Coprime Coarray Interpolation for DOA Estimation via Nuclear Norm Minimization

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ISCAS 2016

Outline

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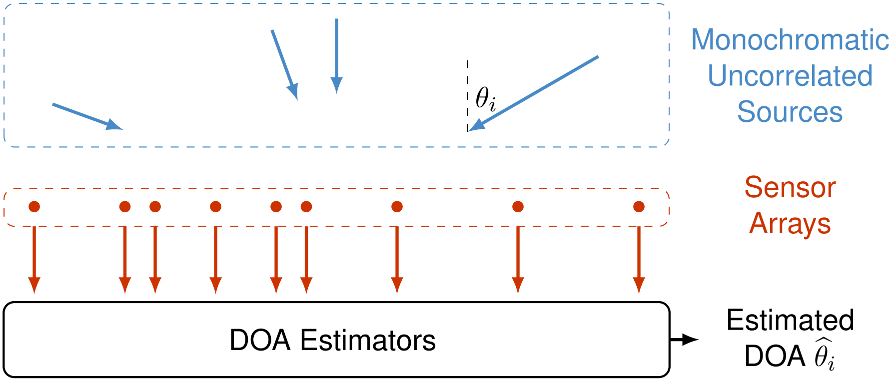
Direction-of-arrival (DOA) estimation[1]



[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.

ULA and sparse arrays

ULA (not sparse)

- Identify at most $N-1$ uncorrelated sources, given N sensors.[1]
- Can only find fewer sources than sensors.

Sparse arrays

1. Minimum redundancy arrays[2]
2. Nested arrays[3]
3. Coprime arrays[4]
4. Super nested arrays[5]

- Identify $O(N^2)$ uncorrelated sources with $O(N)$ physical sensors.
- More sources than sensors!

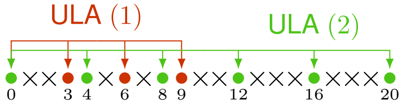
[1] Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*, 2002.
[2] Moffet, *IEEE Trans. Antennas Propag.*, 1968.
[3] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010.
[4] Vaidyanathan and Pal, *IEEE Trans. Signal Proc.*, 2011.
[5] Liu and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2016.

Coprime arrays[1]

The coprime array with $(M, N) = 1$ is the union of

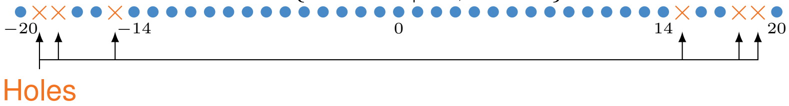
1. an N-element ULA with spacing $M\lambda/2$ and
2. a $2M$-element ULA with spacing $N\lambda/2$.

Physical array $\mathbb{S}$ $(M = 3, N = 4)$:



Difference coarray

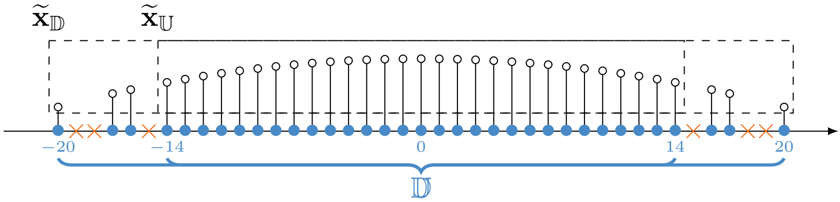
$$\mathbb{D} = \{n_1 - n_2 \mid n_1, n_2 \in \mathbb{S}\}$$



[1] Vaidyanathan and Pal, *IEEE Trans. Signal Proc.*, 2011.

The spatial smoothing MUSIC Algorithm[1]

1. Sample covariance matrix: $\widetilde{\mathbf{R}}_{\mathbb{S}} = \frac{1}{K}\sum_{k=1}^{K} \widetilde{\mathbf{x}}_{\mathbb{S}}(k)\widetilde{\mathbf{x}}_{\mathbb{S}}^H(k)$.
2. Sample autocorrelation function on the difference coarray: $\widetilde{\mathbf{x}}_{\mathbb{D}}$.



3. Hermitian Toeplitz matrix $\widetilde{\mathbf{R}}$ (indefinite matrix).

$$\widetilde{\mathbf{R}} = \begin{bmatrix} \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_0 & \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_{-1} & \cdots & \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_{-14} \\ \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_1 & \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_0 & \cdots & \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_{-13} \\ \vdots & \vdots & \ddots & \vdots \\ \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_{14} & \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_{13} & \cdots & \langle\widetilde{\mathbf{x}}_{\mathbb{U}}\rangle_0 \end{bmatrix}$$

4. MUSIC on $\widetilde{\mathbf{R}}$ resolves $(|\mathbb{U}|-1)/2 = O(N^2)$ uncorrelated sources.

[1] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010; Liu and Vaidyanathan, *IEEE Signal Proc. Letter*, 2015.

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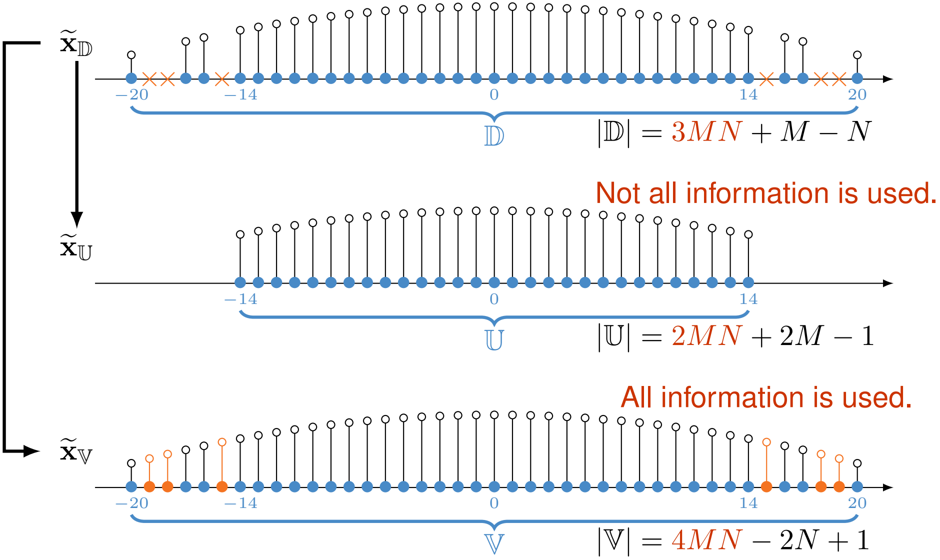
Why coarray interpolation?



Previous work

1. Spatial smoothing MUSIC[1]: No coarray interpolation.
2. Positive-definite Toeplitz matrix completion[2]: Not always feasible.
3. Coarray interpolation (ICA-AI)[3]: Non-convex optimization.
4. Sparse support recovery techniques[4]: Predefined dense grid and parameters.
5. Gridless DOA estimator via low-rank recovery[5]: Not used for interpolation, but for denoising.

[1] Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2010; Liu and Vaidyanathan, *IEEE Signal Proc. Letter*, 2015.

[2] Abramovich, Spencer, and Gorokhov, *IEEE Trans. Signal Proc.*, 1999.

[3] Friedlander and Weiss, *IEEE Trans. Aero. Elec. Sys.*, 1992; Tuncer, Yasar, and Friedlander, *Radio Science*, 2007.

[4] Zhang, Amin, and Himed, *IEEE ICASSP*, 2013; Pal and Vaidyanathan, *IEEE Trans. Signal Proc.*, 2015;

[5] Pal and Vaidyanathan, *IEEE Signal Proc. Letter*, 2014.

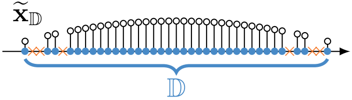
The proposed method (via nuclear norm minimization)

$$\widetilde{\mathbf{R}}_{\mathbb{V}}^{\star} = \underset{\widetilde{\mathbf{R}}_{\mathbb{V}} \in \mathbb{C}^{|\mathbb{V}^+| \times |\mathbb{V}^+|}}{\arg\min} \quad \|\widetilde{\mathbf{R}}_{\mathbb{V}}\|_* \quad \text{s. t.}$$

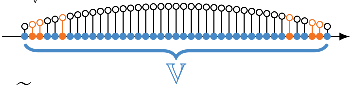
$$\widetilde{\mathbf{R}}_{\mathbb{V}} = \widetilde{\mathbf{R}}_{\mathbb{V}}^H,$$

$$\langle \widetilde{\mathbf{R}}_{\mathbb{V}} \rangle_{n_1, n_2} = \langle \widetilde{\mathbf{x}}_{\mathbb{D}} \rangle_{n_1 - n_2},$$

$$n_1, n_2 \in \mathbb{V}^+ = \{n \mid n \in \mathbb{V}, n \geq 0\}.$$



$\widetilde{\mathbf{x}}_{\mathbb{V}}$ ≈ autocorrelation functions



$\widetilde{\mathbf{R}}_{\mathbb{V}}$ ≈ covariance matrices

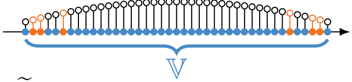
Low rank, Hermitian Toeplitz

- $\widetilde{\mathbf{R}}_{\mathbb{V}}$ has a low-rank structure for sufficient number of snapshots. The nuclear norm $\|\cdot\|_*$ (sum of singular values) is a convex relaxation of the matrix rank.
- $\widetilde{\mathbf{R}}_{\mathbb{V}}$ is Hermitian.
- $\widetilde{\mathbf{R}}_{\mathbb{V}}$ is a Toeplitz matrix with some known entries.

Advantages over the previous work

1. All the information is used.
2. Gridless.
3. Always feasible, even though $\widetilde{\mathbf{R}}_{\mathbb{V}}^{\star}$ can be indefinite.
4. Convex program.
5. It is possible to resolve beyond the limit of $\mathbb{U}$.

$\widetilde{\mathbf{x}}_{\mathbb{V}} \approx$ autocorrelation functions



$\widetilde{\mathbf{R}}_{\mathbb{V}} \approx$ covariance matrices

Low rank, Hermitian Toeplitz

Coarray interpolation

$$\widetilde{\mathbf{R}}_{\mathbb{V}}^{\star} = \underset{\widetilde{\mathbf{R}}_{\mathbb{V}} \in \mathbb{C}^{|\mathbb{V}^{+}| \times |\mathbb{V}^{+}|}}{\arg\min} \quad \|\widetilde{\mathbf{R}}_{\mathbb{V}}\|_{*}$$

subject to

$$\widetilde{\mathbf{R}}_{\mathbb{V}} = \widetilde{\mathbf{R}}_{\mathbb{V}}^{H},$$

$$\langle \widetilde{\mathbf{R}}_{\mathbb{V}} \rangle_{n_1, n_2} = \langle \widetilde{\mathbf{x}}_{\mathbb{D}} \rangle_{n_1 - n_2}.$$

MUSIC

$$\widetilde{\mathbf{R}}_{\mathbb{V}}^{\star} = \widetilde{\mathbf{U}} \boldsymbol{\Lambda} \widetilde{\mathbf{U}}^{H},$$

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{\mathbf{U}}_s & \widetilde{\mathbf{U}}_n \end{bmatrix},$$

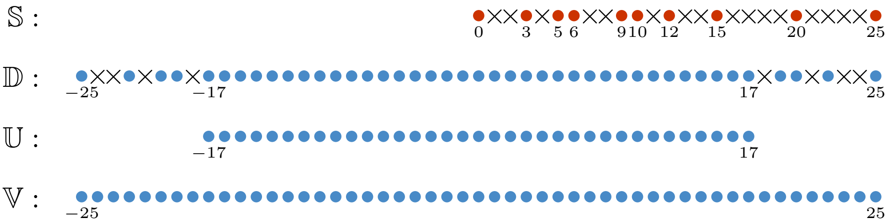
$$P_{\text{MUSIC}}(\bar{\theta}) = \frac{1}{\left\| \widetilde{\mathbf{U}}_n^{H} \mathbf{v}_{\mathbb{V}^{+}}(\bar{\theta}) \right\|_2^2}$$

Outline

Simulation parameters

A coprime array with $M = 3$ and $N = 5$: (10 sensors)

$$\begin{aligned}
\mathbb{S} &= \{0, 3, 5, 6, 9, 10, 12, 15, 20, 25\}, & |\mathbb{S}| &= 10, & |\mathbb{S}| - 1 &= 9, \\
\mathbb{D} &= \{-25, -22, -20, -19, \\
&\quad -17, \ldots, 17, 19, 20, 22, 25\}, & |\mathbb{D}| &= 43, & (|\mathbb{D}| - 1)/2 &= 21, \\
\mathbb{U} &= \{-17, \ldots, 17\}, & |\mathbb{U}| &= 35, & (|\mathbb{U}| - 1)/2 &= 17, \\
\mathbb{V} &= \{-25, \ldots, 25\}, & |\mathbb{V}| &= 51.
\end{aligned}$$



Simulation parameters (Cont.)

1. The maximum number of resolvable uncorrelated sources:
   - using $\mathbb{U}$: 17,
   - using $\mathbb{D}$: 21.
2. Equal-power uncorrelated sources.
3. 0 dB SNR and 500 snapshots.
4. Root-mean-squared error:

$$E = \sqrt{\frac{1}{D}\sum_{i=1}^{D}\left(\widehat{\bar{\theta}}_i - \bar{\theta}_i\right)^2},$$

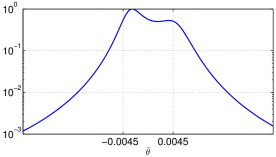
where

- $\{\widehat{\bar{\theta}}_i\}_{i=1}^{D}$ is the estimated normalized DOA, and
- $\{\bar{\theta}_i\}_{i=1}^{D}$ is the true normalized DOA.

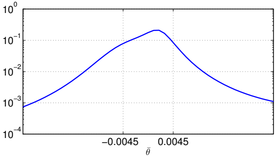
Example 1: Two closely spaced sources (10 sensors)

$P(\bar{\theta})$

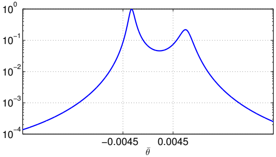
SS-MUSIC[1]



Co-LASSO[4]



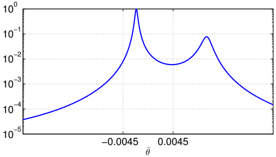
Spline[3]



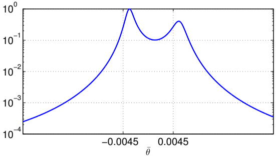
E	SS-MUSIC	Co-LASSO	Spline
E	0.00093588	0.23299	0.0019951

$P(\bar{\theta})$

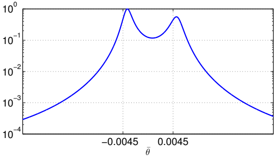
ICA-AI[3]



P.D. Toeplitz Completion[2]



Nuclear norm (Proposed)

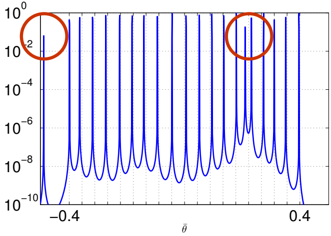


E	ICA-AI	P.D. Toeplitz Completion	Nuclear norm (Proposed)
E	0.0046232	0.0011188	0.00073202

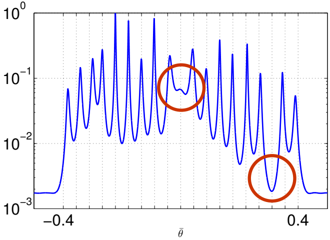
[1-5] See page 10 for all the references

Example 2: $D = 19$ sources (10 sensors)

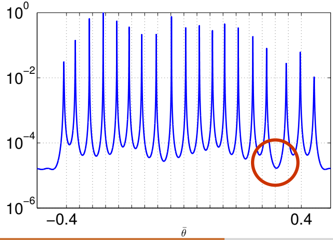
Co-LASSO ($E = 0.0054924$)



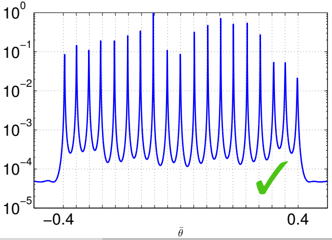
Spline ($E = 0.017665$)



ICA-AI ($E = 0.018319$)



Nuclear norm ($E = 0.003458$)



Outline

Concluding remarks

- Coarray interpolation via nuclear norm minimization:
  - The correlation information on the difference coarray is fully utilized.
  - The estimation error is reduced.
  - More sources than $(|\mathbb{U}| - 1)/2$ (the limit of SS MUSIC using $\widetilde{\mathbf{x}}_{\mathbb{U}}$) can be resolved.
- In the future, it will be interesting to incorporate the matrix denoising idea into the interpolation approach.

Thank you!