Tensor MUSIC in Multidimensional Sparse Arrays

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Outline

1. Introduction
   - Motivation
   - Tensors

2. Contribution: Tensor MUSIC in Multidimensional Sparse Arrays
   - Coarray tensor
   - Tensor MUSIC spectrum

3. Numerical Examples

4. Concluding Remarks
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Harmonic Retrieval in Planar Array Processing

Ultimate Goal

Estimate source profiles (azimuth, elevation, range, Doppler, etc.) from sensor measurements efficiently and accurately.

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Sparse Array Processing\textsuperscript{2,3}

### Uniform Linear Arrays (ULAs)

- ULA with $N$ sensors and sensor separation $\lambda/2$.

- Identify at most $N - 1$ sources using $N$ sensors.

\[
\begin{array}{c}
\text{\underbrace{\quad \cdots \quad}} \\
\hline \\
\lambda/2
\end{array}
\]

### Linear Sparse Arrays

- Nested array with $N_1$, $N_2$ and min. separation $\lambda/2$.

- Identify $O(N^2)$ uncorrelated sources using $O(N)$ sensors.

\[
\begin{array}{c}
\underbrace{\quad \cdots \quad} & \underbrace{\quad \cdot \quad} & \underbrace{\quad \cdot \quad} & \underbrace{\quad \cdot \quad} \\
\hline \\
\lambda/2(N_1 + 1)\lambda/2
\end{array}
\]


Tensor Model\textsuperscript{4,5, etc.}

**Motivation**

**Measurements**

**Vector Model**

\[ \mathbf{x} = \]

Spatial/temporal relations are **mixed** \( \times \)

**Tensor Model**

\[ \mathbf{\chi} = \]

Spatial/temporal relations are **separated** \( \checkmark \)

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Main Goal of this Work

Proposed Scheme

Input → Sparse arrays → Tensor Models → Tensor MUSIC → Azimuth, Elevation, Doppler, etc.

Related work:

- ULA, tensors, and MUSIC ⇒ DOA and polarization\(^6,\)\(^7\).
- Nested arrays, tensors, and MUSIC ⇒ azimuth, elevation, and polarization\(^8\).

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Notations

Tensor $\mathcal{A}$

Outer product $\mathcal{A} \odot \mathcal{B}$

Inner product $\langle \mathcal{A}, \mathcal{B} \rangle$

$n$-mode product $\mathcal{A} \times_n \mathbf{U}$

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Tensor Decomposition\textsuperscript{10}

- **CANDECOMP/PARAFAC (CP) decomposition:**
  \[ X \approx \sum_{r=1}^{R} a_r \circ b_r \circ c_r. \]

- **High-order SVD (HOSVD):**
  \[ X \approx g \times_1 A \times_2 B \times_3 C. \]

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Sparse Array Processing

**Vector Model:**

\[ \hat{x}_S(k) \rightarrow \hat{R}_S \rightarrow \hat{x}_D \rightarrow \hat{R} \rightarrow \text{MUSIC} \]

Physical array \( S \) \rightarrow Difference coarray \( D \)

**Tensor Model (Proposed):**

\[ \hat{\chi}_S(k) \rightarrow \hat{R}_S \rightarrow \hat{\chi}_D \rightarrow \hat{R} \rightarrow \text{Tensor MUSIC} \]

Existing \( \rightarrow \) Proposed

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Some Discussions on the Coarray Tensor $\tilde{\mathbf{R}}$

**Vector model**

$$\langle \tilde{\mathbf{R}} \rangle_{p_1, p'_1} = \langle \tilde{\mathbf{x}} \rangle D_{m_1},$$

$$p_1 - p'_1 = m_1.$$

- $\tilde{\mathbf{R}}$ avoids implementing spatial smoothing in tensors.
- $\tilde{\mathbf{R}}$ admits the (tensor) MUSIC algorithm.

**Tensor model**

$$\langle \tilde{\mathbf{R}} \rangle_{p_1, p_2, \ldots, p_R, p'_1, p'_2, \ldots, p'_R} = \langle \tilde{\mathbf{x}} \rangle D_{m_1, m_2, \ldots, m_R},$$

$$p_r - p'_r = m_r,$$

$$r = 1, 2, \ldots, R.$$

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### Tensor MUSIC

#### MUSIC

1. **Eigen-decomposition:**
   \[
   \tilde{\mathbf{R}} = \tilde{\mathbf{U}} \Lambda \tilde{\mathbf{U}}^H.
   \]

2. **Signal and noise subspace:**
   \[
   \tilde{\mathbf{U}} = \begin{bmatrix} \tilde{\mathbf{U}}_s & \tilde{\mathbf{U}}_n \end{bmatrix}
   \]

3. **MUSIC spectrum:**
   \[
   P(\bar{\theta}) = \frac{1}{\| \tilde{\mathbf{U}}_n^H \mathbf{v}(\bar{\theta}) \|^2}
   \]
   \[
   \mathbf{v}(\bar{\theta}) : \text{steering vectors.}
   \]

#### Tensor MUSIC

1. **HOSVD:**
   \[
   \tilde{\mathbf{R}} = \tilde{\mathbf{K}} \times_1 \tilde{\mathbf{U}}_1 \times_2 \tilde{\mathbf{U}}_2 \cdots \times_R \tilde{\mathbf{U}}_R \times_{R+1} \tilde{\mathbf{U}}_1^* \times_{R+2} \tilde{\mathbf{U}}_2^* \cdots \times_{2R} \tilde{\mathbf{U}}_R^*.
   \]

2. **Signal and noise subspace:**
   \[
   \tilde{\mathbf{U}}_r = \begin{bmatrix} \tilde{\mathbf{U}}_{r,s} & \tilde{\mathbf{U}}_{r,n} \end{bmatrix}
   \] is a unitary matrix.

3. **Tensor MUSIC spectrum**
   \[
   P_{HOSVD}(\bar{\mu}) = \frac{1}{\| \mathbf{V}(\bar{\mu}) \times_1 \tilde{\mathbf{U}}_{1,n}^* \tilde{\mathbf{U}}_{1,n}^H \times_R \tilde{\mathbf{U}}_{R,n}^* \tilde{\mathbf{U}}_{R,n}^H \|_F^2}
   \]
   \[
   \mathbf{V}(\bar{\mu}) : \text{steering tensors.}
   \]

---

Our observation: $P_{\text{HOSVD}}(\bar{\mu})$ is a separable MUSIC spectrum

$$P_{\text{HOSVD}}(\bar{\mu}) = \prod_{r=1}^{R} P_r(\bar{\mu}^{(r)}), \quad P_r(\bar{\mu}^{(r)}) = \frac{1}{\| \tilde{U}_{r,n}^H v_{\tilde{U}^r_r} (\bar{\mu}^{(r)}) \|^2}$$

$P_{\text{HOSVD}}(\bar{\mu})$ has cross-terms

- Actual
- $P_{\text{HOSVD}}(\bar{\mu})$
Proposed Tensor MUSIC spectrum via CP

**CP**

\[
\tilde{\mathcal{R}} = \sum_{\ell=1}^{D} \tilde{a}_{\ell}^{(1)} \circ \tilde{a}_{\ell}^{(2)} \circ \cdots \circ \tilde{a}_{\ell}^{(R)} \circ \tilde{a}_{\ell}^{(1)*} \circ \tilde{a}_{\ell}^{(2)*} \circ \cdots \circ \tilde{a}_{\ell}^{(R)*} .
\]

**Signal and noise subspace**

Signal subspace \( \mathcal{S} = \text{span}\{ \tilde{a}_{\ell}^{(1)} \circ \tilde{a}_{\ell}^{(2)} \circ \cdots \circ \tilde{a}_{\ell}^{(R)} \}_{\ell=1}^{D} \),

Noise subspace \( \mathcal{N} = \mathcal{S}^\perp \).

**Tensor MUSIC spectrum**

\[
P_{\text{CP}} (\bar{\mu}) = \frac{1}{\| \text{proj}_{\mathcal{N}} \mathbf{V}_U^+ (\bar{\mu}) \|_F^2}.
\]
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Numerical Examples

Tensor Dimension $R = 2$

- 10 sensors (or samples) in each dimension
- Coprime array/sampling with $M = 3$ and $N = 5$
- 1000 snapshots, 0dB SNR, and $D = 5$ equal-power sources.

\[
P_{ULA,HOSVD} (\bar{\mu}) \quad P_{Coprime,HOSVD} (\bar{\mu}) \quad \text{Proposed} \quad P_{Coprime,CP} (\bar{\mu})
\]

Low resolution \(\times\) \quad High resolution \(\checkmark\) \quad High resolution \(\checkmark\)

Cross terms \(\times\) \quad Cross terms \(\times\) \quad No cross terms \(\checkmark\)
Numerical Examples

**Tensor Dimension $R = 3$**

- 10 sensors (or samples) in each dimension,
- Coprime array/sampling with $M = 3$ and $N = 5$,
- 1000 snapshots, 0dB SNR, and $D = 5$ equal-power sources.

\[
\begin{align*}
P_{ULA,HOSVD} (\bar{\mu}) & \quad P_{Coprime,HOSVD} (\bar{\mu}) \\
\text{Proposed} & \quad P_{Coprime,CP} (\bar{\mu})
\end{align*}
\]

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Parameter estimation using
1. Sparse arrays / non-uniform sampling,
2. Tensor models, and
3. MUSIC.

Tensor MUSIC using HOSVD on $\tilde{\mathcal{R}}$:
1. Product of MUSIC spectra
2. Cross-terms

Tensor MUSIC using CP on $\tilde{\mathcal{R}}$:
1. No cross-terms