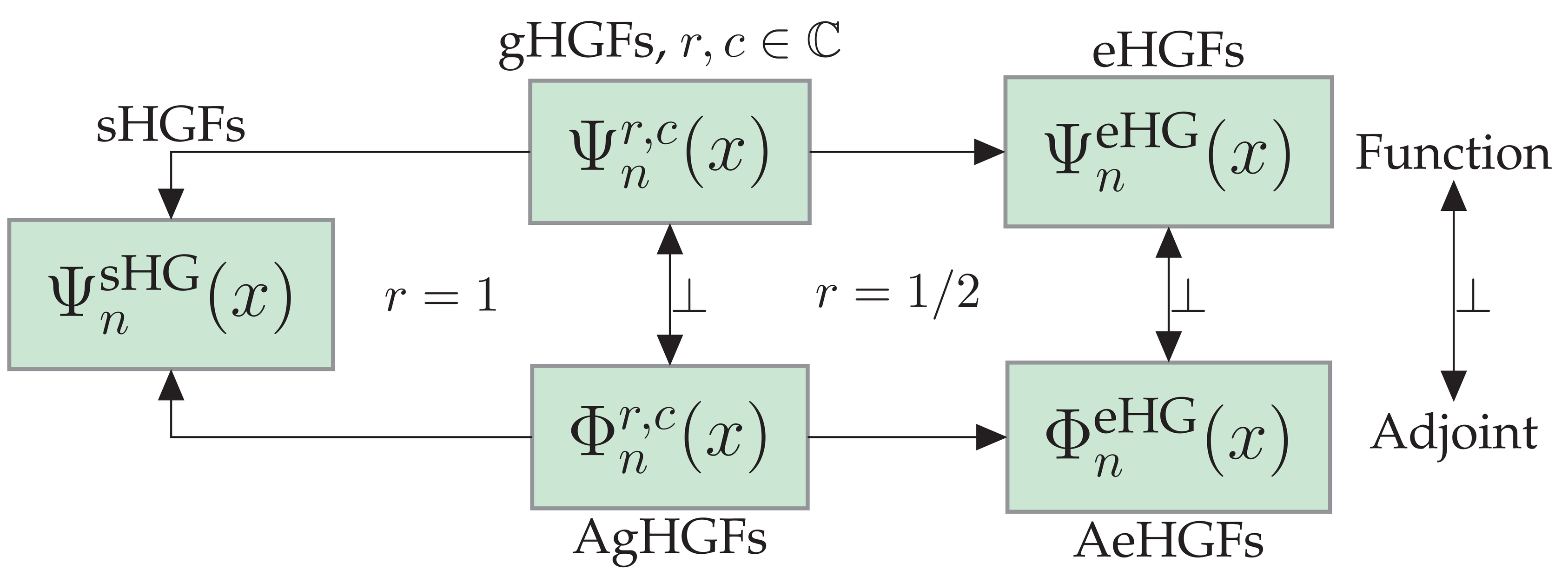
Soo-Chang Pei[1], Chun-Lin Liu[2], and Yun-Chiu Lai[3]
Department of Electrical Engineering[1], Graduate Institute of Communication Engineering[1,2,3],
National Taiwan University, Taipei, Taiwan, 10617
pei@cc.ee.ntu.edu.tw[1], r99942052@ntu.edu.tw[2], d96942020@ntu.edu.tw[3]

Abstract

A new transform, called the generalized fractional Fourier transform (gFrFT), is proposed. Originally, the eigenfunctions of the fractional Fourier transform (FrFT) are known as the Hermite Gaussian functions (HGFs). Besides, in optics, the HGFs are generalized to be the generalized Hermite Gaussian functions (gHGFs) and their adjoint functions (AgHGFs). Therefore, we can define the gFrFT by the eigenvalues of the FrFT and the eigenfunctions (gHGFs/AgHGFs) in the analysis or synthesis step. Four types of the gFrFT are defined and discussed. The integral forms of the gFrFTs are derived and they are closely related to some popular transforms, such as the Fourier transform (FT), the FrFT, and the complex linear canonical transform (CLCT). We can also extend the FT and the FrFT to the standard and elegant versions. Finally, some properties of the gFrFT are discussed.

Standard/Elegant/Generalized HGFs

- sHGFs $\Psi_n^{\text{sHG}}(x) = \left(\frac{\sqrt{2}}{w_0 2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n\left(\frac{\sqrt{2}x}{w_0}\right) e^{-\frac{x^2}{w_0^2}}$ are self-orthogonal, i.e. $\langle \Psi_m^{\text{sHG}}(x), \Psi_n^{\text{sHG}}(x)\rangle = \delta_{m,n}$ [1].
- eHGFs $\Psi_n^{\text{eHG}}(x) = \left(\frac{\sqrt{c}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\sqrt{c}x) e^{-cx^2}$ are biorthogonal with AeHGFs $\Phi_n^{\text{eHG}}(x) = \left(\frac{\sqrt{c^*}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\sqrt{c^*}x)$, the adjoint of eHGFs [4].
- gHGFs $\Psi_n^{r,c}(x) = \left(\frac{\sqrt{2rc}}{2^n n! \sqrt{\pi}}\right)^{\frac{1}{2}} H_n(\sqrt{2rc}x) e^{-cx^2}$ [3].
  AgHGFs $\Phi_n^{r,c}(x) = \left(\frac{\sqrt{2r^*c^*}}{2^n n! \sqrt{\pi}}\right)^{1/2} H_n(\sqrt{2r^*c^*}x) e^{(c^*-2r^*c^*)x^2}$.



Definitions

- By [2], the three-step definition for the gFrFT Type-I, denoted by $\mathfrak{F}_{\alpha,r,c}^{(\text{I})}$
  1. *Analysis*: $a_n^{r,c} = \int_{\mathbb{R}} f(x) \left(\Phi_n^{r,c}(x)\right)^* dx$.
  2. *Eigenvalue multiplication*: $b_n^{r,c} = e^{-jn\alpha} a_n^{r,c}$.
  3. *Synthesis*: $F_{\alpha,r,c}^{(\text{I})}(u) = \mathfrak{F}_{\alpha,r,c}^{(\text{I})}\{f\}(u) = \sum_{n=0}^{\infty} b_n^{r,c} \Psi_n^{r,c}(u)$.
- Four types of the gFrFT:

gFrFT Type	I	II	III	IV
Analysis kernels	$\Phi_n^{r,c}(x)$	$\Psi_n^{r,c}(x)$	$\Psi_n^{r,c}(x)$	$\Phi_n^{r,c}(x)$
Eigenvalues	$e^{-jn\alpha}$	$e^{-jn\alpha}$	$e^{-jn\alpha}$	$e^{-jn\alpha}$
Synthesis kernels	$\Psi_n^{r,c}(x)$	$\Phi_n^{r,c}(x)$	$\Psi_n^{r,c}(x)$	$\Phi_n^{r,c}(x)$

The Integral Form

- By definition and using the Mehler's formula, we have the integral form for the gFrFT Type-I: $F_{\alpha,r,c}^{(\text{I})}(u) =$

$$N_1 e^{c(-1+r(1+j\cot\alpha))u^2} \int_{\mathbb{R}} f(x) e^{c(1-r(1-j\cot\alpha))x^2} e^{-j2rcxu\csc\alpha} dx,$$

  where $N_1 = \sqrt{rc(1-j\cot\alpha)/\pi}$. Other types can be derived in the same way.
- If we substitute r and c with their conjugates and exchange the x/u-domain complex Gaussian functions, the type-I kernel becomes the type-II kernel.

The Connection to Other Transforms

- Fourier transform in angular frequency: $\alpha = \pi/2, r = 1, c = 1/2$.
- Fourier transform in frequency: $\alpha = \pi/2, r = 1, c = \pi$.
- We extend these transforms into standard/generalized/elegant cases:

	$r = 1$	r	$r = 1/2$
$c \in \mathbb{R}^+, \alpha = \pi/2$	sFT	gFT	eFT
$c \in \mathbb{R}^+, \alpha$	sFrFT	gFrFT	eFrFT

- The gFrFT Type-I is related to the complex linear canonical transform (CLCT) with

$$\mathbf{M}_{\alpha,r,c}^{(\text{I})} = \begin{bmatrix} \cos\alpha + j\frac{r-1}{r}\sin\alpha & \frac{1}{2rc}\sin\alpha \\ 2c(\frac{1}{r}-2)\sin\alpha & \cos\alpha - j\frac{r-1}{r}\sin\alpha \end{bmatrix}.$$

- We can decompose $\mathbf{M}_{\alpha,r,c}^{(\text{I})}$ into the combination of scaling, chirp multiplication, and the FrFT:

$$\begin{bmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -k_1 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ k_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1^{-1} \end{bmatrix},$$

  where $\sigma_1 = \sqrt{2rc}$, $k_1 = j(r-1)/r$.
- The decomposition of the gFrFT operators ($\sigma_2 = \sigma_1^*$ and $k_2 = k_1^*$)

$$\mathfrak{F}_{\alpha,r,c}^{(\text{I})} = \mathcal{S}_{\sigma_1}^{-1}\mathcal{C}_{k_1}^{-1}\mathcal{F}_\alpha\mathcal{C}_{k_1}\mathcal{S}_{\sigma_1}, \quad \mathfrak{F}_{\alpha,r,c}^{(\text{II})} = \mathcal{S}_{\sigma_2}^{-1}\mathcal{C}_{k_2}^{-1}\mathcal{F}_\alpha\mathcal{C}_{k_2}\mathcal{S}_{\sigma_2},$$
$$\mathfrak{F}_{\alpha,r,c}^{(\text{III})} = \mathcal{S}_{\sigma_1}^{-1}\mathcal{C}_{k_1}^{-1}\mathcal{F}_\alpha\mathcal{C}_{k_2}\mathcal{S}_{\sigma_2}, \quad \mathfrak{F}_{\alpha,r,c}^{(\text{IV})} = \mathcal{S}_{\sigma_2}^{-1}\mathcal{C}_{k_2}^{-1}\mathcal{F}_\alpha\mathcal{C}_{k_1}\mathcal{S}_{\sigma_1}.$$

- With $\mathbf{M}_{\alpha,r,c}^{(\text{I})}$ and the decomposition of gFrFT operators, it is trivial to obtain the mathematical properties.
- It is very likely to decompose the CLCT into the gFrFT.

References

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