Coprime DFT Filter Bank Design: Theoretical Bounds and Guarantees

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Outline

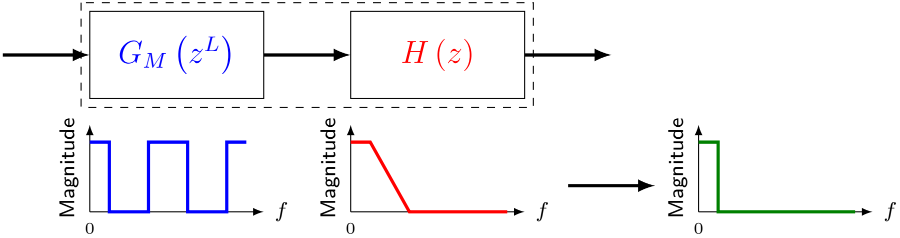
Motivation

- Coprime DFT filter banks: [1]
  - Enhanced degrees of freedom: $O(MN)$ based on $O(M+N)$ samples.
  - Applications in Direction-of-arrival estimation [2], Beamforming [3], and Spectrum estimation [4].
  - How to design filter taps?

Interpolated FIR filter design [5] [6] [7]:

- Design IFIR $F_i(z) \rightarrow$ design L and two *lowpass* filters $G_M(z)$ and $H(z)$.

$$F_i(z) = G_M\left(z^L\right) H(z)$$



Coprime DFT Filter Banks

$G(z), H(z)$:
FIR Type-I filters
N_g, N_h: filter orders

$$G(z) = \sum_{n=0}^{N_g} g(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{N_h} h(n) z^{-n}$$

↓

$G(z^M), H(z^N)$:
sparse coefficient filters

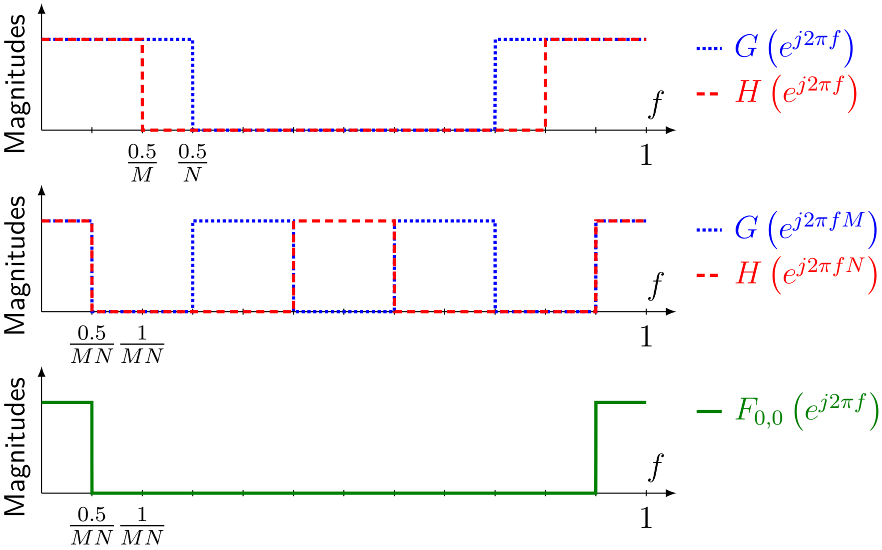
$$G\left(z^M\right) \qquad H\left(z^N\right)$$

↓

$$F_{\ell,k}(z) \leftarrow G\left(z^M W_N^\ell\right) \times H\left(z^N W_M^k\right)$$

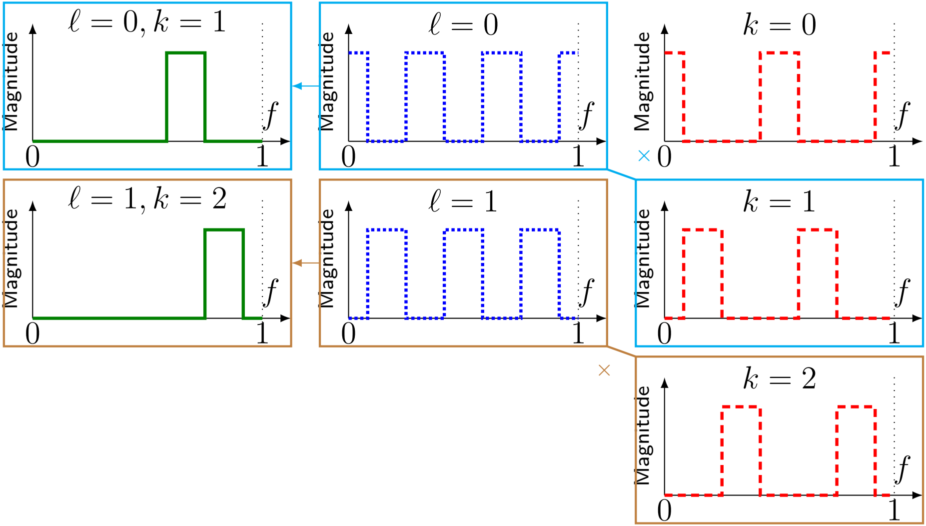
coprime DFT filter banks	DFT filter banks	DFT filter banks
MN-filters	N-filters	M-filters
$\ell = 0, 1, \ldots, N-1,$ $k = 0, 1, \ldots, M-1.$	$\ell = 0, 1, \ldots, N-1.$	$k = 0, 1, \ldots, M-1.$

Coprime DFTFBs, the Ideal Case

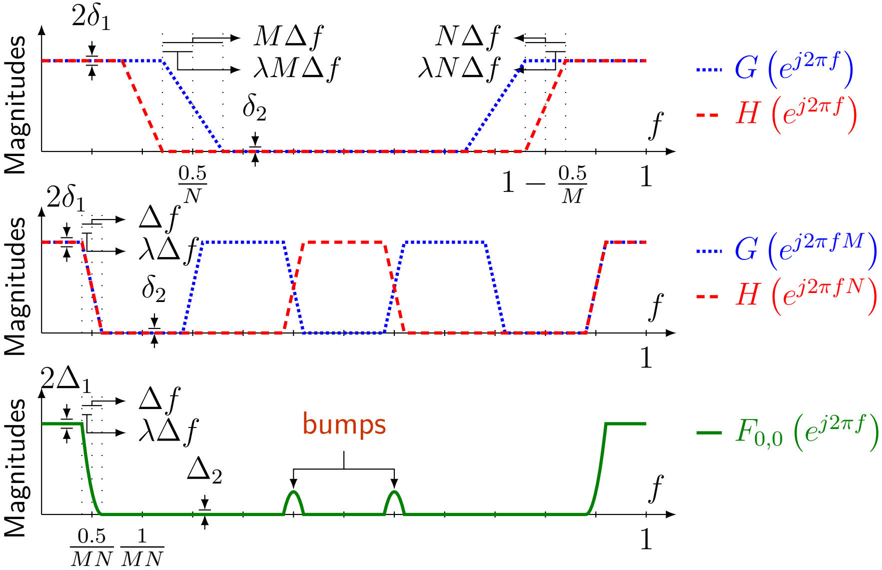


Coprime DFTFBs, the Ideal Case

$$F_{\ell,k}\left(e^{j2\pi f}\right) = G\left(e^{j2\pi f M}e^{-j2\pi \ell/N}\right) \times H\left(e^{j2\pi f N}e^{-j2\pi k/M}\right)$$



Coprime DFTFBs, the Practical Case

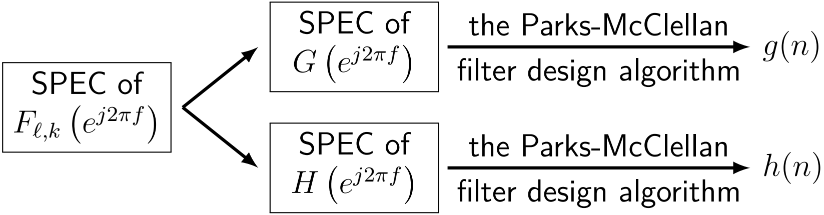


Coprime DFTFB Design Methods

- Goal: Design $g(n)$ and $h(n)$ such that
- $F_{\ell,k}\left(e^{j2\pi f}\right)$ is an approximation of the ideal case.
- Notion of *approximation* in $F_{\ell,k}\left(e^{j2\pi f}\right)$:
  1. Passband ripples Δ_1,
  2. Stopband ripples Δ_2,
  3. Transition band width Δf,
  4. Passband edges and stopband edges.
- Define an appropriate error measure.

Design Method I (Main Concept) [8]

- Divide the design problem into two sub-problems.



Design Method I (Design Equations)

- Passband ripples and stopband ripples for $G\left(e^{j2\pi f}\right)$ and $H\left(e^{j2\pi f}\right)$,

$$\delta_1 = 1 - \sqrt{1-\Delta_1}, \qquad \delta_2 = \frac{\Delta_2}{2-\sqrt{1-\Delta_1}}.$$

- Transition bandwidth Δf,

$$\Delta f \geq \frac{2\log_{10}\left(\frac{1}{10\delta_1\delta_2}\right)}{3\min\left\{MN_g, NN_h\right\}}.$$

- Select λ satisfying

$$\lambda \geq \hat{\lambda}_Q \triangleq \frac{Q_1 - \sqrt{-\ln\left(4\Delta_2\right)}}{Q_1 - Q_2},$$

where $Q_1 \triangleq Q^{-1}\left(1-\delta_1\right)$, $Q_2 \triangleq Q^{-1}\left(\delta_2\right)$, and $Q^{-1}\left(\ \cdot\ \right)$: inverse Q functions.

Design Method II (Motivation)

- Design method I:
  - Heuristic choice of λ.
  - No control over overall amplitude responses $A(e^{j2\pi f})$.
- $A\left(e^{j2\pi f}\right)$: filter bank coverage to the whole spectrum

$$A\left(e^{j2\pi f}\right) = \sum_{\ell=0}^{N-1} \sum_{k=0}^{M-1} \left| F_{\ell k}\left(e^{j2\pi f}\right) \right|,$$

The filter bank satisfying the following criteria is preferred:

1. $\left|F_{00}\left(e^{j2\pi f}\right)\right|$ is close to unity in the passband.
2. $\left|F_{00}\left(e^{j2\pi f}\right)\right|$ is close to zero in the stopband.
3. Overall amplitude responses $A\left(e^{j2\pi f}\right)$ is close to unity at all frequencies.

Design Method II (Problem Formulation)

Optimization Problem

$$\min_{g(n),h(n)} \quad w_1 \left\| \left| F_{00}\left(e^{j2\pi f}\right) \right|_{f\in\left[0,\frac{1}{2MN}\right)\cup\left(1-\frac{1}{2MN},1\right)} - 1 \right\|_p + w_2 \left\| \left| F_{00}\left(e^{j2\pi f}\right) \right|_{f\in\left[\frac{1}{2MN},1-\frac{1}{2MN}\right]} \right\|_p + w_3 \left\| A\left(e^{j2\pi f}\right) - 1 \right\|_p ,$$

- $w_1 + w_2 + w_3 = 1, w_1, w_2, w_3 \geq 0$. Weights among these three factors.
- $\|\cdot\|_p$ denotes the p-norm.

Design Method II (Problem Formulation)

Discretized Optimization Problem (P1)

By taking N_{pt} uniform samples over f (writing as $\mathbf{f}$), we obtain

$$\begin{aligned} \min_{\mathbf{a},\mathbf{b}} \quad & w_1 \left\| \mathbf{J}_p \times \left[(\mathbf{C}_M \mathbf{a}) \odot (\mathbf{C}_N \mathbf{b}) - \mathbf{1} \right] \right\|_p \\ & + w_2 \left\| \mathbf{J}_s \times \left[(\mathbf{C}_M \mathbf{a}) \odot (\mathbf{C}_N \mathbf{b}) \right] \right\|_p \\ & + w_3 \left\| \mathbf{P} \times \left[(\mathbf{C}_M \mathbf{a}) \odot (\mathbf{C}_N \mathbf{b}) \right] - \mathbf{1} \right\|_p , \end{aligned}$$

where "$\odot$" indicates the Hadamard product.

Assumptions:

- $g(n)$ and $h(n)$ are type-I linear phase FIR filters.
- Stopband ripples (δ_2) are much smaller compared to passband responses ($1 \pm \delta_1$).

Design Method II (Details)

- **a** and **b**:

$$\mathbf{a} = \left[g\left(N_g/2\right) \quad 2g\left(N_g/2-1\right) \quad \dots \quad 2g\left(0\right)\right]^T,$$
$$\mathbf{b} = \left[h\left(N_h/2\right) \quad 2h\left(N_h/2-1\right) \quad \dots \quad 2h\left(0\right)\right]^T.$$

- $\mathbf{C}_N$ and $\mathbf{C}_M$: Discrete cosine transform matrices.

$$\mathbf{C}_M = \begin{bmatrix} \cos\left(2\pi M\left[\mathbf{f}\right]_1 \times 0\right) & \dots & \cos\left(2\pi M\left[\mathbf{f}\right]_1 \times \frac{N_g}{2}\right) \\ \vdots & \ddots & \vdots \\ \cos\left(2\pi M\left[\mathbf{f}\right]_{N_{\text{pt}}} \times 0\right) & \dots & \cos\left(2\pi M\left[\mathbf{f}\right]_{N_{\text{pt}}} \times \frac{N_g}{2}\right) \end{bmatrix},$$

$$\mathbf{C}_N = \begin{bmatrix} \cos\left(2\pi N\left[\mathbf{f}\right]_1 \times 0\right) & \dots & \cos\left(2\pi N\left[\mathbf{f}\right]_1 \times \frac{N_h}{2}\right) \\ \vdots & \ddots & \vdots \\ \cos\left(2\pi N\left[\mathbf{f}\right]_{N_{\text{pt}}} \times 0\right) & \dots & \cos\left(2\pi N\left[\mathbf{f}\right]_{N_{\text{pt}}} \times \frac{N_h}{2}\right) \end{bmatrix}.$$

Design Method II (Details)

- $\mathbf{J}_p$ and $\mathbf{J}_s$: selection matrices that choose the passband/stopband.

$$\mathbf{J}_p = \begin{bmatrix} \mathbf{I}_{\frac{N_{\text{pt}}}{2MN}} & \mathbf{O}_{\frac{N_{\text{pt}}}{2MN}\times\left(N_{\text{pt}}-\frac{N_{\text{pt}}}{MN}\right)} & \mathbf{O}_{\frac{N_{\text{pt}}}{2MN}} \\ \mathbf{O}_{\frac{N_{\text{pt}}}{2MN}} & \mathbf{O}_{\frac{N_{\text{pt}}}{2MN}\times\left(N_{\text{pt}}-\frac{N_{\text{pt}}}{MN}\right)} & \mathbf{I}_{\frac{N_{\text{pt}}}{2MN}} \end{bmatrix},$$

$$\mathbf{J}_s = \begin{bmatrix} \mathbf{O}_{\left(N_{\text{pt}}-\frac{N_{\text{pt}}}{MN}\right)\times\frac{N_{\text{pt}}}{2MN}} & \mathbf{I}_{N_{\text{pt}}-\frac{N_{\text{pt}}}{MN}} & \mathbf{O}_{\left(N_{\text{pt}}-\frac{N_{\text{pt}}}{MN}\right)\times\frac{N_{\text{pt}}}{2MN}} \end{bmatrix},$$

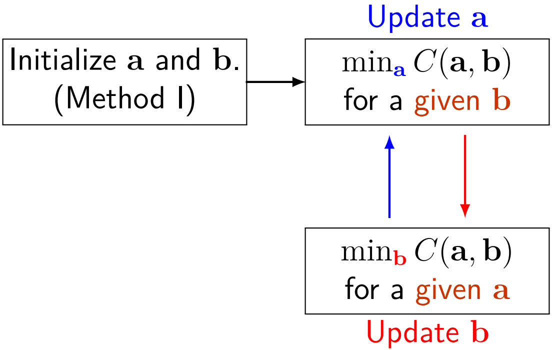
- $\mathbf{P}$: Generate overall amplitude responses.

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{\frac{N_{\text{pt}}}{2MN}} & \mathbf{I}_{\frac{N_{\text{pt}}}{2MN}} & \dots & \mathbf{I}_{\frac{N_{\text{pt}}}{2MN}} \end{bmatrix} \in \{0,1\}^{\frac{N_{\text{pt}}}{2MN}\times N_{\text{pt}}}.$$

- $\mathbf{1}$ all-one column vector.

Design Method II (Solution)

- $(\mathbf{C}_M\mathbf{a}) \odot (\mathbf{C}_N\mathbf{b})$ is a *bilinear form* of $\mathbf{a}$ and $\mathbf{b}$.
- Alternating minimization to $C(\mathbf{a}, \mathbf{b})$ (the cost function in (P1)).
- Design method I as the initial condition.



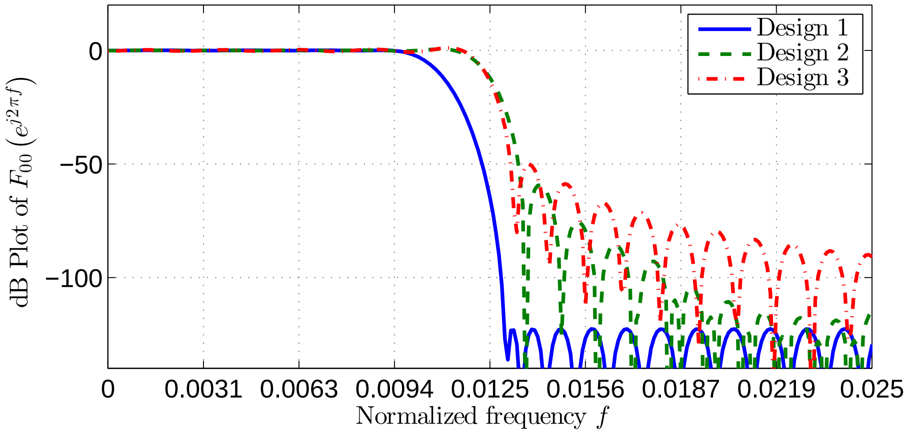
Comparison

Design 1: The example of [8], where $M = 8$, $N = 5$, $N_g = 100$, $N_h = 160$, $\Delta_1 = 0.01$, $\Delta_2 = 0.001$, and $\lambda = \hat{\lambda}_Q = 0.86926$.

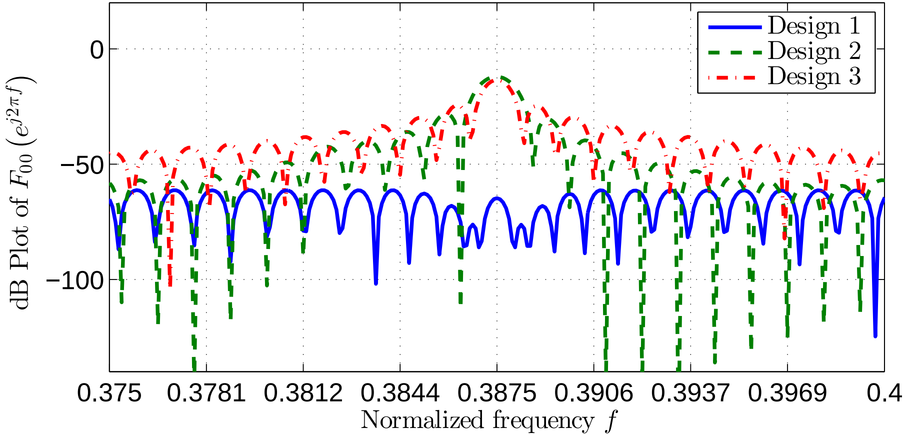
Design 2: $M = 8$, $N = 5$, $N_g = 100$, $N_h = 160$. Solve (P1) by alternating minimization, where Design 1 above is set as the initial point. We choose $N_{\text{pt}} = 2560$, $w_1 = w_2 = w_3 = 1/3$ and $p = 1$.

Design 3: The same as Design 2 except $p = 2$.

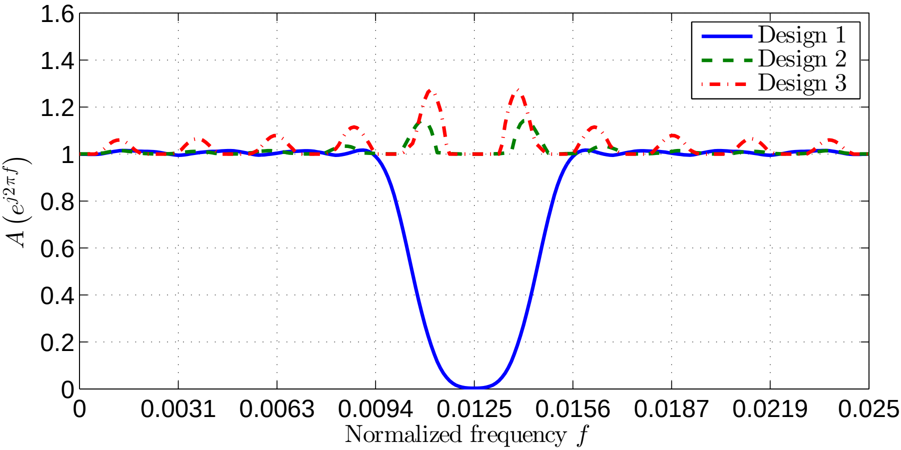
Passband Response



Stopband Response



Overall Amplitude Response



Bump Analysis

Definition: (Bumps)

A bump in coprime DFTFB results from overlapping between the finite transition bands of the sparse coefficient filters $G\left(e^{j2\pi fM}\right)$ and $H\left(e^{j2\pi fN}\right)$.

- Bumps are undesired responses.
- How many bumps are there?
- Where do these bumps located?
- What is the level of bumps?

The Number of Bumps

Lemma

For any $0 \leq \ell \leq N-1, 0 \leq k \leq M-1$, *there exists a unique* $f_0 \in [0,1)$ *such that* $\left|F_{\ell k}\left(e^{j2\pi f}\right)\right| = \left|F_{00}\left(e^{j2\pi(f-f_0)}\right)\right|$.

Theorem: (The number of bumps)

$F_{\ell k}\left(e^{j2\pi f}\right)$ contains exactly two bumps for any $0 \leq \ell \leq N-1$, $0 \leq k \leq M-1$.

The Bump Locations

Theorem: (The bump locations)

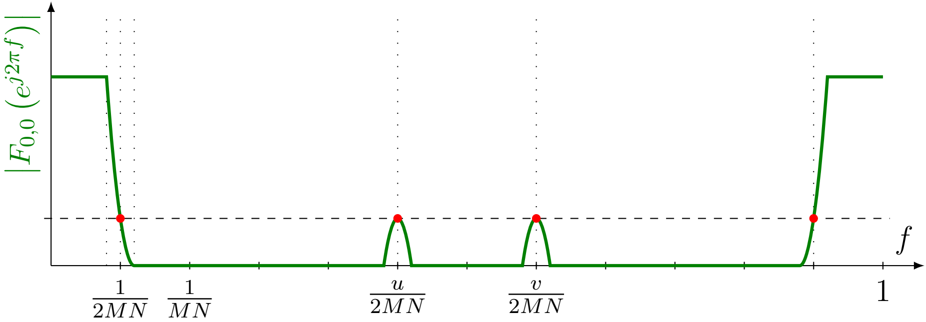
The two bumps of $F_{00}\left(e^{j2\pi f}\right)$ are located around $f = u/(2MN)$ and $f = v/(2MN)$ with

$$u = 2Mn_{+} - 1 = 2Nm_{+} + 1 \notin \{-1, 0, 1\},$$
$$v = 2Mn_{-} + 1 = 2Nm_{-} - 1 \notin \{-1, 0, 1\},$$

where $m_{\pm} \in \{0, 1, \ldots, M-1\}$, $n_{\pm} \in \{0, 1, \ldots, N-1\}$, and $Mn_{\pm} - Nm_{\pm} = \pm 1$. Also, the amplitude response of $F_{00}\left(e^{j2\pi f}\right)$ satisfies

$$\left|F_{00}\left(e^{\frac{j\pi}{MN}}\right)\right| = \left|F_{00}\left(e^{\frac{-j\pi}{MN}}\right)\right| = \left|F_{00}\left(e^{\frac{j\pi u}{MN}}\right)\right| = \left|F_{00}\left(e^{\frac{j\pi v}{MN}}\right)\right|.$$

The Bump Locations (Illustration)



Hold true for *any* coprime DFT filter bank design!

The Bump Level

Theorem

Assume the stopband ripples for $G\left(e^{j2\pi f}\right)$ *and* $H\left(e^{j2\pi f}\right)$ *are* ϵ_1 *and* ϵ_2*, respectively. The bump level in coprime DFTFB is bounded by*

$$L \leq \left| F_{00}\left(e^{\frac{j\pi p}{MN}}\right) \right| \leq U,$$

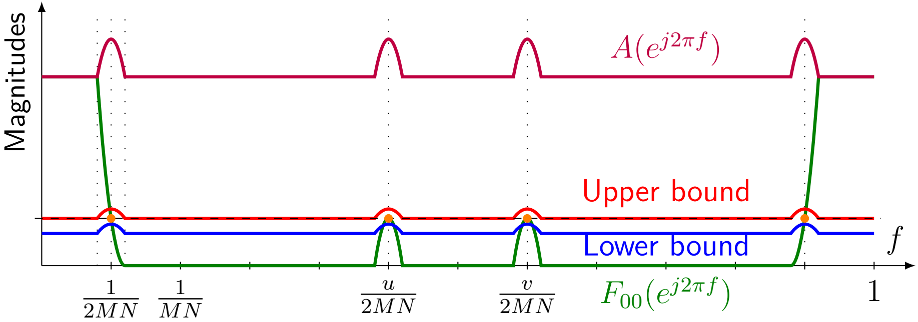
where

$$L = \frac{1}{4}\left(A\left(e^{\frac{j\pi p}{MN}}\right) - \epsilon\right), \qquad U = \frac{1}{4}A\left(e^{\frac{j\pi p}{MN}}\right),$$

$$\epsilon = 2\left(N-2\right)\epsilon_1 + 2\left(M-2\right)\epsilon_2 + \left(M-2\right)\left(N-2\right)\epsilon_1\epsilon_2,$$

$p \in \{\pm 1, u, v\}$.

The Bump Level (Illustration)



Hold true for *any* coprime DFT filter bank design!

Conclusion

- Practical coprime DFT filter bank design
  - The design method in [8] eliminates bumps but neglects overall amplitude responses.
  - Our proposed method provides trade-offs among passband responses, stopband responses, and overall amplitude responses.
- Theoretical bump analysis (true for any comprime DFT filter bank design)
  - Exactly two bumps in one filter.
  - Bump locations can be determined from M and N uniquely.
  - Bump levels are approximately $1/4$ of overall amplitude responses.

References

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