Tensor MUSIC in Multidimensional Sparse Arrays

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Outline

1 Introduction
- Motivation
- Tensors

2 Contribution: Tensor MUSIC in Multidimensional Sparse Arrays
- Coarray tensor
- Tensor MUSIC spectrum

3 Numerical Examples

4 Concluding Remarks

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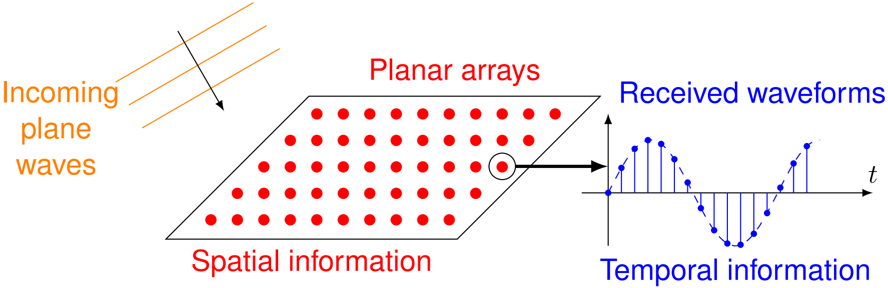
Harmonic Retrieval in Planar Array Processing[1]



Utimate Goal

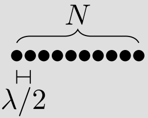
Estimate source profiles (azimuth, elevation, range, Doppler, etc.) from sensor measurements efficiently and accurately.

[1] Harry L. Van Trees. *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*. Wiley Interscience, 2002.

Sparse Array Processing[2,3]

Uniform Linear Arrays (ULAs)

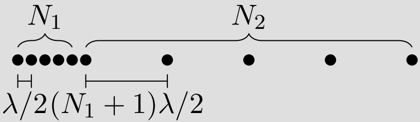
- ULA with N sensors and sensor separation $\lambda/2$.



- Identify at most $N-1$ sources using N sensors. ✗

Linear Sparse Arrays

- Nested array with N_1, N_2 and min. separation $\lambda/2$.

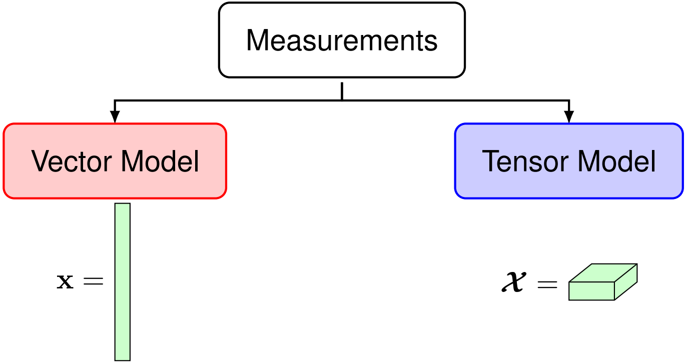


- Identify $O(N^2)$ uncorrelated sources using $O(N)$ sensors. ✓

[2] Alan T Moffet. "Minimum-redundancy linear arrays". In: *IEEE Trans. Antennas Propag.* 16.2 (1968), pp. 172–175.

[3] Piya Pal and P. P. Vaidyanathan. "Nested Arrays: A Novel Approach to Array Processing With Enhanced Degrees of Freedom". In: *IEEE Trans. Signal Process.* 58.8 (2010), pp. 4167–4181.

Tensor Model[4,5, etc.]



Spatial/temporal relations are mixed ✗

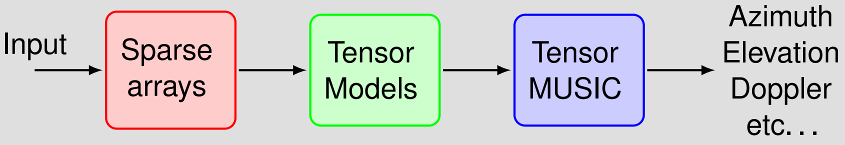
Spatial/temporal relations are separated ✓

[4] M. Haardt, F. Roemer, and G. Del Galdo. "Higher-Order SVD-Based Subspace Estimation to Improve the Parameter Estimation Accuracy in Multidimensional Harmonic Retrieval Problems". In: *IEEE Trans. Signal Process.* 56.7 (2008), pp. 3198–3213.

[5] D. Nion and N.D. Sidiropoulos. "Tensor Algebra and Multidimensional Harmonic Retrieval in Signal Processing for MIMO Radar". In: *IEEE Trans. Signal Process.* 58.11 (2010), pp. 5693–5705.

Main Goal of this Work

Proposed Scheme



Related work:

- ULA, tensors, and MUSIC $\Rightarrow$ DOA and polarization[6,7].
- Nested arrays, tensors, and MUSIC $\Rightarrow$ azimuth, elevation, and polarization[8].

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[6] Sebastian Miron, Nicolas Le Bihan, and Jerome I Mars. “Vector-Sensor MUSIC for Polarized Seismic Sources Localization”. In: *EURASIP Journal on Advances in Signal Processing* 2005.1 (2005), pp. 74–84.

[7] M. Boizard et al. “Numerical performance of a tensor MUSIC algorithm based on HOSVD for a mixture of polarized sources”. In: *Proc. European Signal Process. Conf.* 2013, pp. 1–5.

[8] Keyong Han and A. Nehorai. “Nested Vector-Sensor Array Processing via Tensor Modeling”. In: *IEEE Trans. Signal Process.* 62.10 (2014), pp. 2542–2553.

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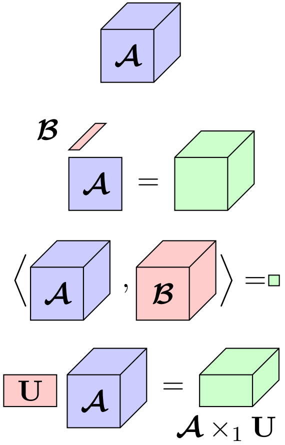
Notations[9]

Tensor $\mathcal{A}$

Outer product $\mathcal{A} \circ \mathcal{B}$

Inner product $\langle \mathcal{A}, \mathcal{B} \rangle$

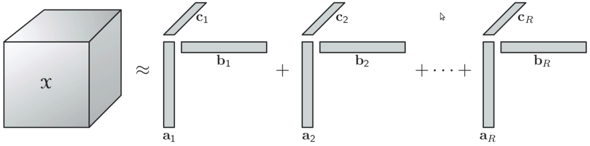
n-mode product $\mathcal{A} \times_n \mathbf{U}$



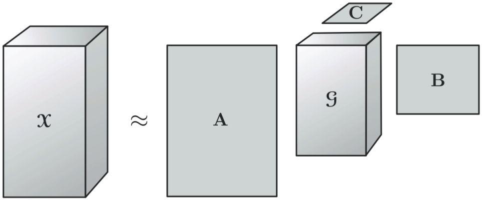
[9] Tamara G. Kolda and Brett W. Bader. "Tensor Decompositions and Applications". In: *SIAM Review* 51.3 (2009), pp. 455–500.

Tensor Decomposition[10]

- CANDECOMP/PARAFAC (CP) decomposition:
  $\boldsymbol{\mathcal{X}} \approx \sum_{r=1}^{R} \mathbf{a}_r \circ \mathbf{b}_r \circ \mathbf{c}_r$.



- High-order SVD (HOSVD): $\boldsymbol{\mathcal{X}} \approx \boldsymbol{\mathcal{G}} \times_1 \mathbf{A} \times_2 \mathbf{B} \times_3 \mathbf{C}$.



[10] Tamara G. Kolda and Brett W. Bader. "Tensor Decompositions and Applications". In: *SIAM Review* 51.3 (2009), pp. 455–500.

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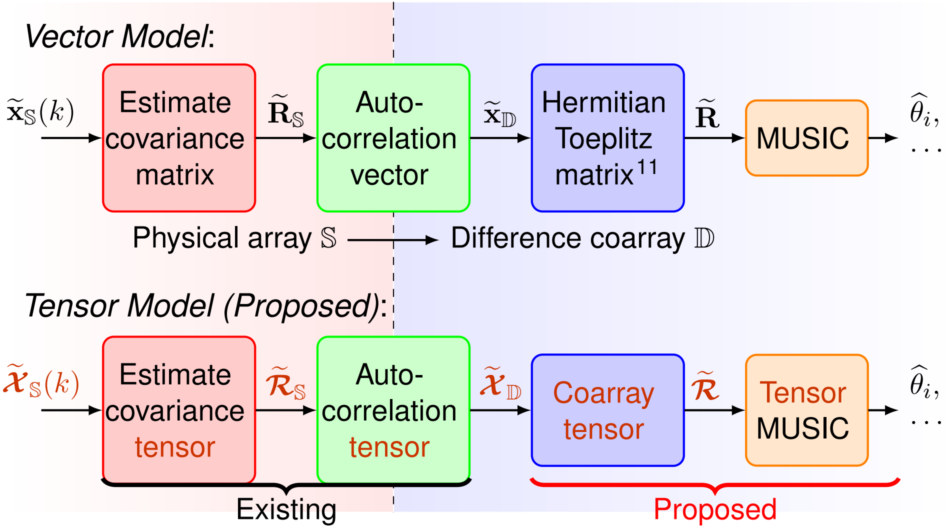
Sparse Array Processing



[11] S.U. Pillai, *et al.* "A new approach to array geometry for improved spatial spectrum estimation". Proc. IEEE 73.10 (1985); C.-L. Liu and P. P. Vaidyanathan. "Remarks on the Spatial Smoothing Step in Coarray MUSIC". IEEE SPL 22.9 (2015).

Some Discussions on the Coarray Tensor $\widetilde{\mathcal{R}}$

Vector model[12]

$$\langle \widetilde{\mathbf{R}} \rangle_{p_1,p_1'} = \langle \widetilde{\mathbf{x}}_{\mathbb{D}} \rangle_{m_1},$$
$$p_1 - p_1' = m_1.$$

Tensor model

$$\langle \widetilde{\mathcal{R}} \rangle_{p_1,p_2,\ldots,p_R,p_1',p_2',\ldots,p_R'}$$
$$= \langle \widetilde{\mathcal{X}}_{\mathbb{D}} \rangle_{m_1,m_2,\ldots,m_R},$$
$$p_r - p_r' = m_r,$$
$$r = 1, 2, \ldots, R.$$

- $\widetilde{\mathcal{R}}$ avoids implementing spatial smoothing in tensors.
- $\widetilde{\mathcal{R}}$ admits the (tensor) MUSIC algorithm.

[12] S.U. Pillai, *et al.* "A new approach to array geometry for improved spatial spectrum estimation". Proc. IEEE 73.10 (1985); C.-L. Liu and P. P. Vaidyanathan. "Remarks on the Spatial Smoothing Step in Coarray MUSIC". IEEE SPL 22.9 (2015).

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Tensor MUSIC

MUSIC

1. Eigen-decomposition: $\widetilde{\mathbf{R}} = \widetilde{\mathbf{U}}\widetilde{\mathbf{\Lambda}}\widetilde{\mathbf{U}}^H$.
2. Signal and noise subspace: $\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{\mathbf{U}}_s & \widetilde{\mathbf{U}}_n \end{bmatrix}$
3. MUSIC spectrum: $P(\bar{\theta}) = \dfrac{1}{\|\widetilde{\mathbf{U}}_n^H \mathbf{v}(\bar{\theta})\|^2}$

   $\mathbf{v}(\bar{\theta})$: steering vectors.

Tensor MUSIC[13]

1. HOSVD:
$$\widetilde{\mathcal{R}} = \widetilde{\mathcal{K}} \times_1 \widetilde{\mathbf{U}}_1 \times_2 \widetilde{\mathbf{U}}_2 \cdots \times_R \widetilde{\mathbf{U}}_R \times_{R+1} \widetilde{\mathbf{U}}_1^* \times_{R+2} \widetilde{\mathbf{U}}_2^* \cdots \times_{2R} \widetilde{\mathbf{U}}_R^*.$$
2. Signal and noise subspace: $\widetilde{\mathbf{U}}_r = \begin{bmatrix} \widetilde{\mathbf{U}}_{r,s} & \widetilde{\mathbf{U}}_{r,n} \end{bmatrix}$ is a unitary matrix.
3. Tensor MUSIC spectrum
$$P_{HOSVD}(\bar{\boldsymbol{\mu}}) = \frac{1}{\|\boldsymbol{\mathcal{V}}(\bar{\boldsymbol{\mu}}) \times_1 \widetilde{\mathbf{U}}_{1,n}\widetilde{\mathbf{U}}_{1,n}^H \ldots \times_R \widetilde{\mathbf{U}}_{R,n}\widetilde{\mathbf{U}}_{R,n}^H\|_F^2}$$
   $\boldsymbol{\mathcal{V}}(\bar{\boldsymbol{\mu}})$: steering tensors.

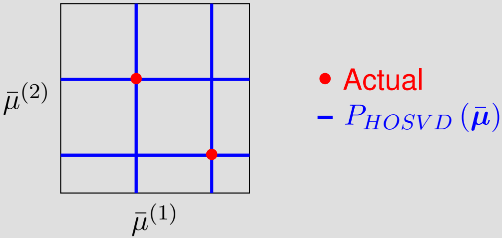
[13] M. Boizard et al. "Numerical performance of a tensor MUSIC algorithm based on HOSVD for a mixture of polarized sources". In: *Proc. European Signal Process. Conf.* 2013, pp. 1–5.

Problem with tensor MUSIC via HOSVD

Our observation: $P_{HOSVD}(\bar{\boldsymbol{\mu}})$ is a separable MUSIC spectrum

$$P_{HOSVD}(\bar{\boldsymbol{\mu}}) = \prod_{r=1}^{R} P_r(\bar{\mu}^{(r)}), \qquad P_r(\bar{\mu}^{(r)}) = \frac{1}{\|\widetilde{\mathbf{U}}_{r,n}^{H} \mathbf{v}_{\mathbb{U}_r^+}(\bar{\mu}^{(r)})\|_2^2}$$

$P_{HOSVD}(\bar{\boldsymbol{\mu}})$ has cross-terms



Proposed Tensor MUSIC spectrum via CP

CP

$$\widetilde{\boldsymbol{\mathcal{R}}} = \sum_{\ell=1}^{D} \widetilde{\mathbf{a}}_\ell^{(1)} \circ \widetilde{\mathbf{a}}_\ell^{(2)} \circ \cdots \circ \widetilde{\mathbf{a}}_\ell^{(R)} \circ \widetilde{\mathbf{a}}_\ell^{(1)*} \circ \widetilde{\mathbf{a}}_\ell^{(2)*} \circ \cdots \circ \widetilde{\mathbf{a}}_\ell^{(R)*}.$$

Signal and noise subspace

Signal subspace $\mathcal{S} = \mathrm{span}\{\widetilde{\mathbf{a}}_\ell^{(1)} \circ \widetilde{\mathbf{a}}_\ell^{(2)} \circ \cdots \circ \widetilde{\mathbf{a}}_\ell^{(R)}\}_{\ell=1}^{D}$,

Noise subspace $\mathcal{N} = \mathcal{S}^\perp$.

Tensor MUSIC spectrum

$$P_{CP}\left(\bar{\boldsymbol{\mu}}\right) = \frac{1}{\left\|\mathrm{proj}_{\mathcal{N}}\, \boldsymbol{\mathcal{V}}_{\mathbb{U}^+}\left(\bar{\boldsymbol{\mu}}\right)\right\|_F^2}.$$

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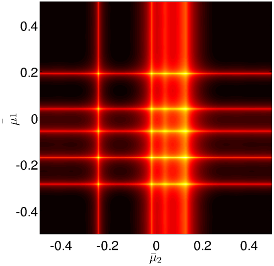
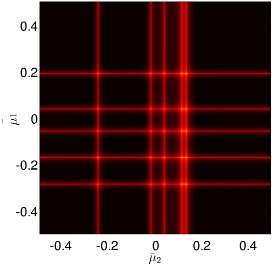
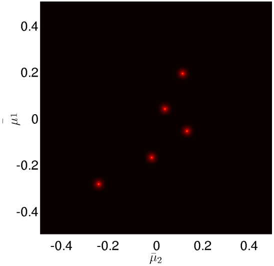
Tensor Dimension $R = 2$

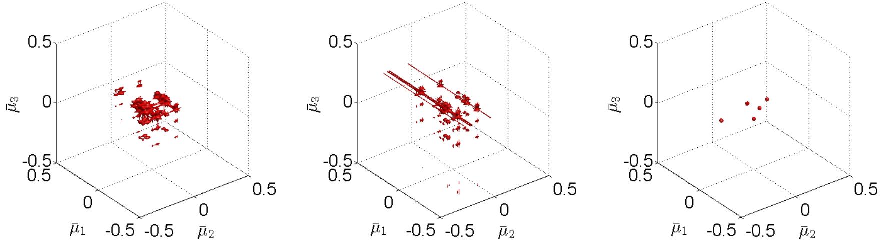
- 10 sensors (or samples) in each dimension
- Coprime array/sampling with $M = 3$ and $N = 5$
- 1000 snapshots, 0dB SNR, and $D = 5$ equal-power sources.

$P_{ULA,HOSVD}(\bar{\boldsymbol{\mu}})$	$P_{Coprime,HOSVD}(\bar{\boldsymbol{\mu}})$	Proposed $P_{Coprime,CP}(\bar{\boldsymbol{\mu}})$
		
Low resolution ✗ Cross terms ✗	High resolution ✓ Cross terms ✗	High resolution ✓ No cross terms ✓

Tensor Dimension $R = 3$

- 10 sensors (or samples) in each dimension,
- Coprime array/sampling with $M = 3$ and $N = 5$,
- 1000 snapshots, 0dB SNR, and $D = 5$ equal-power sources.

$P_{ULA,HOSVD}(\bar{\boldsymbol{\mu}})$	$P_{Coprime,HOSVD}(\bar{\boldsymbol{\mu}})$	Proposed $P_{Coprime,CP}(\bar{\boldsymbol{\mu}})$



Low resolution ✗ Cross terms ✗	High resolution ✓ Cross terms ✗	High resolution ✓ No cross terms ✓

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Concluding Remarks

- Parameter estimation using
  1. Sparse arrays / non-uniform sampling,
  2. Tensor models, and
  3. MUSIC.
- Tensor MUSIC using HOSVD on $\widetilde{\mathcal{R}}$:
  1. Product of MUSIC spectra
  2. Cross-terms
- Tensor MUSIC using CP on $\widetilde{\mathcal{R}}$:
  1. No cross-terms