Maximally Economic Sparse Arrays and Cantor Arrays

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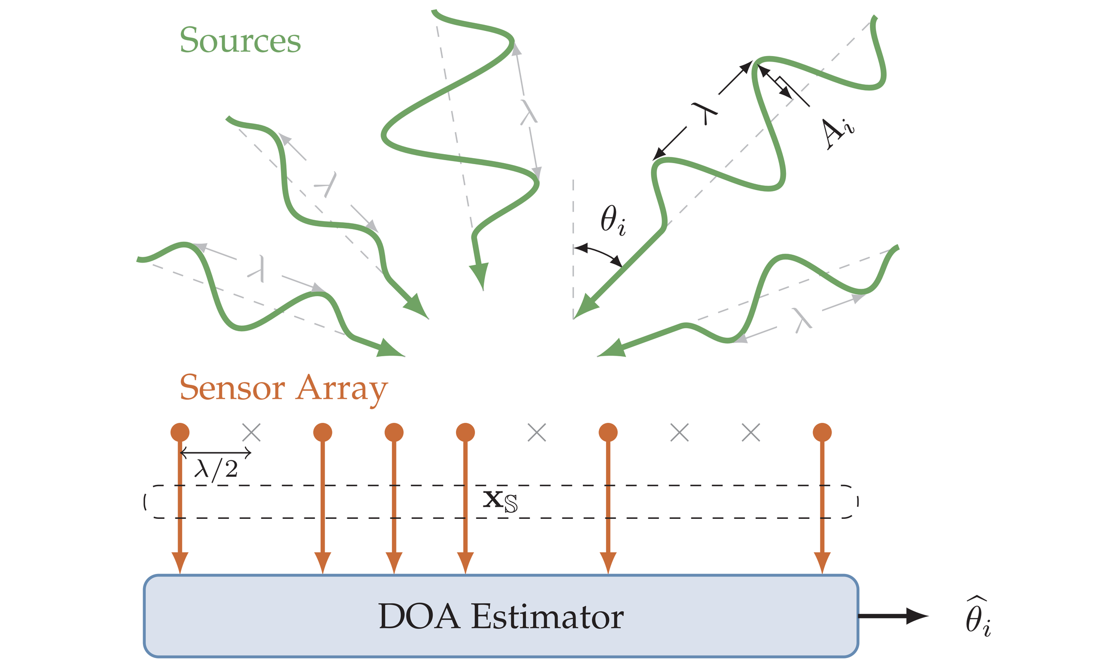
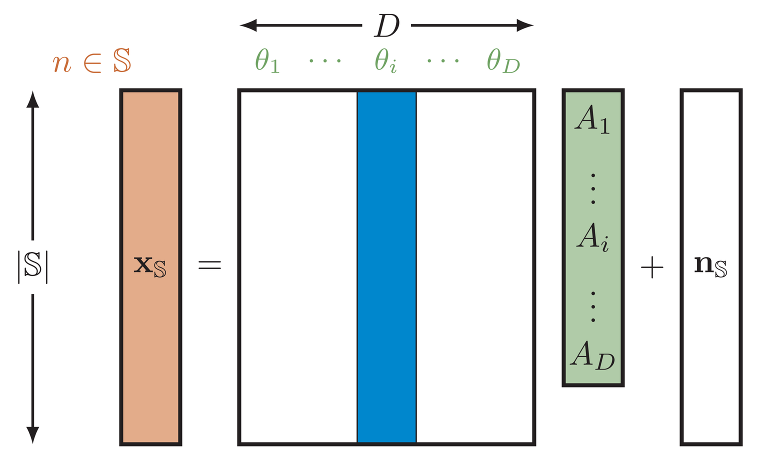
Main Contributions

- ⋆ New design for **symmetric arrays with hole-free difference coarrays**.
- ⋆ The **essentialness property** and **maximally economic sparse arrays**.
- ⋆ (Fractal) Cantor arrays: **New definition**, **the difference coarray**, and **maximal economy**.

Direction-of-Arrival Estimation



The Data Model



Steering vector: $\mathbf{v}_{\mathbb{S}}(\theta_i) = [\exp(j\pi n \sin\theta_i)]_{n\in\mathbb{S}}$,

- ⋆ Stochastic (or unconditional) model with uncorrelated sources and noise.

$$\mathbf{s} = [A_1, \ldots, A_D, \mathbf{n}_{\mathbb{S}}^T]^T,$$

$$\mathbb{E}[\mathbf{s}] = \mathbf{0}, \quad \mathbb{E}[\mathbf{s}\mathbf{s}^H] = \operatorname{diag}(p_1, \ldots, p_D, p_n\mathbf{I}).$$

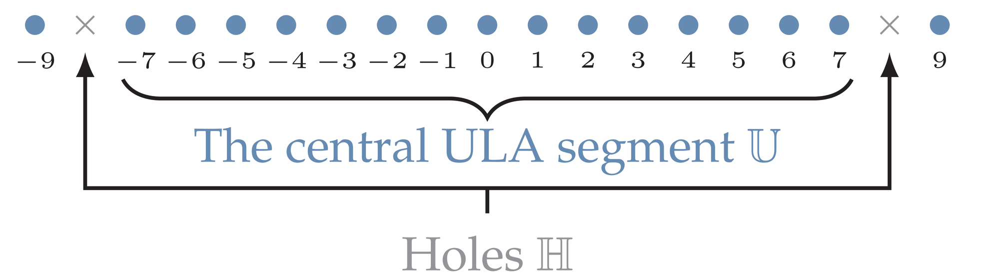
- ⋆ DOAs are deterministic, but unknown.
- ⋆ The number of sources D is known.

Covariance matrix on $\mathbb{S}$ → Autocorrelation vector on $\mathbb{D}$ [3, 4]

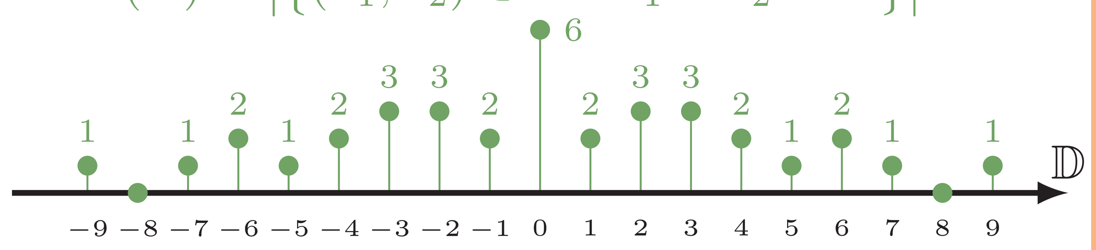
The Difference Coarray $\mathbb{D}$

Physical array $\mathbb{S}$
N sensors

$$\mathbb{D} \triangleq \{n_1 - n_2 : n_1, n_2 \in \mathbb{S}\}$$



Weight function
$$w(m) = |\{(n_1, n_2) \in \mathbb{S}^2 : n_1 - n_2 = m\}|$$



Desired Properties

- ⋆ Hole-free difference coarray ($\mathbb{D} = \mathbb{U}$) ❶
  Coarray MUSIC [3, 4], performance analysis [5].
- ⋆ Large difference coarray ($|\mathbb{D}| > \mathcal{O}(N)$) ❷
  More uncorrelated sources than sensors [5].

Symmetric Arrays

- ⋆ Advantages: Simplified array design, implementation, and calibration; DOA estimators [6]. ❸

Theorem: New Design for Symmetric Arrays with Hole-Free $\mathbb{D}$

(a): Minimum redundancy array, 9 elements

(b): The reversed version of (a), 9 elements

(c): The union of (a) and (b), 16 elements

(1): Remove 4 and 25 from (c), 14 elements

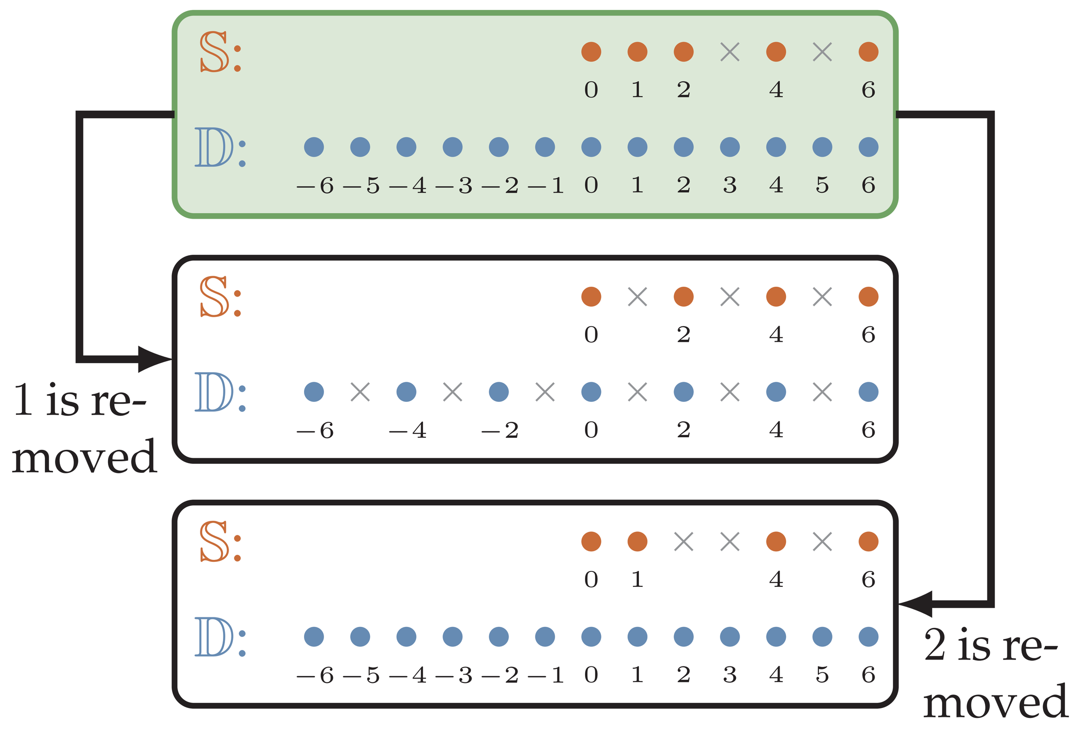
Array (1) is less expensive than array (c)

The Essentialness Property

Definition

The sensor located at $n \in \mathbb{S}$ is said to be essential with respect to $\mathbb{S}$ if the difference coarray changes when the sensor at n is deleted from the array.

Example



1 is essential; 2 is inessential.

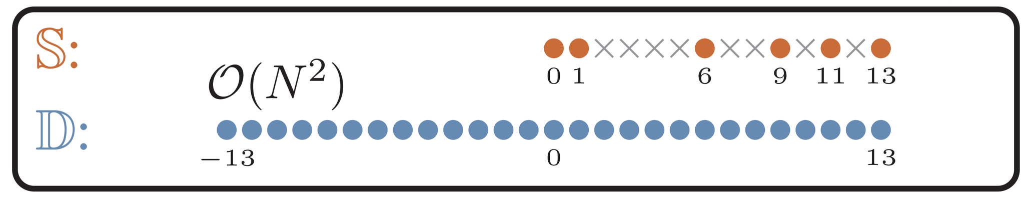
Maximally Economic Sparse Arrays

Definition

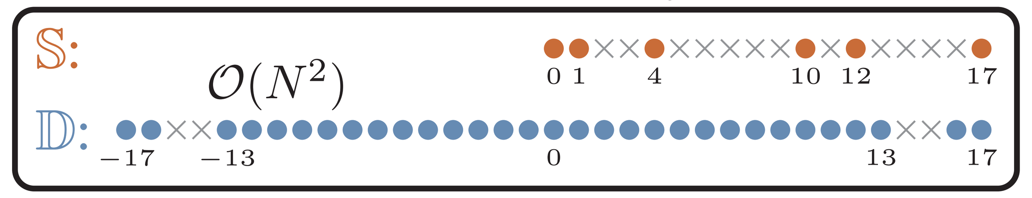
A array $\mathbb{S}$ is maximally economic if all the sensors in $\mathbb{S}$ are essential. ❹

Theorem
These Arrays are Maximally Economic

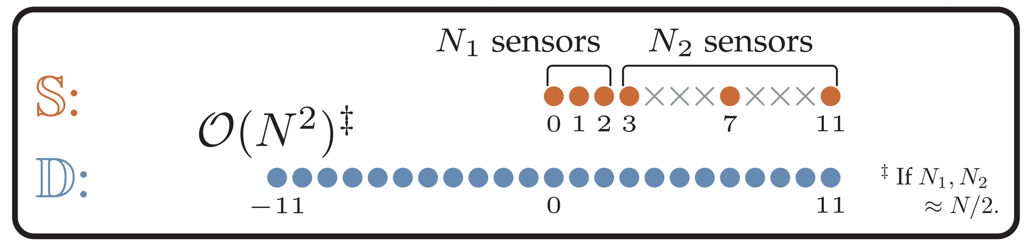
Minimum redundancy arrays [1] ❶❷❹



Minimum hole arrays [2] ❷❹



The nested arrays with $N_2 \geq 2$ [3] ❶❷❹



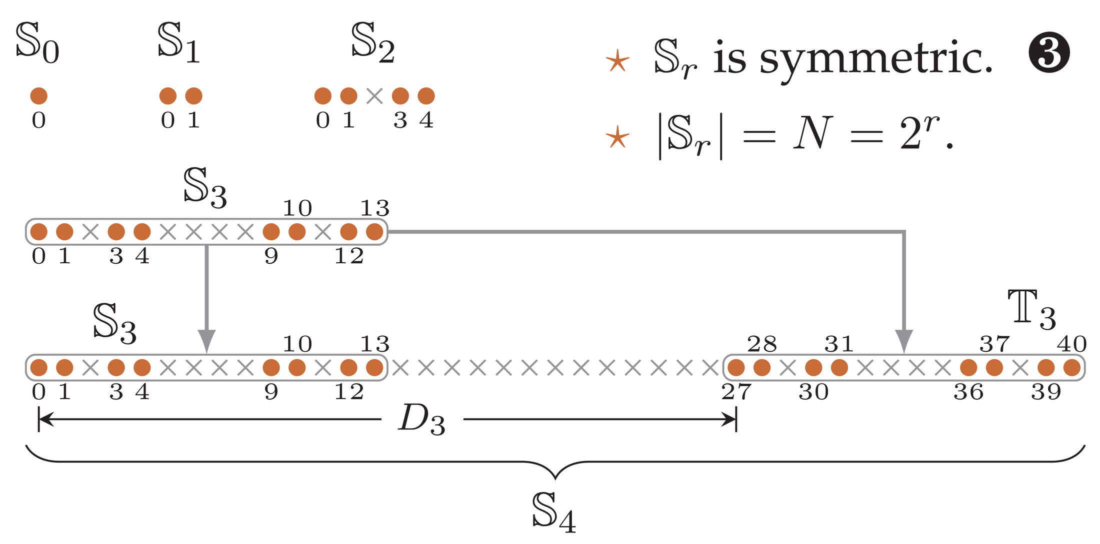
The Cantor Array $\mathbb{S}_r$, $r \geq 0$ ❶❷❸❹

New Definition

$$\mathbb{S}_r \triangleq \begin{cases} \{0\}, & \text{if } r = 0, \\ \mathbb{S}_{r-1} \cup \mathbb{T}_{r-1}, & \text{if } r \geq 1, \end{cases}$$

where $\mathbb{T}_r \triangleq \{n + D_r : n \in \mathbb{S}_r\}$, $D_r \triangleq 2A_r + 1$, and $A_r \triangleq \max(\mathbb{S}_r) - \min(\mathbb{S}_r)$ is the aperture of $\mathbb{S}_r$.

Example



- ⋆ $\mathbb{S}_r$ is symmetric. ❸
- ⋆ $|\mathbb{S}_r| = N = 2^r$.

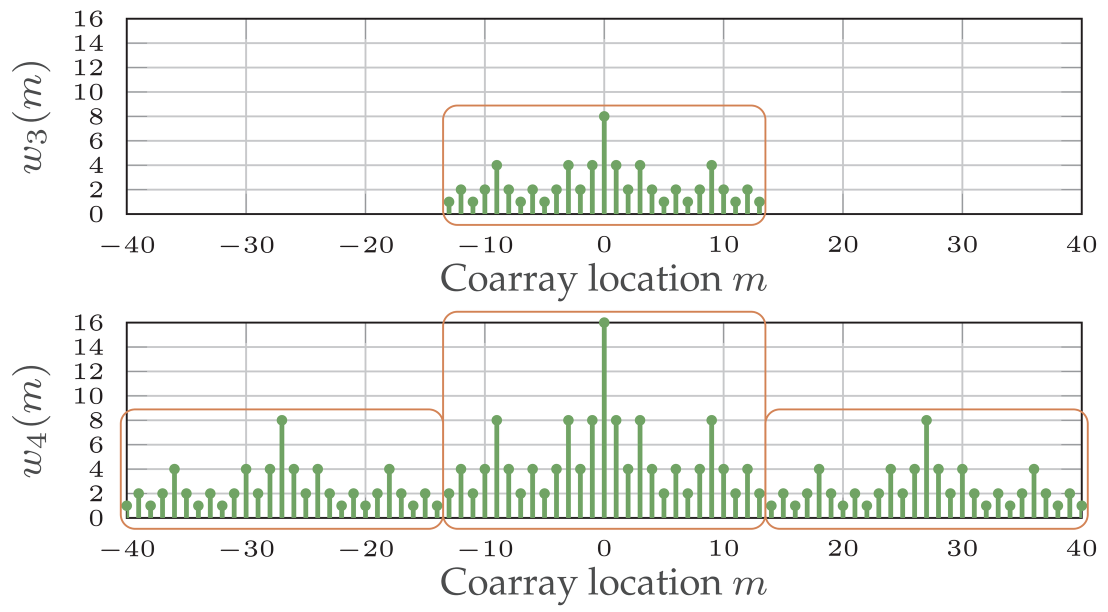
Old Definition from the Cantor Set [7, 8]

Theorem: $\mathbb{D}_r$ for the Cantor Array

- ⋆ $\mathbb{D}_r$ is hole-free. (New) ❶
- ⋆ $|\mathbb{D}_r| = 3^r = N^{\log_2 3} \approx N^{1.585} > \mathcal{O}(N)$. (New) ❷

The Weight Function

$$w_r(m) = \begin{cases} 2w_{r-1}(m), & \text{if } |m| \leq A_{r-1}, \\ w_{r-1}(m \pm D_{r-1}), & \text{if } |m \pm D_{r-1}| \leq A_{r-1}, \\ 0, & \text{otherwise.} \end{cases}$$



Theorem

The Cantor Array is Maximally Economic

Lemma

If $n_1, n_2 \in \mathbb{S}$ and $w(n_1 - n_2) = 1$, then n_1 and n_2 are both essential with respect to $\mathbb{S}$.

Example

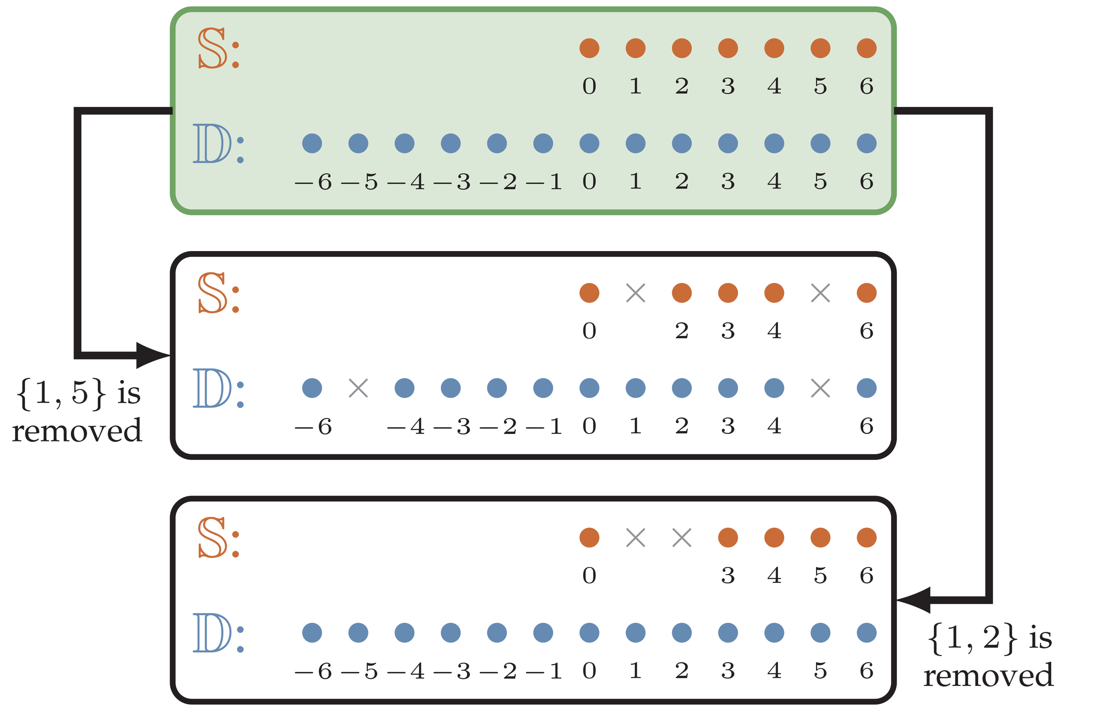
$\mathbb{S}_3 = \{0, 1, 3, 4, 9, 10, 12, 13\}$ is maximally economic since

$$w_3(13-0) = w_3(12-1) = w_3(10-3) = w_3(9-4) = 1.$$

In general, $\mathbb{S}_r$ is maximally economic. (New) ❹

Ongoing Work

The k-Essentialness Property



The Essentialness Property and DOA Estimators

- ⋆ Some DOA estimators [4] rely on the central ULA segment $\mathbb{U}$, instead of the difference coarray $\mathbb{D}$.

References

[1] Moffet, *IEEE Trans. Antennas Propag.*, 1968.
[2] Taylor and Golomb, *Rulers, Part I*, 1985.
[3] Pal and Vaidyanathan, *IEEE Trans. Signal Process.*, 2010.
[4] Liu and Vaidyanathan, *IEEE Signal Process. Lett.*, 2015.
[5] Liu and Vaidyanathan, *Digit. Signal Process.*, 2017; Wang and Nehorai, *IEEE Trans. Signal Process.*, 2017; Koochakzadeh and Pal, *IEEE Signal Process. Lett.*, 2016.
[6] Haupt, *IEEE Trans. Signal Process.*, 1994; Friedlander and Weiss, *IEEE Trans. Antennas Propag.*, 1994; Xu, Roy, and Kailath, *IEEE Trans. Signal Process.*, 1994; Hoctor and Kassam, *IEEE Trans. Image Process.*, 1996.
[7] Falconer, *Fractal Geometry: Mathematical Foundations and Applications, 2nd ed*, 2005.
[8] Puente-Baliarda and Pous, *IEEE Trans. Antennas Propag.*, 1996.