EE5027: Adaptive Signal Processing
Kalman Filters in a Nutshell

Chun-Lin Liu (劉俊麟)

chunlinliu@ntu.edu.tw

Assistant Professor
Department of Electrical Engineering
Graduate Institute of Communication Engineering
National Taiwan University

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Reference

1. S. Haykin, *Adaptive Filter Theory*, Fourth Edition, Prentice Hall, 2001.

The State-Space Model

A process equation: $\mathbf{x}(n+1) = \mathbf{F}(n+1,n)\mathbf{x}(n) + \mathbf{v}_1(n),$

A measurement equation: $\mathbf{y}(n) = \mathbf{C}(n)\mathbf{x}(n) + \mathbf{v}_2(n).$

- $\mathbf{x}(n) \in \mathbb{C}^M$: The state vector at time n
- $\mathbf{F}(n+1,n) \in \mathbb{C}^{M\times M}$: The transition matrix from time n to time $n+1$
- $\mathbf{v}_1(n) \in \mathbb{C}^M$: Process noise
- $\mathbf{y}(n) \in \mathbb{C}^N$: The observation (measurement)
- $\mathbf{C}(n) \in \mathbb{C}^{N\times M}$: The measurement matrix
- $\mathbf{v}_2(n) \in \mathbb{C}^N$: Measurement noise

The State-Space Model

- The process noise $\mathbf{v}_1(n)$ satisfies

$$\mathbb{E}[\mathbf{v}_1(n)] = \mathbf{0}, \qquad \mathbb{E}[\mathbf{v}_1(n)\mathbf{v}_1^H(k)] = \begin{cases} \mathbf{Q}_1(n), & \text{if } n = k \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{1}$$

- The measurement noise $\mathbf{v}_2(n)$ satisfies

$$\mathbb{E}[\mathbf{v}_2(n)] = \mathbf{0}, \qquad \mathbb{E}[\mathbf{v}_2(n)\mathbf{v}_2^H(k)] = \begin{cases} \mathbf{Q}_2(n), & \text{if } n = k \\ \mathbf{0}, & \text{otherwise.} \end{cases} \tag{2}$$

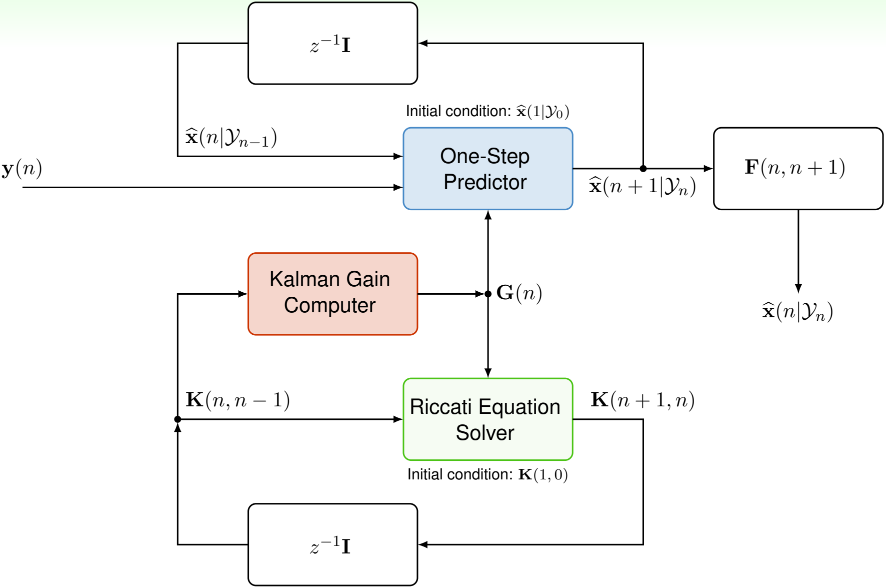
- $\mathbf{v}_1(n)$ and $\mathbf{v}_2(n)$ are uncorrelated ($\mathbb{E}[\mathbf{v}_1(n)\mathbf{v}_2^H(k)] = \mathbf{0}$).
- The initial state $\mathbf{x}(0)$ is uncorrelated with both $\mathbf{v}_1(n)$ and $\mathbf{v}_2(n)$ ($\mathbb{E}[\mathbf{x}(0)\mathbf{v}_1^H(k)] = \mathbf{0}$ and $\mathbb{E}[\mathbf{x}(0)\mathbf{v}_2^H(k)] = \mathbf{0}$).
- If $\mathbf{F}(n+1, n)$ is invertible, then $\mathbf{F}(n, n+1) = \mathbf{F}^{-1}(n+1, n)$.

Problem Statement

Given the state-space model, the observations, and the initial conditions, estimate the state vector with respect to time.

- State-space model: $\mathbf{F}(n, n+1)$, $\mathbf{C}(n)$, $\mathbf{Q}_1(n)$, $\mathbf{Q}_2(n)$.
- Observations: $\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(n)$.
- Initial conditions: $\widehat{\mathbf{x}}(1|\mathcal{Y}_0)$ and $\mathbf{K}(1, 0)$.
- $\widehat{\mathbf{x}}(n+1|\mathcal{Y}_n)$: Predicted estimate of the state vector at time $n+1$, given the observations $\mathcal{Y}_n = \{\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(n)\}$.
- $\widehat{\mathbf{x}}(n|\mathcal{Y}_n)$: Filtered estimate of the state vector at time n, given the observations $\mathcal{Y}_n = \{\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(n)\}$.

A Block Diagram for Kalman Filters



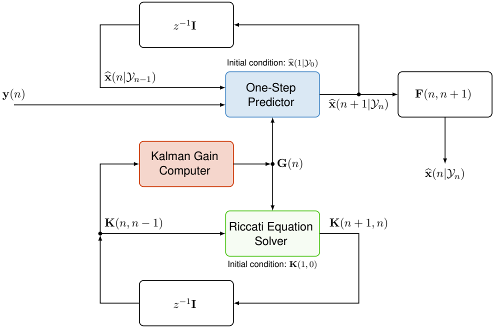
One-Step Predictor (For the State Vectors)

1. The innovations process of $\mathbf{y}(n)$

$$\boldsymbol{\alpha}(n) = \mathbf{y}(n) - \mathbf{C}(n)\widehat{\mathbf{x}}(n|\mathcal{Y}_{n-1}). \quad (3)$$

2. The minimum mean-square-error estimate of $\mathbf{x}(n+1|\mathcal{Y}_n)$ based on the innovations process

$$\widehat{\mathbf{x}}(n+1|\mathcal{Y}_n) = \mathbf{F}(n+1,n)\widehat{\mathbf{x}}(n|\mathcal{Y}_{n-1}) + \mathbf{G}(n)\boldsymbol{\alpha}(n). \quad (4)$$



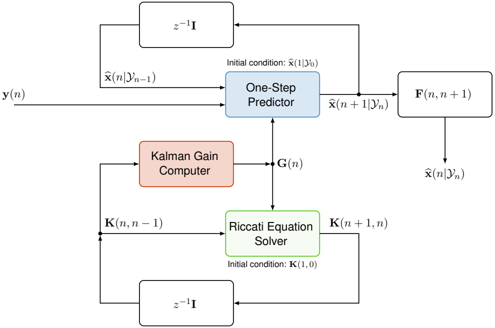
Kalman Gain

1. The correlation matrix of the innovations process $\boldsymbol{\alpha}(n)$

$$\mathbf{R}(n) = \mathbb{E}[\boldsymbol{\alpha}(n)\boldsymbol{\alpha}^H(n)] = \mathbf{C}(n)\mathbf{K}(n, n-1)\mathbf{C}^H(n) + \mathbf{Q}_2(n). \quad (5)$$

2. Kalman gain $\mathbf{G}(n)$

$$\mathbf{G}(n) = \mathbf{F}(n+1, n)\mathbf{K}(n, n-1)\mathbf{C}^H(n)\mathbf{R}^{-1}(n). \quad (6)$$



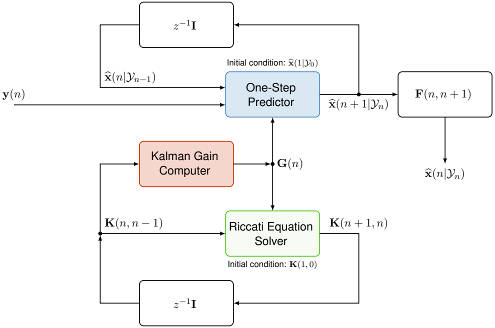
Riccati Equation (For the Error Correlation Matrix)

1. The correlation matrix of $\varepsilon(n) \triangleq \mathbf{x}(n) - \widehat{\mathbf{x}}(n|\mathcal{Y}_n)$:

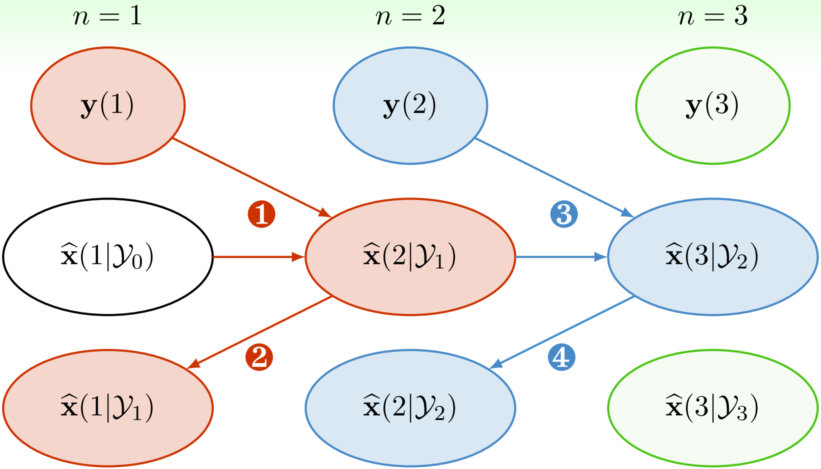
$$\mathbf{K}(n) = \mathbf{K}(n, n-1) - \mathbf{F}(n, n+1)\mathbf{G}(n)\mathbf{C}(n)\mathbf{K}(n, n-1). \tag{7}$$

2. The correlation matrix of $\varepsilon(n+1, n) \triangleq \mathbf{x}(n+1) - \widehat{\mathbf{x}}(n+1|\mathcal{Y}_n)$:

$$\mathbf{K}(n+1, n) = \mathbf{F}(n+1, n)\mathbf{K}(n)\mathbf{F}^H(n+1, n) + \mathbf{Q}_1(n). \tag{8}$$



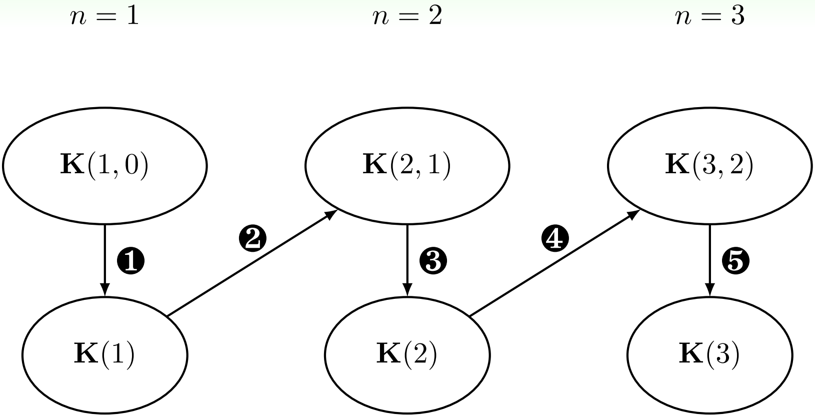
The Estimated State Vectors in Kalman Filters



❶, ❸: One-step predictor

❷, ❹: Filtering ($\widehat{\mathbf{x}}(n|\mathcal{Y}_n) = \mathbf{F}(n, n+1)\widehat{\mathbf{x}}(n+1|\mathcal{Y}_n)$)

The Error Correlation Matrices in Kalman Filters



❶+❷, ❸+❹: Riccati equation solver