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Multi-objective design and tolerance allocation for single- and multi-level systems

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Abstract In this work we develop a method to perform simultaneous design and tolerance allocation for engineering problems with multiple objectives. Most studies in existing literature focus on either optimal design with constant tolerances or the optimal tolerance allocation for a given design setup. Simultaneously performing both design and tolerance allocation with multiple objectives for hierarchical systems increases problem dimensions and raises additional computational challenges. A design framework is proposed to obtain optimal design alternatives and to rank their performances when variations are present. An optimality influence range is developed to aid design alternatives selections with an influence signal-to-noise ratio that indicates the accordance of objective variations to the Pareto set and an influence area that quantifies the variations of a design . An additional tolerance design scheme is implemented to ensure that design alternatives meet the target tolerance regions. The proposed method is also extended to decomposed multi-level systems by integrating traditional sensitivity analysis for uncertainty propagation with analytical target cascading. This work enables decision-makers to select their best design alternatives on the Pareto set using three measures with different purposes. Examples demonstrate the effectiveness of the method on both single- and multi-level systems.

Keywords Tolerance allocation · Multi-objective optimization · Robust design · Sensitivity analysis · Multi-level systems

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Introduction

Design is a multi-objective decision-making process considering manufacturing, cost, aesthetics, usability and many other product attributes. Often time decisions have to be made under various operating and environmental uncertainties. For example, consider the design of a three-story structure composed of several beam elements in supporting a load. The design objectives of this structure are minimizing the overall weight while at the same time minimizing the maximal displacement when carrying the load. These two objectives are contradicting to each other since a lighter structure, without changing the material, generally means smaller cross section areas, and therefore larger displacement. Designers are not only require to provide specifications of all beams in the structure, but also determine the tolerances to these specs due to uncertainties in operation conditions, in material properties, and in manufacturing processes. Impacts of these uncertainties on products' performances in-use result in perceivable product qualities to consumers. An optimal design therefore needs to be determined while ensuring performance requirements are met under uncertainty with the minimal manufacturing cost.

Motivated by the robust design concept in Fowlkes and Creveling (1995), this work aims to obtain the optimal design first and then assign the optimal tolerance to these selected parameters. The optimal design and tolerance allocation for multi-objective engineering problems can be formulated as in Eq. (1). n_f objectives are optimized simultaneous in Eq. (1b) while design variables $\mathbf{x}$ are subjected to variations.

$$\text{given } \mathbf{\Delta f}^T = \{\Delta f_1^T, \ldots, \Delta f_{n_f}^T\} \tag{1a}$$

$$\text{minimize } \mathbf{f}(\mathbf{X}) = \{f_1(\mathbf{X}), \ldots, f_{n_f}(\mathbf{X})\} \tag{1b}$$

$$\text{minimize tolerance cost} \tag{1c}$$

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$$\text{with respect to } \{\mathbf{x}, \mathbf{\Delta x}\} \tag{1d}$$

$$\text{subject to } \mathbf{X}(\mathbf{x}, \mathbf{\Delta x}) \in \mathcal{F} \tag{1e}$$

The uncertainties $\mathbf{X}$ are uniquely determined via input design variables $\mathbf{x}$ and tolerances $\mathbf{\Delta x}$ as shown in Eq. (1e). The input uncertainties $\mathbf{X}$ due to manufacturing or assembly tolerances result in output performance variations including objectives $\mathbf{f}$ and constraints $\mathbf{g}$. A set of maximal acceptable objective variations, $\mathbf{\Delta f}^T$ in Eq. (1a), are given to the designers. $\mathbf{\Delta f}^T$ and the inequality constraints $\mathbf{g}$ form the feasible space $\mathcal{F}$. The tolerance design cost in Eq. (1c) is the cost to meet the desired performance variation ranges. In this study, we consider constraints being satisfied in the worst-case scenario such that $\max\{g(\mathbf{X})\} \leq 0$.

Discussions on Eq. (1) in the literature can be categorized into different focuses, namely tolerance synthesis, tolerance allocation, robust design, and multi-objective design optimization. Table 1 summarizes the domains of interests between this work and similar studies in the literature. In what follows, we will use an output performance characteristic y as a nonlinear function of input variables $\mathbf{x}$ such that $y = f(\mathbf{x})$. The uncertainties $\mathbf{X}$ within the tolerance region result in the variation in y.

Tolerance synthesis considers the impact of uncertainties $\mathbf{X}$ on objective functions, Eq. (1b), or on constraints, Eq. (1e). The choices on the tolerance models determine the method in accessing the impacts of these uncertainties on the output performances. When tolerances are specified as intervals (Parkinson 2000; Wu and Rao 2004), a sensitivity matrix is generally applied (Lin and Zhang 2001), resulting in either the worst case formulation in Eq. (2a) or the statistical analysis in Eq. (2b).

$$\Delta y = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial x_i} \Delta x_i \right| \tag{2a}$$

$$\Delta y = \left[\sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \right)^2 \Delta x_i^2 \right]^{1/2} \tag{2b}$$

Equations in (2), also referred to as tolerance stackup in the literature, imply the function f is relatively linear and may yield large errors when apply to nonlinear problems. Methods such as Monte Carlo simulation or advanced sampling techniques have then be used for tolerance analysis with significant computational expenses; for example, see Early and Thompson (1989),Skowronski and Turner (1997). When tolerances are modeled as random distributions (Martosell et al. 2007; Xu et al. 2005), predominately Gaussian, probability functions of $y = f(\mathbf{X})$ are generally studied. Martosell et al. use Monte Carlo sampling in obtaining these probability functions (Martosell et al. 2007). Xu et al. (2005) and Savage et al. (2006) use the first order reliability method in estimating the propagated variations.

With the tolerance analysis methods in the literature, one can study the best assignments to set the tolerance level of each design variable or parameter such that the overall product performance is satisfactory. From Table 1, tolerance allocation treats input tolerances as design variables and minimizes the tolerance cost in Eq. (1c). Various tolerance cost models have been proposed with the main concept being that the manufacturing cost generally increases with smaller tolerances (Cheng and Maghsoodloo 1995; Yeo et al. 1998). Common cost models include reciprocal model Eq. (3a), reciprocal-power model Eq. (3b), exponential model Eq. (3c) and combinations of above, for example the exponential-reciprocal model Eq. (3d). A, B, C, and r in Eq. (3) are parameters to be fitted with on-site data.

$$\text{Tolerance Cost} = A + B/\mathbf{\Delta} \tag{3a}$$

$$= A + B/\mathbf{\Delta}^r \tag{3b}$$

$$= A + B/\exp[C\mathbf{\Delta}] \tag{3c}$$

$$= A + B/\exp[C\mathbf{\Delta}]\mathbf{\Delta}^r \tag{3d}$$

Cost minimization in tolerance allocation results in a nonlinear programming problem that can be solved by methods such as standard optimization techniques (Ye and Salustri 2003) or genetic algorithms (Martosell et al. 2007; Xue and Ji 2004; Chen and Fischer 2000). Zhou et al. (2001) develop a tailored algorithms using number theoretical method in obtaining the global optimal solutions for NLP in tolerance allocation. Jordaan and Ungerer (2002) use response surface to improve the efficiency of assigning dimensional tolerances.

In addition to tolerance cost, some research in tolerance allocation also considers the loss function as a monetary expression for the cost of product off-the-target quality loss and the variation of the performances from consumers' perspectives. Assuming a target performance y^{target} is also given to the designer. The loss function was first proposed by Taguchi et al. (1989) to quantify product quality degradation. Loss functions of various forms have been used in tolerance allocation, for example see Vasseur et al. (1997), Choi et al. (2000), Caleb Li (2004), Jeang (2007), Cheng and Maghsoodloo (1995)

$$\text{Quality Loss Cost} = k(y - y^{\text{target}})^2 \tag{4}$$

The multi-objective formulation in Eq. (1b) has been studied extensively in optimization literature; however, the application of multi-objective on multi-level complex systems received relatively less attentions. Many studies use weighted sum to combine several objective into one for complex systems (Tappeta and Renaud 1997; McAllister and Simpson 2003; Berrichi et al. 2009; Subramaniam et al. 2001). Li and Haimes (1987) developed a hierarchical generating method with envelop analysis in generating Pareto set of complex systems. Multi-objective genetic algorithms have also been

Table 1 Taxonomy on related research

	Eq. (1a)	Eq. (1b)	Eq. (1c)	Eq. (1d)	Eq. (1e)
Tolerance synthesis		✓			✓
Tolerance allocation			✓	✓	✓
Robust design		✓		✓	✓
Multi-objective optimization		✓		✓	✓

used for Pareto generation in robust design problems (Giassi et al. 2004) and in manufacturing processes (Cheng et al. 2009; Turkcan and Selim Akturk 2003).

Albeit a large quantity of studies have been conducted, they are either insufficient in handling Eq. (1) or have different focus than what we intended to achieve in this work. Recently Li and coworkers developed a sensitivity analysis based uncertainty propagation approach in obtaining the robust optimal design (Li and Azarm 2008). Given acceptable objective ranges, a portion of the original deterministic Pareto set without considering uncertainty will be identified as the robust results. This method is later extended to multi-objective design with tolerance allocation for uncertainty reduction in single and multi-level systems (Li et al. 2009, 2010). However, their method is focused on uncertainty reduction of a given design point on the Pareto set. Design alternatives are not compared and as a result one will have difficulty apply existing methods to select their optimum. Our method, on the other hand, provides decision-makers design alternatives on the Pareto set with different ranking systems such that they can use their own judgements in selecting their optimal design. In what follows, we will describe three main steps of our method in section "Methodology" section, namely the Pareto generation and uncertainty analysis step in section "Pareto generation and uncertainty analysis", the uncertainty reduction with tolerance design step in section "Uncertainty reduction with tolerance design", the uncertainty impact analysis in section "Uncertainty impact analysis via optimality influence range". In section "Extensions to multi-level system", we will extend the proposed method to complex multi-level systems. An anchor design example is demonstrated in section "Anchor design examples" with concluding remarks in section "Conclusion".

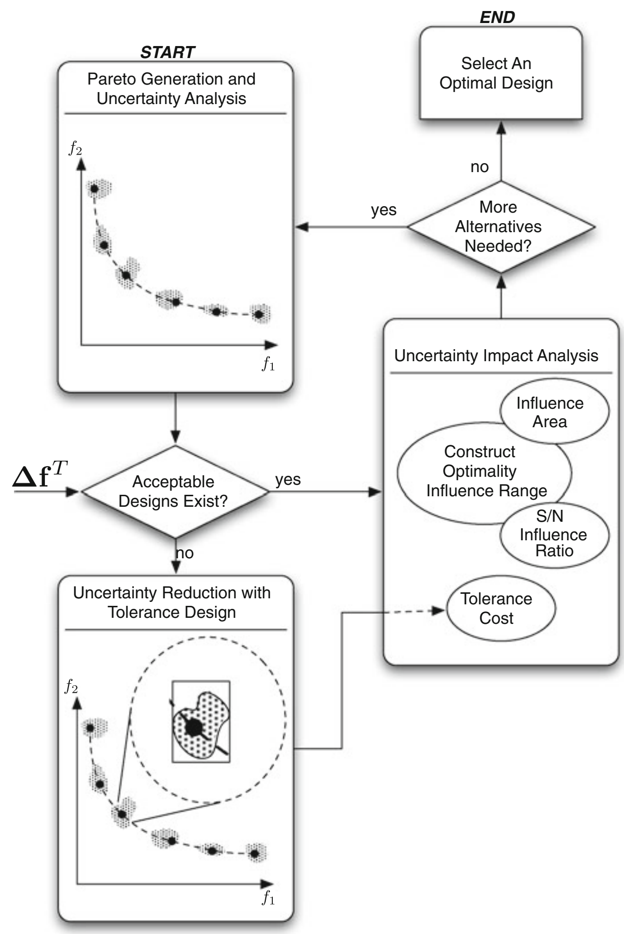


Fig. 1 Flowchart of the proposed method

Methodology

One of the important concepts in multi-objective optimization is that multiple optimal solutions are generally obtained resulting in a Pareto set. Decision-makers are then able to 'pick' his/her own optimum based on personal preferences or past experiences. Although variations change the Pareto set, the same concept should be maintained: *designers should be able to access different design alternatives and investigate their performances and cost in compliance with given acceptable variation ranges.*

Figure 1 illustrates our proposed approach in multi-objective optimization with tolerance allocation when uncertainties exist in design variables or parameters. Shadows in these figures represent uncertainties in reality to be considered. A worst-case Pareto set is first generated using the Eq. (5) with input uncertainty ranges considered as parameters. To ensure a possibly non-convex Pareto set be obtained correctly, we use constraint method in Pareto set generation. The worst-case constraint violations can be approximated as $\mathbf{g}(\mathbf{x}) + |\partial \mathbf{g}/\partial \mathbf{x}|\ \mathbf{\Delta x}$.

$$\begin{aligned} &\text{given} \quad \mathbf{\Delta x} \\ &\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{f}(\mathbf{x}) = \{f_1(\mathbf{x}), \ldots, f_{n_f}(\mathbf{x})\} \\ &\text{subject to } \max\{\mathbf{g}(\mathbf{x}, \mathbf{\Delta x})\} \leq 0 \end{aligned} \tag{5}$$

The performance variation of the points on the Pareto set, denoted as Pareto points, will then be investigated via uncertainty analysis. We introduce the optimality influence range to quantify objective functions variations with details to be discussed in section "Uncertainty impact analysis via optimality influence range". Results of the optimality influence range are the influence areas as well as signal-to-noise influence ratio to determine the performance of a Pareto point under uncertainty. If one or more Pareto points and their influence ranges are acceptable, we will provide selection sequence for decision-makers. However, if the variations of the Pareto points are not acceptable, we then go to the next stage to tolerance design.

$$\begin{aligned} &\text{given} \quad \mathbf{x}^* \in \mathcal{P} \\ &\underset{\mathbf{\Delta x}}{\text{minimize}} \quad \text{Cost}(\mathbf{\Delta x}) \\ &\text{subject to } \mathbf{\Delta f} \leq \mathbf{\Delta f}^T \end{aligned} \tag{6}$$

Equation (6) shows the constrained optimization in tolerance design for each point in the Pareto set $\mathcal{P}$. This method is inert to the choice of cost model in Eq. (6) and in this work we use exponential function from the literature. Theoretically Eq. (6) should include original constraints $\mathbf{g}$. However, such an inclusion will result in significant increase in computation cost to calculate sensitivity of all constraints with respect to all design variables for all subsystems as will be seen in section "Extensions to multi-level system". We experienced great computation cost increase with little gain in the tolerance by including original constraints; therefore in this study we remove all constraints $\mathbf{g}$ in Eq. (5) assuming that tolerance reductions only 'shrink' the influences of uncertainties on constraints with initial tolerances being assigned properly. Unless a much larger tolerances are obtained, our proposed method provides a better resolution especially dealing with complex multi-level systems. A larger tolerances also imply poor initial engineering practice in assigning initial tolerances and is factored out in our study.

After tolerance cost for each $\mathbf{x}^* \in \mathcal{P}$ to satisfy $\mathbf{\Delta f}^T$ is determined, it will become an additional judgement for a design. Designers now have three measures in selecting their optimum, namely

Criterion 1: how much tolerance cost to bring the design to desired objective ranges;
Criterion 2: how well the objective performances vary: influence area;
Criterion 3: how well a design remains optimal: signal-to-noise influence ratio.

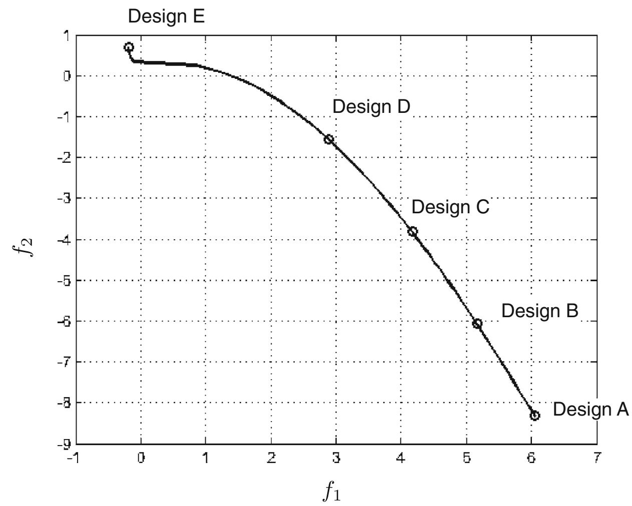


Fig. 2 Pareto set and design alternatives for Eq. (7)

If none of the existing Pareto points are acceptable, more optimal alternatives are generated until at least one is present. The proposed method is general for single- and multi-level systems. Additional challenges raised with multi-level systems will be discussed in section "Extensions to multi-level system". In what follows, we will use an example with two nonlinear objective functions and two design variables and describe each step in more details.

Pareto generation and uncertainty analysis

This step in Pareto generation and uncertainty analysis involves Eqs. (1b-1e) given current design tolerances $\mathbf{\Delta x}$ and acceptable objective variation ranges $\mathbf{\Delta f}^T = [0.03, 0.1]$ in Eq. (1a). We use the bi-obejctive optimization example in Eq. (7) with two design variables as a demonstration of our proposed approach.

$$\begin{aligned} &\text{given } \{\mathbf{\Delta x} = 0.01,\ \mathbf{\Delta f}^T = [0.03, 0.1]\} \\ &\text{min} \quad \{f_1 = R_1 + R_2,\ f_2 = -R_1 R_2\} \\ &\text{w.r.t } \mathbf{x} = [x_1, x_2]^T \end{aligned} \tag{7}$$

$$\begin{aligned} \text{s.t.} \quad & g_1 = \max\{0.8X_1 - X_2 + 0.2\} \leq 0 \\ & g_2 = \max\{-X_1 + X_2 - 1.6\} \leq 0 \\ & R_1 = -0.1 + (4 - 2.1x_1^2 + 0.25x_1^4)x_1^2 \\ & \quad + x_1 x_2 + (4x_2^2 - 4)x_2^2 \\ & R_2 = (x_1 - 1)^2 \\ & -1 \leq \mathbf{x} \leq 1 \end{aligned} \tag{8}$$

The constraint method is used in generating design alternatives on the Pareto set, resulting in five Pareto points as shown in Fig. 2. As can be seen the Pareto set is concave, using weighted-sum method will not get the complex Pareto set. This constraint method in generating Pareto set

Table 2 Five design alternatives on the Pareto set

	$f_1^*(\Delta f_1)$	$f_2^*(\Delta f_2)$	x_1^*	x_2^*
A	6.0609 (0.0523)	−8.3193 (0.1334)	−0.9900	−0.1279
B	5.1692 (0.0611)	−6.0628 (0.1491)	−0.8358	−0.1069
C	4.1688 (0.0643)	−3.8063 (0.1327)	−0.6788	−0.0858
D	2.8879 (0.0593)	−1.5498 (0.0866)	−0.4749	−0.0598
E	−0.1847 (0.0012)	0.7065 (0.0289)	0.1321	0.6987

Table 3 Tolerance cost of design alternatives

	Cost	Δf_1	Δf_2	Δx_1	Δx_2
Design A	62.7348	0.0300	0.0763	0.0058	0.01
Design B	91.4635	0.0300	0.0730	0.0049	0.01
Design C	102.2000	0.0300	0.0618	0.0047	0.01
Design D	85.1231	0.0300	0.0437	0.0051	0.01
Design E	13.4759	0.0012	0.0289	0.0100	0.01

is also used for problems with multi-level structures. Since constraints in Eq. (7) are both linear, worst case robustness can simply obtained using interval arithmetic (Hanss 2004).

Variations of each design alternatives are obtained using linearization with finite differences given input $\mathbf{\Delta x}$ as in Eq. (9). Although Eq. (9) can yield large error for nonlinear objective functions, $\mathbf{\Delta x}$ are generally small for most engineering problems with respect to the function nonlinearity, making Eq. (9) a better compromise between accuracy and efficiency than much more expensive sampling techniques. In addition, the variation ranges obtained via Eq. (9), as shown in Table 2, also enable us in different quantification metrics in section "Uncertainty impact analysis via optimality influence range".

$$\Delta f_j \approx \left| \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \Delta x_i \right| \tag{9}$$

Uncertainty reduction with tolerance design

Step 1 in section "Pareto generation and uncertainty analysis" provides design alternatives on the Pareto set that are robust with their objective variation ranges. If $\mathbf{\Delta f} > \mathbf{\Delta f}^T$ for all Pareto design points, a tolerance design scheme as shown in Eq. (10) is then needed to find the minimal cost in bring all design alternatives within acceptable variation ranges. For demonstration purpose, the exponential cost model in Eq. (10) with A =1,000 and B = 500 is used.

$$\begin{aligned}
&\text{given } \{\mathbf{x}, \mathbf{\Delta f}^T\} \\
&\text{minimize tolerance cost} = \sum_{\mathbf{\Delta x}} A e^{-B\mathbf{\Delta x}} \\
&\text{with respect to } \{\mathbf{\Delta x}\} \\
&\text{subject to } \mathbf{\Delta f} \leq \mathbf{\Delta f}^{\mathrm{T}}
\end{aligned} \tag{10}$$

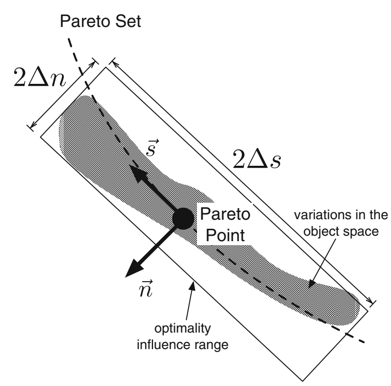


Fig. 3 Optimality influence range

Table 3 lists the tolerance costs for all design alternatives in Fig. 2 to meet target variation range in Eq. (7). As can be seen Design C has the highest tolerance cost while Design E has the lowest. All design alternatives except Design E has Δf_1 reaching the maximal acceptable tolerance 0.03. The input tolerance Δx_1 is reduced in Design A to Design D to meet $\mathbf{\Delta f}^T$. Design E allows both input tolerances to their maximal value 0.01 without violating $\mathbf{\Delta f}^T$.

Based on the result in Table 3 one might expect Design E being the best choice among all design alternatives. However in this work we suggest that performances of design alternatives under uncertainty be considered as well as cost in judging a design. In the following, we will introduce other measures in analyzing the performance of a design under uncertainty.

Uncertainty impact analysis via optimality influence range

Design alternatives on a Pareto set are preferred if a design has good tendency to remain on the Pareto set within the prescribed tolerance regions. In this research we define the optimality influence range in Fig. 3 that quantifies the consequences of design variations on the objectives. For a Pareto point, its objective variations due to $\mathbf{\Delta x}$ in the design space are shown with shadows representing uncertainty. The optimality influence range is a hyper-rectangle that encloses all the objective variations with an angle. Although the objective variations rarely have rectangle shapes due to nonlinearity of the functions, the optimality influence range is able to capture behaviors of objective functions under uncertainty. The unit vector $\vec{s}$ is tangent to the Pareto set on the design point and $\vec{n}$ is the vector perpendicular to $\vec{s}$. Since variations along $\vec{s}$ direction tend to 'stay' on the Pareto set while along $\vec{n}$ tend to deviate from the Pareto set, we define $\vec{s}$ as signal vector and $\vec{n}$ as noise vector. The lengths $\Delta s = \vec{s} \cdot \mathbf{\Delta f}$ and $\Delta n = \vec{n} \cdot \mathbf{\Delta f}$ are the signal variation and the noise variation, respectively.

An important criterion to describe the objective function variations is that differences in objective variations should be captured. In Fig. 4, two design scenarios, A and B, on the Pareto set are shown with their variations in the object space. Both designs have different performances variations and different tendencies to remain on the Pareto set: Scenario B has better optimality 'signal' and less 'noise' than scenario A. In previous work (Li and Azarm 2008), these two design will end up having the same objective variation range (OVR) due to the the fact that mathematical definition of OVR is unable to describe their variations away from the optima. This indicates the inability of OVR in quantifying variations in the objective space due to uncertainties in the variables/parameters. Our proposed optimality influence range extends the concept of OVR with better quantification of the variations of a Pareto point in the objective space. In this work we assume all problems have been properly scaled such that the difference between Fig. 4a and b are not results of improper scaling.

Table 4 Optimality influence ranges of design alternatives

	$\vec{s}$	$\vec{n}$	Δs	Δn
A	(0.36,−0.93)	(0.93,0.36)	0.1433	0.0003
B	(−0.38,0.93)	(0.93,0.38)	0.1612	0.0001
C	(−0.44,0.90)	(0.90,0.44)	0.1475	0.0002
D	(−0.57,0.82)	(0.82,0.57)	0.1049	0.0003
E	(−0.01,1.00)	(1.00,0.01)	0.0289	0.0014

Table 4 list the optimality influence ranges for all design alternatives. Two important criteria can be extracted from Table 4, namely the optimality influence area for the output variations and the optimality influence signal-to-noise ratio for the tendency to remain on the Pareto set. These two criteria will be introduced in sections "Influence area" and "Signal-to-noise influence ratio".

Influence area

Area covered by the optimality influence range, denoted as the influence area, is used to compare two Pareto design alternatives. A design with smaller influence area generally means smaller objective variation and should be preferred by designers. This criterion can simply be computed using Eq. (11) based on the influence range information in Table 4.

$$\text{influence area} = \Delta s \times \Delta n \tag{11}$$

Signal-to-noise influence ratio

The unit vector $\vec{s}$ is defined as the signal direction since variations along $\vec{s}$ tend to stay on the Pareto set while variations along the noise vector $\vec{n}$ move away from the Pareto set. With the signal and noise unit vector being defined, we introduce the optimality influence signal-to-noise (S/N) ratio in

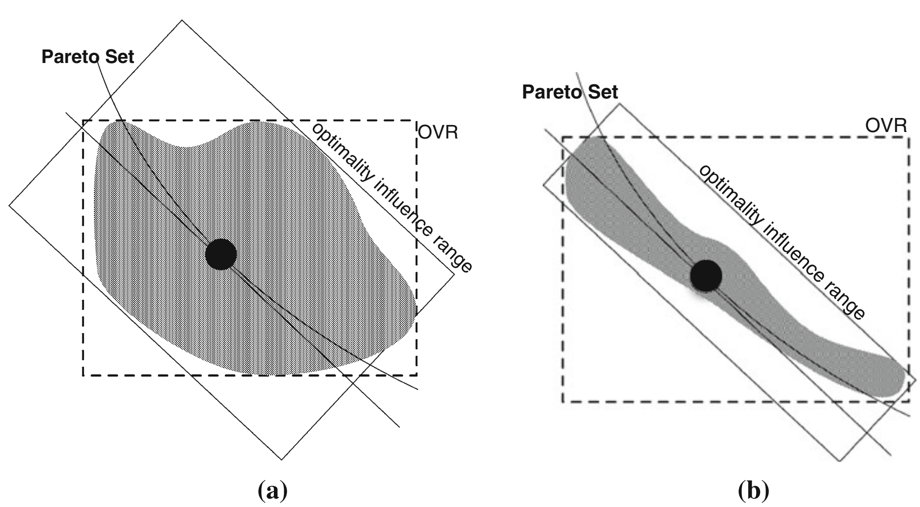


Fig. 4 Comparisons between OVR in Li and Azarm (2008) and influence range. **a** Scenario A, **b** scenario B

Table 5 Influence area and S/N ratio of design alternatives

	Area ($\times 10^{-4}$)	SN ($\times 10^{3}$)
Design A	0.4750	0.4324
Design B	0.1097	2.3685
Design C	0.3513	0.6191
Design D	0.3102	0.3549
Design E	0.4042	0.0206

Table 6 Design alternatives on the Pareto using ATC

	$f_1^*(\Delta f_1)$	$f_2^*(\Delta f_2)$	x_1	x_2
A	6.0608 (0.0530)	−8.3193 (0.1360)	−0.9900	−0.1279
B	5.1691(0.0617)	−6.0628 (0.1503)	−0.8358	−0.1069
C	4.2131 (0.0631)	−3.8962 (0.1298)	−0.6860	−0.1100
D	3.2683 (0.0169)	−1.5498 (0.1133)	−0.6019	0.9900
E	−0.1350 (0.0095)	0.4179 (0.0254)	0.2366	0.6918

Eq. (12) as the third criterion in selecting design alternatives.

$$\text{influence S/N ratio} = \frac{\Delta s}{\Delta n} \tag{12}$$

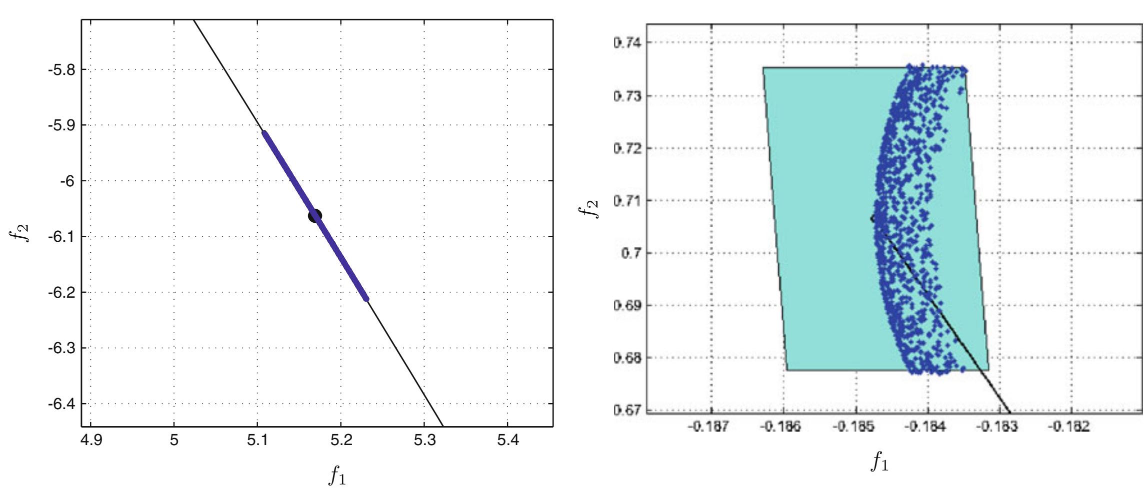
The influence S/N ratio differs from allowable increase/decrease in linear programming literature in that we focus on the compliance of a design to remain optimal (Neufville 1990). By doing so, we allow the design to be varied and a design considering good performance is one that remains on the Pareto set under uncertainty. The allowable increase/decrease, on the other hand, focus on the limit of uncertainty by which the optimal design starts to change. A design with a large S/N ratio tends to remain on the Pareto set: they remain optimal but at different design point. Therefore our S/N ratio provides a better performance indication of how design behaves under uncertainty and how much attention designers should pay to alter the design.

Table 5 lists the influence area and S/N ratio of all design alternatives A to E. As can be seen Design B has the smallest influence area with the largest S/N ratio. Design A, on the other hand, has the largest influence area while E has the worst signal-to-noise ration given the same input variations. Figure 5a and b shows the influence range of Design B and E, respectively. As can be seen the variations in $\mathbf{\Delta x}$ result in good accordance to the Pareto set for B but a much larger objective variations in E, resulting in a large influence area and poor S/N ratio. If E is selected, one might expect a large portion of design outcomes that are not 'optimal'.

Short summary

Three measures in selecting design alternatives to meet Eq. (7) have been constructed. Decision-makers can now use these criteria in choosing their design based on their own criteria. With data from Tables 3, 4, and 5, comparisons between design alternatives can be made. Figure 6 list three criteria side-to-side to show their relative importance. Each criterion list design alternatives from left to right as A to E. Directions of arrows in three criteria means whether the criterion is the larger the better ↑ or the smaller the better ↓.

A designer with cost as the number one concern might choose Design E. Unfortunately Design E is also the one with the worst influence S/N ratio. On the other hand, if whether the system remains optimal is the number one interest, one will eventually select Design B over the rest of the design alternatives. In this method we develop selection criteria for designers and leave the final design choice to the decision-maker.



(a) **(b)**

Fig. 5 Optimality influence ranges. **a** design B, **b** design E

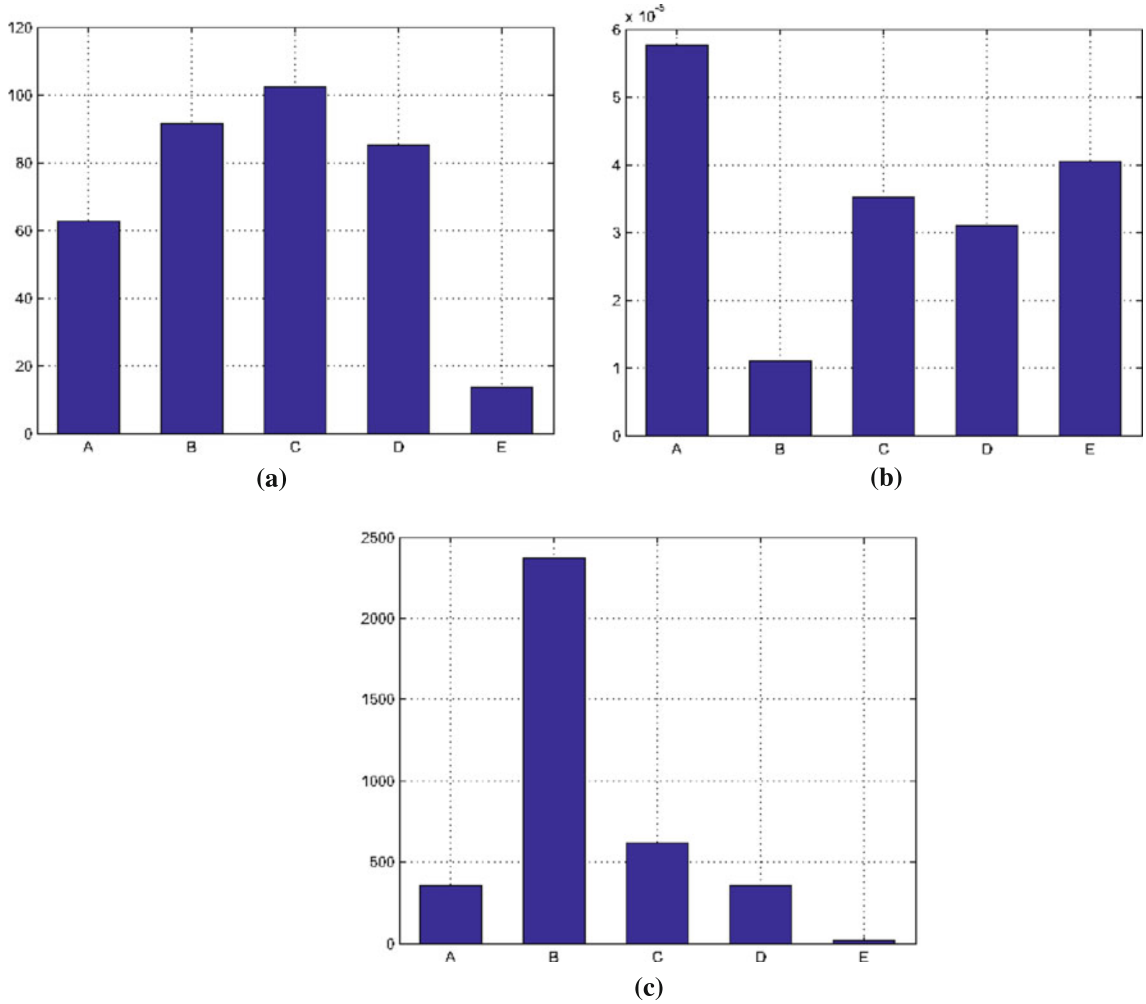


Fig. 6 Comparisons of design alternatives. **a** cost (↓), **b** influence area (↓), **c** influence S/N ratio (↑)

Table 7 Optimality influence range, area, S/N ratio, and tolerance cost of ATC design alternatives

	$\vec{s}$	$\vec{n}$	Δs	Δn	Area($\times 10^{-4}$)	S/N Ratio($\times 10^{4}$)	Cost	Δf_1	Δf_2	Δx_1	Δx_2
A	(0.36,−0.93)	(0.93,0.36)	0.15	$1.00e^{-4}$	0.2141	0.0996	65.8670	0.0300	0.0769	0.0057	0.0100
B	(−0.38,0.93)	(0.93,0.38)	0.16	$1.39e^{-5}$	0.0225	1.1735	94.7718	0.0300	0.0731	0.0049	0.0100
C	(−0.44,0.90)	(0.90,0.44)	0.14	$1.53e^{-5}$	0.0220	0.9461	93.3756	0.0300	0.0610	0.0049	0.0100
D	(−0.57,0.82)	(0.82,0.57)	0.10	$5.04e^{-2}$	51.7992	0.0002	16.3149	0.0119	0.1000	0.0100	0.0093
E	(−0.01,1.00)	(1.00,0.01)	0.03	$9.30e^{-3}$	2.3740	0.0003	13.4759	0.0095	0.0254	0.0100	0.0100

Table 8 Anchor design parameters

E	Ft_{allow}	ρ	σ_{allow}	Δm	$\Delta\delta_1$
70 GPa	400 N	270 kg/m^3	127 MPa	0.04 Kg	0.075 m

Although we use an analytical example to demonstrate our proposed method step by step, this method can readily be applied to much more complex systems. In what follows, we will discuss the challenges and solution methods in extending the proposed work to complex multi-level systems.

Extensions to multi-level system

Large-scale design problems are high dimensional and deeply-coupled in nature. The complexity of such large-scale systems prevents designers from solving them as a whole.

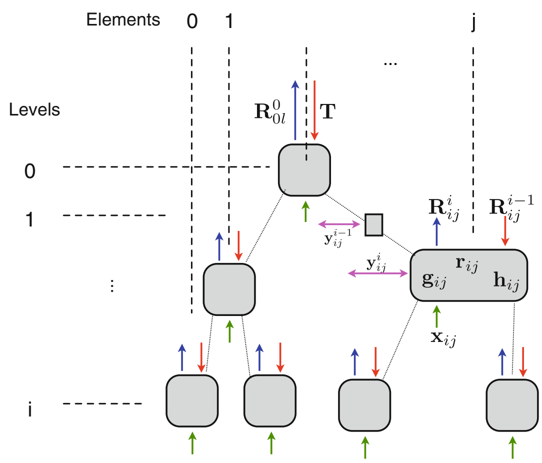


Fig. 7 Hierarchical structure of ATC

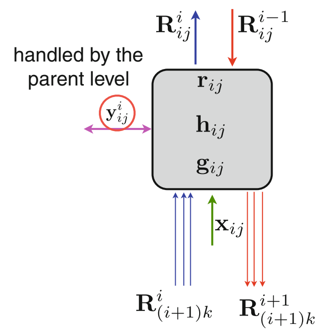


Fig. 8 Subsystem ij within ATC

Analytical target cascading (ATC) is a systematic approach in solving decomposed large-scale systems that has solvable subsystems (Kim et al. 2003; Michelena et al. 2003). Let a large-scale system be decomposed into subsystems of different hierarchies (levels) as shown in Fig. 7, which has i levels with j elements at each level. Design targets $\mathbf{T}$ are assigned to the top-level system. Design variables of the element ij, also called local variables, are $\mathbf{x}_{ij}$. Local constraints of element ij are $\mathbf{g}_{ij}$ and $\mathbf{h}_{ij}$ for inequality and equality constraints, respectively. $\mathbf{R}^i_{ij}$ are the responses from the element ij to the $(i-1)$th level while $\mathbf{R}^{i-1}_{ij}$ are the targets to the element ij from the $(i-1)$th level where $\mathbf{R}^i_{ij} = \mathbf{r}_{ij}(\mathbf{x}_{ij})$. $\mathbf{y}^i_{ij}$ are linking variables between elements at the same level i. $\mathbf{S}_k$ is a binary selection matrix to define which linking variables in $\mathbf{y}^i_{(i+1)j}$ are elements of $\mathbf{y}^i_{(i+1)k}$ in child k.

$$
\begin{aligned}
&\underset{\{\bar{\mathbf{x}}_{ij},\mathbf{y}^i_{(i+1)j},\varepsilon^R_{ij},\varepsilon^y_{ij}\}}{\text{minimize}} \|\mathbf{R}^0_{0l}-\mathbf{T}\| + \sum_{i=0}^{N-1}\sum_{j\in E_i}\varepsilon^R_{ij} + \sum_{i=0}^{N-1}\sum_{j\in E_i}\varepsilon^y_{ij} \\
&\text{s. t.} \sum_{k\in\mathcal{C}_{ij}} \|\mathbf{w}^R_{(i+1)k}\circ(\mathbf{R}^i_{(i+1)k}-\mathbf{R}^{i+1}_{(i+1)k})\|_2^2 \le \varepsilon^R_{ij} \\
&\sum_{k\in\mathcal{C}_{ij}} \|\mathbf{S}_k\mathbf{w}^y_{(i+1)j}\circ(\mathbf{S}_k\mathbf{y}^i_{(i+1)j}-\mathbf{y}^{i+1}_{(i+1)k})\|_2^2 \le \varepsilon^y_{ij} \qquad (13) \\
&\mathbf{g}_{ij}(\bar{\mathbf{x}}_{\mathbf{ij}}) \le 0, \quad \mathbf{h}_{ij}(\bar{\mathbf{x}}_{\mathbf{ij}}) = 0, \quad \mathbf{R}^i_{ij} = \mathbf{r}_{ij}(\bar{\mathbf{x}}_{\mathbf{ij}}), \\
&\bar{\mathbf{x}}_{\mathbf{ij}} = \left[\mathbf{x}^i_{ij}, \mathbf{y}^i_{ij}, \mathbf{R}^i_{(i+1)k_1}, \ldots, \mathbf{R}_{(i+1)k^i_{\mathcal{C}_{ij}}}\right]^T \\
&\forall j\in E_i,\ i=0,1,\ldots,N
\end{aligned}
$$

The decomposed problem in Fig. 7 tries to solve each subproblem individually while at the same time ensures the consistency between subsystems. Equation (13) shows the overall optimization formulation of ATC as a whole. The objective of Eq. (13) includes minimizing the Euclidean norm of the difference between targets and responses as well as minimizing the consistency between levels and between components. The feasibility of Eq. (13) requires satisfying the relaxed consistency constraints to the original undecomposed problem. This relaxation can be imposed as designers' preferences by assigning different weights $\mathbf{w}^R_{(i+1)k}$ and $\mathbf{w}^y_{(i+1)j}$ via weighting update method in Michalek and Papalambros (2005).

Figure 8 shows the information flows in and out of each subsystem ij in Fig. 7. In addition to the local design variables $\mathbf{x}_{ij}$, the responses from the $(i+1)$th level, the targets passed down from the $(i-1)$th level , and the linking variables are all the inputs to subsystem ij. Outputs from ij are the responses to the upper level, the targets to the lower level, and the linking variables values.

The goal for the element ij is to match the target from the $(i-1)$th level while keeping the consistency between itself and the $(i+1)$th level. After assuming all equality constraints are removed explicitly or implicitly, we can obtain the optimization problem for element ij as shown in Eq. (14). The consistency constraints in Eq. (14) have been moved to objective function by applying monotonicity principals with respect to ε^R_{ij} and ε^y_{ij}. $\bar{\mathbf{x}}_{ij}$ in Eq. (14) is the vector of all inputs for element ij.

$$
\begin{aligned}
&\underset{\{\bar{\mathbf{x}}_{ij},\mathbf{y}^i_{(i+1)j}\}}{\text{minimize}} \|\mathbf{w}^R_{ij}\circ(\mathbf{R}^i_{ij}-\mathbf{R}^{i-1}_{ij})\|_2^2 + \|\mathbf{S}_j\mathbf{w}^y_{ip}\circ(\mathbf{S}_j\mathbf{y}^{i-1}_{ip}-\mathbf{y}^i_{ij})\|_2^2 \\
&+\sum_{k\in\mathcal{C}_{ij}} \|\mathbf{w}^R_{(i+1)k}\circ(\mathbf{R}^i_{(i+1)k}-\mathbf{R}^{i+1}_{(i+1)k})\|_2^2 \qquad (14) \\
&+\sum_{k\in\mathcal{C}_{ij}} \|\mathbf{S}_k\mathbf{w}^y_{(i+1)j}\circ(\mathbf{S}_k\mathbf{y}^i_{(i+1)j}-\mathbf{y}^{i+1}_{(i+1)k})\|_2^2 \\
&\text{subject to } \mathbf{g}_{ij}(\bar{\mathbf{x}}_{ij}) \le 0, \quad \mathbf{R}^i_{ij} = \mathbf{r}_{ij}(\bar{\mathbf{x}}_{ij}), \\
&\bar{\mathbf{x}}_{\mathbf{ij}} = \left[\mathbf{x}^i_{ij}, \mathbf{y}^i_{ij}, \mathbf{R}^i_{(i+1)k_1}, \ldots, \mathbf{R}_{(i+1)k^i_{\mathcal{C}_{ij}}}\right]^T
\end{aligned}
$$

Let us use the decomposed problem of Eq. (7) and describe the challenges might encountered in each step of our proposed method in Fig. 1 when integrating with ATC. Figure 9 shows the decomposed bi-level structure of Eq. (7) with one top level supersystem and two lower level

Table 9 Anchor design alternatives on the Pareto set

AIO (ATC)	m	δ_1	d_1	d_2	d_3	d_{r1}	d_{r2}
A (A′)	6 (6)	0.0367 (0.0336)	0.0321 (0.0327)	0.0322 (0.0262)	0.0272 (0.0325)	0.0041 (0.0027)	0.0025 (0.0021)
B (B′)	7 (7)	0.0270 (0.0240)	0.0346 (0.0356)	0.0348 (0.0278)	0.0294 (0.0353)	0.0046 (0.0028)	0.0028 (0.0021)
C (C′)	8 (8)	0.0207 (0.0176)	0.0370 (0.0384)	0.0372 (0.0286)	0.0314 (0.0382)	0.0050 (0.0033)	0.0031 (0.0021)
D (D′)	9 (9)	0.0164 (0.0134)	0.0392 (0.0412)	0.0394 (0.0292)	0.0333 (0.0410)	0.0055 (0.0029)	0.0034 (0.0021)
E (E′)	10 (10)	0.0133 (0.0153)	0.0413 (0.0394)	0.0415 (0.0300)	0.0351 (0.0435)	0.0060 (0.0192)	0.0037 (0.0021)

Table 10 Optimality influence range, area, S/N ratio, and tolerance cost of anchor design alternatives

	Δm	$\Delta\delta_1(\times 10^{-4})$	$\vec{s}$	Δs	$\Delta n(\times 10^{-4})$	Area$(\times 10^{-6})$	SN	Cost
A (AIO)	0.0409	5.6581	(−1.00,0.0138)	0.0409	1.5979	6.5325	255.8366	465.9257
A′ (ATC)	0.0410	5.6257	(−1.00,0.0138)	0.0410	2.3957	9.8180	171.0655	481.2533
B (AIO)	0.0440	4.1348	(−1.00,0.0094)	0.0440	1.5220	6.6911	288.8578	461.4271
B′ (ATC)	0.0441	4.1312	(−1.00,0.0094)	0.0441	2.1153	9.3214	208.3197	479.9802
C (AIO)	0.0469	3.0520	(−1.00,0.0065)	0.0469	1.2830	6.0218	365.8516	456.6822
C′(ATC)	0.0470	3.0583	(−1.00,0.0065)	0.0470	1.8060	8.4954	260.4751	470.2197
D (AIO)	0.0494	2.5615	(−1.00,0.0052)	0.0494	1.3798	6.8137	357.8867	452.4890
D′(ATC)	0.0495	2.5966	(−1.00,0.0052)	0.0495	1.7861	8.8385	277.0628	468.5949
E (AIO)	0.0539	1.9631	(−1.00,0.0042)	0.0539	1.2084	6.5144	446.1132	447.6505
E′(ATC)	0.0540	1.9268	(−1.00,0.0042)	0.0540	1.4120	7.6266	382.5386	453.7727

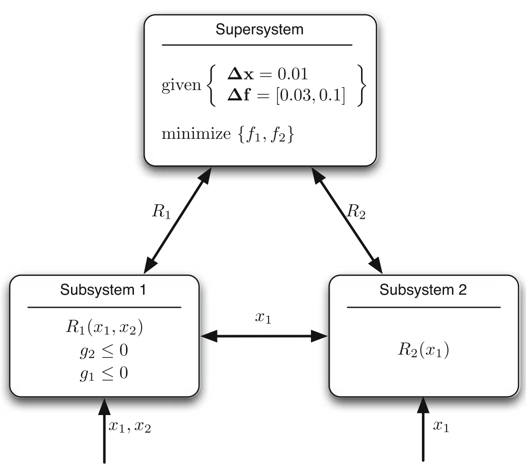


Fig. 9 Bi-level structure of Eq. (7)

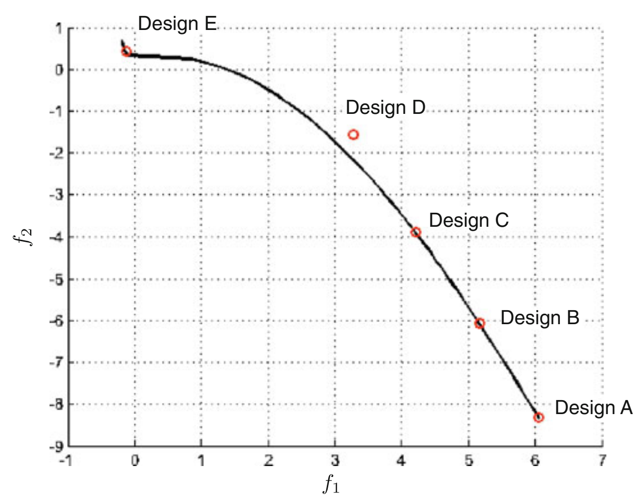


Fig. 10 Design alternatives on the Pareto set using ATC

subsystems. Responses between levels are R_1 and R_2. x_1 is the linking variables between subsystems.

Pareto set generation

The constraint method in generating Pareto set with all-in-one structure can be applied to the top level of the decomposed system. If some objectives are from subsystems, one can either apply the constrained method to these subsystems or consider these objectives are 'rebalanced' up to the top level and apply constrained method on the top system only. Figure 10 shows the ATC design alternatives with values in Table 6 compared with the all-in-one Pareto set. Although such an extensions can be straightforward, direct implementations might cause several difficulties including

1. Design alternatives obtained using ATC might deviate from the all-in-one Pareto set: theoretically ATC will yield identical results as AIO (Michelena et al. 2003),

small inconsistency might cause large deviations between responses and result in unignorable deviations. Design D in Fig. 10 shows such a case. Although one might keep increasing the inconsistency weights in D, the design progression can be very slow.

2. Inconsistency might result in infeasible design alternatives: the choice on the weighting update convergence tolerance is crucial between obtaining a feasible design and reaching an optimal solution. If weighting update tolerance is very small, large weights are required to reduce inconsistency resulting in slow design progressing of the entire system toward the true optimum. On the other hand, small weights comes from larger acceptable convergence tolerance. High inconsistency level will make design infeasible. In this work we use 10^{-4} times the objective variation range as the weighting update tolerance as a balance between inconsistency and optimality.
3. Objective function weights and inconsistency weights needs to be assigned separately: weights between objective functions result in a single 'composite' objective that can be implemented into Eq. (14). One should separate these objective weights from inconsistency weights since they have completely independent functionalities. Inconsistency weights are assigned to obtain a convergent and consistent design while objective weights are used to obtain one Pareto point in the pareto set. Altering objective weights will result in different Pareto solutions and their corresponding inconsistency weights still need to be computed using weighting update method in Michalek and Papalambros (2005). In this work we use constraint method to avoid possibly non-convex Pareto set and also eliminate the need to assign objective weights.

Uncertainty analysis

The linking variables in multi-level systems create additional challenges in analyzing the accumulation of tolerances through levels. In Sobieszczanski-Sobieski (1990), a set of sensitivity equations is formed for systems with linking variables with the solutions being the final sensitivity of the system. These sensitivity equations are used in parallel with ATC in this work. For the element ij in Fig. 8, the system responses $\mathbf{R}_{ij}^{i}$ are functions of input variables $\mathbf{x}_{ij}$, linking variables $\mathbf{y}_{ij}$ and its lower level responses $\mathbf{R}_{(i+1)k}^{i}$. The variation of the output $\mathbf{R}_{ij}^{i}$ from all input uncertainties can be obtained using Eq. (15). This equation is implemented when ATC convergent results are available. Given subsystem models, all partial derivatives in Eq. (15) are known. The variations of the linking variables $\Delta\mathbf{y}_{ij}$, the variation of the lower level responses $\Delta\mathbf{R}_{(i+1)k}^{i}$ and the variation of the output responses for subsystem ij, $\Delta\mathbf{R}_{ij}^{i}$ are unknown to subsystem ij. These unknown quantities are the responses of other subsystems, resulting in a set of sensitivity equations with only one solution. Equation (16) is the sensitivity equation of the ATC problem in Fig. 9 after unknowns are aggregated into a matrix form.

$$\Delta\mathbf{R}_{ij}^{i} = \frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{x}_{ij}}\Delta\mathbf{x}_{ij} + \frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{y}_{ij}}\Delta\mathbf{y}_{ij} + \frac{\partial \mathbf{r}_{ij}}{\partial \mathbf{R}_{(i+1)k}^{i}}\Delta\mathbf{R}_{(i+1)k}^{i} \tag{15}$$

$$\begin{pmatrix} \Delta f_1 \\ \Delta f_2 \\ \Delta R_1 \\ \Delta R_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{\partial f_1}{\partial R_1} & -\frac{\partial f_1}{\partial R_2} \\ 0 & 1 & -\frac{\partial f_2}{\partial R_1} & -\frac{\partial f_2}{\partial R_2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\partial R_1}{\partial x_1} & \frac{\partial R_1}{\partial x_2} \\ \frac{\partial R_2}{\partial x_1} & 0 \end{pmatrix} \times \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \end{pmatrix} \tag{16}$$

Optimality influence range and tolerance cost

Obtaining the optimality influence range requires the unit vector $\vec{s}$ tangent to the Pareto set at the design point. In most practical cases $\vec{s}$ needs to be calculated using finite differences. Although weighted sum approach in Pareto generation have the tangent vector being the weights, weighted sum suffers from the inability to obtain the complete Pareto set and therefore not suggested in this work. Alternatively, one can also use the method described in Utyuzhnikov et al. (2008).

Table 7 and Fig. 11 show the results of optimality influence range as well as results of tolerance cost to meet the desired variation ranges. Similar conclusions to AIO can be made using ATC with Design E being the most economical design and Design B being the design with the best performances at the optimal.

Anchor design examples

In this example we consider the design of an anchor system from Allison et al. (2005) with three cantilever beams of equal length $L = 1$ m with solid circular cross section of diameters d_1, d_2, and d_3, for beam 1, 2, and 3, respectively. Two solid circular rods with diameters d_{r1} and d_{r2} attached to the beam with pin joints as shown in Fig. 12. A downward force F is applied at beam 1 with $F = 1{,}000$ N. This three-level anchor system is capable of distributing extensive loads to prevent failure.

The optimal design of the anchor system tends to find the cross sections of all beams and rods with the minimal overall anchor weight and the minimal beam 1 deflection without static failure. In addition, manufacturing and assembly variations result in dimensional instability such that all cross sections are subjected to variations $\Delta\mathbf{x} = 0.1$ mm. Equation (17) shows the mathematical representation of the anchor design with a known target performance variation

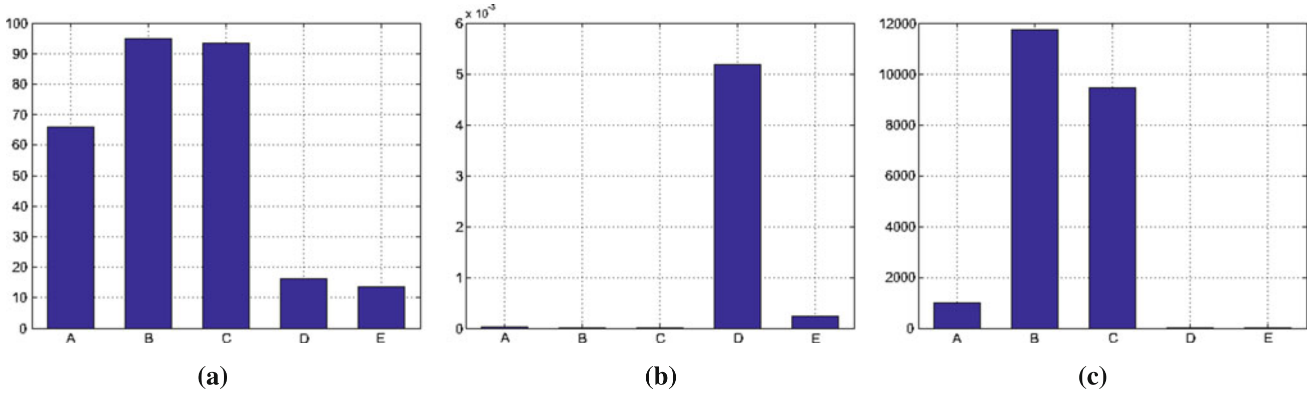
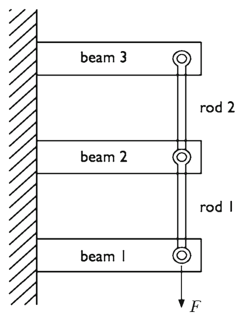


Fig. 11 Comparisons of design alternatives in ATC. **a** cost (↓), **b** influence area (↓), **c** influence S/N ratio (↑)

Fig. 12 Anchor design



range $\Delta m = 0.04\,\text{kg}$ and $\Delta\delta_1 = 7.5\,\text{mm}$.

$$\begin{aligned}
&\min \quad \mathbf{f} = \{m, \delta_1\} \qquad (17)\\
&\text{w.r.t} \quad \{\mathbf{x}, \mathbf{\Delta x}\}\\
&\text{s.t.} \quad g_{1i}(\mathbf{x}, \mathbf{\Delta x}) = \max\{\sigma_{bi}(\mathbf{x}, \mathbf{\Delta x})\} \le \sigma_{\text{allow}} \quad i = 1, 2, 3\\
&\qquad g_{2i}(\mathbf{x}, \mathbf{\Delta x}) = \max\{\sigma_{rj}(\mathbf{x}, \mathbf{\Delta x})\} \le \sigma_{\text{allow}} \quad i = 1, 2\\
&\qquad g_{3i}(\mathbf{x}, \mathbf{\Delta x}) = \max\{Ft_i(\mathbf{x}, \mathbf{\Delta x})\} \le Ft_{\text{allow}} \quad i = 1, 2, 3\\
&\qquad \text{where } \mathbf{x} = [d_1, d_2, d_3, d_{r1}, d_{r2}]
\end{aligned}$$

Inequality constraints are formulated as the worst case considering both beam stress σ_b and rod stress σ_r. Ft are transmitted force at each beams. Material properties are listed in Table 8. The stiffness of beam i, K_{bi}, the stiffness of rod i, K_{ri}. the beam area moment of inertia I_{bi}, and the beam deflections δ_i are computed using

$$I_{bi} = \frac{\pi}{64} d_i^4, \quad A_{rj} = \frac{\pi}{4} d_{rj}^2, \quad K_{bi} = \frac{3EI_{bi}}{L_b^3},$$

$$K_{rj} = \frac{EA_{rj}}{L_{rj}}$$

$$\delta_i = \frac{1}{K_{bi}}(F_i - F_{i+1}), \quad \delta_{i+1} = \delta_i - \frac{F_{i+1}}{K_{ri}},$$

$$\sigma_{bi} = \frac{32K_{bi}\delta_i L_b}{\pi d_i^3}$$

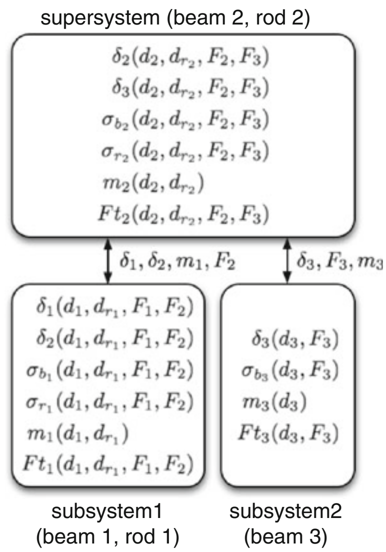


Fig. 13 Anchor design decomposed

$$\sigma_{ri} = \frac{F_{i+1}}{A_{rj}}, \quad m_i = \frac{\pi}{4}(d_i^2 L_b + d_{rj}^2 L_{rj}),$$

$$Ft_i = F_i - F_{i+1}$$

In this design problem, decision-makers are interested in the solutions to the following questions:

Q1). Given current manufacturing tolerances, do acceptable design exist?

Q2). If more than one acceptable design exist, which one is the best?

Q3). If no acceptable design exist, which one is the most beneficial to achieve the target performance variations?

For demonstration purpose, this problem is also decomposed as a bi-level system shown in Fig. 13 with beam 2 and rod 2 being at the top level as the supersystem, beam 1 and rod 1 being the subsystem 1 at the lower level, with beam 3

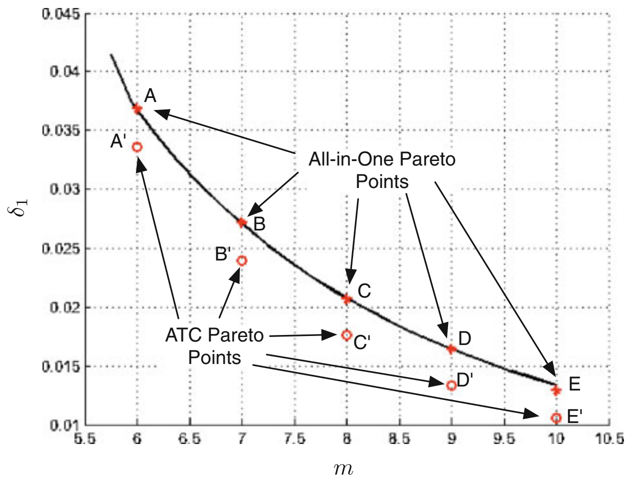


Fig. 14 Anchor design alternatives on the Pareto set

be the subsystem 2 at the lower level. The design problem for each subsystem is shown as Eqs.(18–20).

The top level supersystem tries to achieve the target T in Eq. (18a) while minimizing the inconsistencies between the top and lower subsystems in terms of deflections (18b), forces (18c) and structural mass (18d). The constraint (18i) is created to obtain the Pareto point with minimal anchor mass by varying the values of m_{allow}. $d_{rj} = 0$ and $F_{i+1} = 0$ at the top level.

supersystem

$$\min\ (\delta_1 - T)^2 \tag{18a}$$
$$+w_1(\delta_1-\delta_1^L)^2+w_2(\delta_2-\delta_2^L)^2+w_3(\delta_3-\delta_3^L)^2 \tag{18b}$$
$$+w_4(F_2 - F_2^L)^2 + w_5(F_3 - F_3^L)^2 \tag{18c}$$
$$+w_6(m_1 - m_1^L)^2 + w_7(m_3 - m_3^L)^2 \tag{18d}$$
$$\text{w.r.t}\ \mathbf{x} = [d_2, d_{r2}, F_2, F_3, m_1, m_3, \delta_1] \tag{18e}$$
$$\text{s.t.}\ \ g_1(\mathbf{x}) = \sigma_{b2} - \sigma_{\text{allow}} \le 0 \tag{18f}$$
$$g_2(\mathbf{x}) = \sigma_{r2} - \sigma_{\text{allow}} \le 0 \tag{18g}$$
$$g_3(\mathbf{x}) = Ft_2 - Ft_{\text{allow}} \le 0 \tag{18h}$$
$$g_4(\mathbf{x}) = m_1 + m_2 + m_3 - m_{\text{allow}} \le 0 \tag{18i}$$

The lower level subsystem 1 has the consistency objective function in Eqs. (19a) and (19b) with design variables being the size of beam 1 and rod 1. Subsystem 1 also have inconsistency as objectives in Eqs. (20a) and (20b) with stress and force constraints in Eqs. (20d) and (20e), respectively.For both all-in-one and ATC models, exponential cost models as in Eq. (10) with A = 100 and B = 1,500 are used.

Results and comparisons

Using constraint method, five Pareto points are generated using both all-in-one (AIO) and ATC formulations with results shown in Fig. 14. In all cases, both ATC convergence and weighting update method convergence requirements are set to 10^{-6}. The corresponding values of anchor mass and mean 1 deflections are listed in Table 9 along with design variable values at these optimal design alternatives.

subsystem 1

$$\min\ w_1(\delta_1 - \delta_1^U)^2 + w_2(\delta_2 - \delta_2^U)^2 \tag{19a}$$
$$+w_4(F_2 - F_2^U)^2 + w_6(m_1 - m_1^U)^2 \tag{19b}$$
$$\text{w.r.t}\ \mathbf{x} = [d_1, d_{r1}, F_2] \tag{19c}$$
$$\text{s.t.}\ \ g_1(\mathbf{x}) = \sigma_{b1} - \sigma_{\text{allow}} \le 0 \tag{19d}$$
$$g_2(\mathbf{x}) = \sigma_{r1} - \sigma_{\text{allow}} \le 0 \tag{19e}$$
$$g_3(\mathbf{x}) = Ft_1 - Ft_{\text{allow}} \le 0 \tag{19f}$$

subsystem 2

$$\min\ w_3(\delta_3 - \delta_3^U)^2 + w_5(F_3 - F_3^U)^2 \tag{20a}$$
$$+w_7(m_3 - m_3^U)^2 \tag{20b}$$
$$\text{w.r.t}\ \mathbf{x} = [d_3, F_3] \tag{20c}$$
$$\text{s.t.}\ \ g_1(\mathbf{x}) = \sigma_{b3} - \sigma_{\text{allow}} \le 0 \tag{20d}$$
$$g_2(\mathbf{x}) = Ft_3 - Ft_{\text{allow}} \le 0 \tag{20e}$$

As can be seen in Fig. 14 the tradeoff between overall structural weight and beam 1 deflection form a convex Pareto set. The design alternatives A to E using AIO formulation and A′ to E′ using ATC formulation have inconsistency between them. This inconsistency between subsystems result in different design alternatives.

None of the design alternatives in Table 9 satisfy the $\boldsymbol{\Delta}\mathbf{f}^T$ requirement and therefore a tolerance design stage is necessary for target variation reduction. Table 10 lists the optimality influence ranges and the corresponding tolerance cost to meet the target variations. As can be seen the AIO and ATC, although solving different mathematical problems, yield similar results. Based on Tables 9 and 10, we can then have the answers to our original design questions:

Answer 1). Given current manufacturing tolerances, none of the design alternatives are acceptable.

Answer 2). The comparisons between five design alternatives show that Design E using AIO or Design E′ using ATC have the smallest influence area and biggest influence S/N ratio with comparable cost. Therefore suggested as the optimal design alternative.

Conclusions

In this work we demonstrate the integration of design under uncertainty with tolerance allocation for single- and multi-level systems with multiple objective functions to be optimized simultaneously. Our goal is to enable designers to choose among various design alternatives on the Pareto set. In addition, we also investigate the integration of Pareto generation and sensitivity equations with analytical target cascading for design of hierarchical systems. The proposed work can be applied to the dimensioning and tolerancing code in ASME Y14.5 (ASME Standards Committee Y14 2009) and its mathematical representations in ASME Y14.5.1M (ASME Standards Committee Y14 2004).

The scales among different objectives are an important issue as they are in standard optimization techniques. Scaling is often used to resolve numerical difficulties with large differences in the values of computed quantities (Papalambros and Wilde 2000). In this work, proper scaling will not only ensuring the search of optima in Pareto generation, they also make the resulting Pareto set more reasonably presented using constrained methods. However, scaling will not affect the result by the proposed method in that we compare design alternatives that have identical scales. In other words, if one select improper objective function scales, he/she will still end up with the same conclusions compared with proper scaling. We have added one paragraph in the concluding remark to convey this point.

The ‘true optimal solutions to Eq. (1) can be obtained via iteratively applying the proposed method in Fig. 1. By doing so, one has implicitly conducing alternating variable optimization that is “usually very inefficient and unreliable” Fletcher (2001). We intends to follow the concept of robust design in engineering such that decisions are made first to determine the optimal values of all design variables and then assign the optimal tolerances to these determined values. This two-step decision-making process resembles the current engineering practice. Although such a design may not be mathematically optimal, they generally leads to the optimal design one can obtain within given resources and time. However, if one prefers the true optimal design, he/she can iteratively perform the proposed work until a convergent result is obtained.

Based on the work, we found that integrating existing methods might look straightforward, one must be cautious about details in implementation. The numerical error in target matching might accumulate to large system deviations especially for design of complex hierarchical engineering systems. Engineers will need to trade-off the solution accuracy with the computation cost in obtaining such solutions.

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