APPLIED MATHEMATICS

Part 3:

Vector Integral Calculus

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Contents

Chapter 9

Vector Integral Calculus and Integral Theorems

9.1 Line Integrals

Line integral of $\vec{F}(\vec{r})$ over a curve $C : \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$,

$$\int_c \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt$$

where $\vec{r}(a)$ is the initial point, $\vec{r}(b)$ is the terminal point of the integral.

Line integral of $\vec{F}(\vec{r})$ over a closed curve C,

$$\oint_c \vec{F}(\vec{r}) \cdot d\vec{r}.$$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\begin{aligned} \Longrightarrow \int_c \vec{F}(\vec{r}) \cdot d\vec{r} &= \int_c (F_1 dx + F_2 dy + F_3 dz) \\ &= \int_a^b (F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}) dt \end{aligned}$$

If $\vec{F}$ = force and $\vec{r}$ = displacement $\Longrightarrow \int \vec{F} \cdot d\vec{r}$ = work

<u>*Ex*</u> :

$$\vec{F}(\vec{r}) = z\hat{i} + x\hat{j} + y\hat{k}$$

(a) $\vec{r} = \cos t\hat{i} + \sin t\hat{j} + 3t\hat{k} \quad (t = 0 \sim 2\pi)$

(b) $\vec{r} = \hat{i} + t\hat{k} \quad (t = 0 \sim 6\pi)$

(a)

$$\begin{aligned} & \int_c \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} dt \\ = & \int_0^{2\pi} (3t\hat{i} + \cos t\hat{j} + \sin t\hat{k}) \cdot (-\sin t\hat{i} + \cos t\hat{j} + 3\hat{k}) dt \\ = & \int_0^{2\pi} (-3t \sin t + \cos^2 t + 3 \sin t) dt \\ = & 7\pi \end{aligned}$$

(b)

$$\int_c \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^{2\pi} (t\hat{i} + \hat{j}) \cdot (\hat{k}) dt = 0 \quad \square$$

Properties of Line Integral

$$\int_c k\vec{F}\cdot d\vec{r} = k\int_c \vec{F}\cdot d\vec{r}$$

$$\int_c (\vec{F}+\vec{G})\cdot d\vec{r} = \int_c \vec{F}\cdot d\vec{r} + \int_c \vec{G}\cdot d\vec{r}$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_{c_1} \vec{F}\cdot d\vec{r} + \int_{c_2} \vec{F}\cdot d\vec{r} \qquad \text{where} \quad C = C_1 \cup C_2$$

9.2 Line Integrals Independent of Path

$$\int_{c_1} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{c_2} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{c_3} \vec{F}(\vec{r}) \cdot d\vec{r}$$

The lime integrals may be independent of path of integration,which means as long as A and B are fixed, the integral is all the same no matter which path is chosen.

Theorem 1

$\int \vec{F}(\vec{r}) \cdot d\vec{r}$ with continuous F_1, F_2 and F_3 in D is independent of path in D if and only if $\vec{F} = \nabla f$.

Proof :

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}, \qquad a \le t \le b$$

$$\vec{F} = \nabla f = (\frac{\partial f}{\partial x})\hat{i} + (\frac{\partial f}{\partial y})\hat{j} + (\frac{\partial f}{\partial z})\hat{k}$$

$$\begin{aligned} \Longrightarrow \int_c \vec{F} \cdot d\vec{r} &= \int_A^B (\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz) \\ &= \int_a^b \frac{df}{dt}dt = \int_A^B df = f(B) - f(A) \end{aligned}$$

$\therefore$ the result is independent of the path, depends only the initial and terminal points A and B. $\square$

<u>*Ex*</u> :

$$\vec{F}(\vec{r}) = 3x^2\hat{i} + 2yz\hat{j} + y^2\hat{k}, \qquad A : (0, 1, 2), \quad B : (1, -1, 7)$$

If the potential f exists,

$$\frac{\partial f}{\partial x} = 3x^2 \quad \Longrightarrow f(x, y, z) = x^3 + g(y, z)$$

$$\frac{\partial f}{\partial y} = 2yz = \frac{\partial g}{\partial y} \quad \Longrightarrow g(y, z) = y^2 z + h(z)$$

$$\frac{\partial f}{\partial z} = y^2 = \frac{\partial g}{\partial z} = y^2 + \frac{dh}{dz} \quad \Longrightarrow h(z) = c \text{ (constant)}$$

$\therefore$ the potential $f(x, y, z) = x^2 + y^2 z + c$ exists.

$\therefore$ $\int_A^B \vec{F} \cdot d\vec{r} = f(B) - f(A) = (1 + 7 + c) - (2 + c) = 6$ $\square$

Theorem 2

$\int \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of path in D if and only if $\oint_c \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ in D.

Proof :

$$\oint_c \vec{F}(\vec{r}) \cdot d\vec{r} = 0$$

$$\Longrightarrow \oint_{c_1} \vec{F}(\vec{r}) \cdot d\vec{r} - \oint_{c_2} \vec{F}(\vec{r}) \cdot d\vec{r} = 0$$

$$\Longrightarrow \int_{c_1} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_{c_2} \vec{F}(\vec{r}) \cdot d\vec{r}$$

$\therefore \int \vec{F} \cdot d\vec{r}$ is independent of path. $\square$

Note:

Since the work W done by a force F along a path C is $W = \int_c \vec{F}(\vec{r}) \cdot \vec{r}$, if the integral is independent of path, then

$$\oint \vec{F}(\vec{r}) \cdot \vec{r} = 0, \quad \text{and}$$

$$\vec{F} = \nabla f.$$

$\Longrightarrow$

$\vec{F}$ is called *conservative* if $\vec{F}$ is a force vector.

$\vec{F}$ is called *irrotational* if $\vec{F}$ is a velocity vector.

"Exact" Differential

$$\int_c \vec{F}(\vec{r}) \cdot d\vec{r} = \int_c (F_1 dx + F_2 dy + F_3 dz)$$

$(F_1 dx + F_2 dy + F_3 dz)$ is called "exact" on an exact differential in D if

$$F_1 dx + F_2 dy + F_3 dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

$$\Longrightarrow F_1 = \frac{\partial f}{\partial x}, \quad F_2 = \frac{\partial f}{\partial y}, \quad F_3 = \frac{\partial f}{\partial z}$$

i.e. $\vec{F} = \nabla f$

"Simply Connected" Domain D

If every closed curve in D can be continuously shrunk to any point in D without leaving D, then D is called a simply connected domain.
e.g. interior of a sphere, a cube, a sphere with finitely many points removed.

Note:
yes: domain between two concentric spheres
no: interior of a torus or

Theorem 3

$$\int_c \vec{F}(\vec{r}) \cdot \vec{r} = \int_c (F_1 dx + F_2 dy + F_3 dz) \quad \text{———} \quad (*)$$

where F_1, F_2 and F_3 and their first derivatives are continuous in D then

(1) if (*) is independent of path,
i.e. $\vec{F} = \nabla f$ or $F_1 dx + F_2 dy + F_3 dz$ is exact
$\Longrightarrow \nabla \times \vec{F} = 0$ in D,

(2) if $\nabla \times \vec{F} = 0$ in D and D is simply connected
$\Longrightarrow$ (*) is independent of path in D. $\square$

<u>*Ex*</u> :

$$I = \int_c [\underbrace{(2xyz^2)}_{= F_1} dx + \underbrace{(x^2 z^2 + z \cos yz)}_{= F_2} dy + \underbrace{(2x^2 yz + y \cos yz)}_{= F_3} dz]$$

$$\nabla \times \vec{F} = [(F_3)_y - (F_2)_z]\, \hat{i} + [(F_1)_z - (F_3)_x]\, \hat{j} + [(F_2)_x - (F_1)_y]\, \hat{k}$$

$$\begin{cases} (F_3)_y = 2x^2 z + \cos yz - yz \sin yz = (F_2)_z \\ (F_1)_z = 4xyz = (F_3)_x \\ (F_2)_x = 2xz^2 = (F_1)_y \end{cases}$$

$$\therefore \nabla \times \vec{F} = 0$$

$$\vec{F} = \nabla f$$

$$\frac{\partial f}{\partial x} = 2xyz^2 \quad \Longrightarrow f(x,y,z) = x^2yz^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = x^2z^2 + z\cos yz \quad \Longrightarrow x^2z^2 + \frac{\partial g}{\partial y} = x^2 + z\cos yz$$

$$\Longrightarrow g(y,z) = \sin yz + h(z)$$

$$\frac{\partial f}{\partial z} = 2x^2yz + y\cos yz \Longrightarrow 2x^2yz + y\cos yz + \frac{dh}{dz} = 2x^2yz + y\cos yz$$

$$\Longrightarrow h(z) = c$$

$$\Longrightarrow f(x,y,z) = x^2yz^2 + \sin yz + c$$

$\therefore$ If C is any curve from $A : (0,0,1)$ to $B : (1, \frac{\pi}{4}, 2)$

$$\int_c \vec{F} \cdot d\vec{r} = \int_A^B df = f(B) - f(A) = (\pi + \sin\frac{\pi}{2} + c) - (c) = \pi + 1 \quad \square$$

<u>*Ex*</u> : Consider a two-domensional domain D and a vector function

$$\vec{F} = \left(\frac{-y}{x^2+y^2}\right)\hat{i} + \left(\frac{x}{x^2+y^2}\right)\hat{j}$$

$$\nabla \times \vec{F} = 0 \quad \text{except at } (x,y) = (0,0)$$

$\therefore$ If D excludes $(0,0)$, then $\int_c \vec{F}\cdot d\vec{r}$ is independent of path.

However, if D includes $(0,0)$, then $\nabla \times \vec{F} = 0$ only in D',

where $D' \cup \{(0,0)\} = D$.

$\therefore$ D' is not simply connected.

$\therefore$ $\int_c \vec{F}\cdot d\vec{r}$ is dependent of path. □

9.3 Double Integrals (Review of Calculus)

$$\begin{cases} \text{definite integral} & \int_a^b \\ \text{indefinite integral} & \int \end{cases}$$

- $\int f(x)dx$

- $\int_c \vec{f}(\vec{r}) \cdot d\vec{r}$

- $\iint\limits_R f(x,y)dxdy$

- $\iiint\limits_V f(x,y,z)dxdydz$

. . . .

- $\iint\limits_S \vec{f} \cdot \vec{n} dA$

Definition:

Double integral over region R:

$$\iint_R f(x,y)dxdy \quad \text{or} \quad \iint_R f(x,y)dA$$

$$= \lim_{N\to\infty} \sum_{k=1}^{N} f(x_k, y_k)\Delta A_k$$

Properties of Double Integral:

$$\iint_R kf dA = k \iint_R f dA \qquad k = \text{constant}$$

$$\iint_R (f+g)dA = \iint_R f dA + \iint_R g dA$$

$$\iint_R f dA = \iint_{R_1} f dA + \iint_{R_2} f dA, \qquad R = R_1 \bigcup R_2$$

Mean Value Theorem for Double Integral:

$$\iint_R f(x,y)dA = f(x_0,y_0)A,$$

where A = area of R and $(x_0, y_0) \in R$.

Evaluation of double integral:

(1) If R can be described by

$$\begin{cases} a \le x \le b \\ g(x) \le y \le h(x) \end{cases}$$

$$\Longrightarrow \iint_R f(x,y)dxdy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)dy\right] dx$$

(2) If R can be described by

$$\begin{cases} c \le y \le d \\ p(y) \le x \le q(y) \end{cases}$$

$$\Longrightarrow \iint_R f(x,y)dxdy = \int_c^d \left[\int_{p(y)}^{q(y)} f(x,y)dx\right] dy$$

(3) If R cannot be represented by those inequalities in (1) and (2), but can be subdivided into portions which can be represented by those inequalities,then

Applications of double integral:

$A = \iint\limits_R dxdy \quad \Longrightarrow$ area of region R

$V = \iint\limits_R f(x,y)dxdy \quad \Longrightarrow$ volume between the surface $z = f(x,y)$ and xy plane

Change of Variables in Double Integrals:

Recall that $\int_a^b f(x)dx = \int_{\alpha=u(a)}^{\beta=u(b)} f(x(u))\dfrac{dx}{du}du.$

For double integral, if $x = x(u,v)$ and $y = y(u,v)$

$$\iint\limits_R f(x,y)dxdy = \iint\limits_{R^*} f\left(x(u,v), y(u,v)\right) \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\equiv |J|} dudv$$

where R^* is region in uv plane, and

$$\text{Jacobian } J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

<u>*Ex*</u> :

$$\iint_R y^2 dxdy \qquad R : 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1$$

(1)

$$\begin{aligned}
\iint_R y^2 dxdy &= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} y^2 dy \right] dx \\
&= \int_0^1 \frac{1}{3} y^3 \Big|_{y=0}^{\sqrt{1-x^2}} dx \\
&= \frac{1}{3} \int_0^1 [1-x^2]^{3/2} dx \\
&\cdots\cdots\cdots \\
&= \frac{\pi}{16}
\end{aligned}$$

(2)

$$\begin{aligned}
\iint_R y^2 dxdy &= \int_0^1 \left[\int_0^{\sqrt{1-y^2}} y^2 dx \right] dy \\
&= \int_0^1 y^2 [x] \Big|_{x=0}^{\sqrt{1-y^2}} dy \\
&= \int_0^1 y^2 \sqrt{1-y^2} dy \\
&= \left[-\frac{y^2(1-y^2)^{3/2}}{4} + \frac{y(1-y^2)^{1/2}}{8} + \frac{1}{8} \sin^{-1} y \right]_0^1 \\
&= \frac{1}{8} \times \frac{\pi}{2} = \frac{\pi}{16}
\end{aligned}$$

(3) $x = r\cos\theta$

$y = r\sin\theta$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

$$\begin{aligned} \therefore \iint_R f(x,y)dxdy &= \iint_{R^*} r^2\sin^2\theta \cdot |r|\, drd\theta \\ &= \int_0^{\pi/2}\left[\int_0^1 r^3\sin^2\theta dr\right]d\theta \\ &= \int_0^{\pi/2}\left[\frac{1}{4}r^4\right]\Bigg|_{r=0}^{1}\sin^2\theta d\theta \\ &= \int_0^{\pi/2}\frac{1}{4}\sin^2\theta d\theta \\ &\cdots\cdots\cdots \\ &= \frac{\pi}{16} \quad \square \end{aligned}$$

9.4 Green's Theorem in the Plane

Consider a closed region R with boundary C consists of finitely many smooth curves. If $F_1(x, y)$, $F_2(x, y)$ and their first derivatives are continuous

$$\Longrightarrow \iint_R (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) dxdy = \oint_C (F_1 dx + F_2 dy) \qquad \text{———} \ (1)$$

The direction of integration along C is such that R is always on the left hand side.

Equation (1) can be written in vectorial form as

$$\iint_R (\nabla \times \vec{F}) \cdot \hat{k} \ dxdy = \oint_C \vec{F} \cdot d\vec{r},$$

where $\vec{F} = F_1 \hat{\imath} + F_2 \hat{\jmath}$.

Proof of Green's theorem:

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \oint_C (F_1 dx + F_2 dy)$$

If R can be represented in the forms:

$a \leq x \leq b$ $\qquad$ $c \leq y \leq d$

$u(x) \leq y \leq v(x)$ $\qquad$ $p(y) \leq x \leq q(y)$

For those regions which cannot be represented by the above forms, the domain is divided into subregions each of them can be represented by the required forms.

$$
\begin{aligned}
\iint\limits_R \frac{\partial F_1}{\partial y} dxdy &= \int_a^b \left[\int_{u(x)}^{v(x)} \frac{\partial F_1}{\partial y} dy \right] dx \\
&= \int_a^b \left[F_1(x, v(x)) - F_1(x, u(x)) \right] dx \\
&= -\int_a^b F_1(x, u(x)) dx - \int_b^a F_1(x, v(x)) dx \\
&= -\oint_c F_1(x, y) dx \qquad \text{———} \ (*)
\end{aligned}
$$

$$
\begin{aligned}
\iint\limits_R \frac{\partial F_2}{\partial x} dxdy &= \int_c^d \left[\int_{p(y)}^{q(y)} \frac{\partial F_2}{\partial x} dx \right] dy \\
&= \int_c^d \left[F_2(q(y), y) dy - F_2(p(y), y) \right] dy \\
&= \int_c^d F_2(q(y), y) dy + \int_d^c F_2(p(y), y) dy \\
&= \oint_c F_2(x, y) dy \qquad \text{———} \ (**)
\end{aligned}
$$

(*) and (**) $\Longrightarrow$

$$
\iint\limits_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = \oint_c (F_1 dx + F_2 dy) \quad \square
$$

<u>*Ex*</u> : $F_1 = y^2 - 7y,\ F_2 = 2xy + 2x$

$$
\begin{aligned}
&\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy \\
=\ &\iint_R [(2y+2) - (2y-7)]dxdy \\
=\ &\iint_R 9dxdy = 9\pi
\end{aligned}
$$

$$
\begin{aligned}
&\oint_C (F_1 dx + F_2 dy) \\
=\ &\int_C \vec{F} \cdot d\vec{r} \\
=\ &\int_C \vec{F} \cdot \vec{r'} \cdot dt \\
=\ &\int_0^{2\pi} \left[(y^2 - 7y)\hat{i} + (2xy + 2x)\hat{j}\right] \cdot \left[-\sin t\hat{i} + \cos t\hat{j}\right] dt \\
=\ &\int_0^{2\pi} \left[(\sin^2 t - 7\sin t)(-\sin t) + (2\cos t \sin t + 2\cos t)(\cos t)\right] dt \\
=\ &\int_0^{2\pi} \left[-\sin^3 t + 7\sin^2 t + 2\cos^2 t \sin t + 2\cos^2 t)\right] dt \\
=\ &7\pi + 2\pi = 9\pi \quad \square
\end{aligned}
$$

<u>*Ex*</u> : Area of a plane region

(a) If $F_1 \equiv 0$, $F_2 \equiv x$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint_R (1-0)dxdy$$
$$= \oint_C (F_1 dx + F_2 dy) = \oint_C xdy$$

$\therefore$ $\boxed{A = \iint_R dxdy = \oint_c xdy}$

(b) If $F_1 = -y$, $F_2 = 0$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dxdy = \iint_R (0+1)dxdy$$
$$= \oint_C (F_1 dx + F_2 dy) = \oint_C (-y)dx$$

$\therefore$ $\boxed{A = \iint_R dxdy = -\oint_C ydx}$

Combine (a) and (b),

$\therefore$ $\boxed{A = \frac{1}{2}\oint_C (xdy - ydx)}$ □

For example, for an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, or $x = a\cos t, y = b\sin t$,

$$
\begin{aligned}
A &= \oint_c x dy \\
&= \int_0^{2\pi} x \frac{du}{dt} dt \\
&= \int_0^{2\pi} (a\cos t)(b\cos t) dt \\
&= ab \int_0^{2\pi} \cos^2 t dt \\
&= \pi ab \quad \square
\end{aligned}
$$

<u>*Ex*</u> : Area of a plane region in polar coordinates $x = r\cos\theta, y = r\sin\theta$

$$\begin{cases} dx = \dfrac{\partial x}{\partial r}dv + \dfrac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta \\ \\ dy = \dfrac{\partial y}{\partial r}dv + \dfrac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta \end{cases}$$

$$\begin{aligned} A &= \frac{1}{2}\oint_C (xdy - ydx) \\ &= \frac{1}{2}\oint_C [(r\cos\theta)(\sin\theta dr + r\cos\theta d\theta) - (r\sin\theta)(\cos\theta dr - r\sin\theta d\theta)] \\ &= \frac{1}{2}\oint_C [(r\cos\theta\sin\theta - r\sin\theta\cos\theta)dr + (r^2\cos^2\theta + r^2\sin^2\theta)d\theta] \\ &= \frac{1}{2}\int_0^2 \pi r^2 d\theta \quad \square \end{aligned}$$

For example, a cardioid is represented by $r = a(1 - \cos\theta), \quad 0 \le \theta \le 2\pi$.

$$A = \frac{a^2}{2}\int_0^2 \pi(1-\cos\theta)^2 d\theta = \frac{3\pi}{2}a^2 \quad \square$$

<u>*Ex*</u> : Transformation of a double integral of the Laplacian of a function into a line integral of its normal derivative.

$$\iint_R \nabla^2 w dx dy = \oint_C \frac{\partial w}{\partial n} ds,$$

where $w = w(x, y)$, $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial x^2}$.

<u>*Proof*</u> :

Let $F_1 = -\dfrac{\partial w}{\partial y}$, $\quad F_2 = \dfrac{\partial w}{\partial x}$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx dy = \iint_R \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) dx dy$$

$$\begin{aligned} \oint_C (F_1 dx + F_2 dy) &= \oint_C \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds}\right) ds \\ &= \oint_C \left(-\frac{\partial w}{\partial y}\frac{dx}{ds} + \frac{\partial w}{\partial x}\frac{dy}{ds}\right) ds \end{aligned}$$

Since

$$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j}$$

$$\hat{t} = \frac{d\vec{r}}{ds} = \frac{\partial x}{\partial s}\hat{i} + \frac{dy}{ds}\hat{j} \quad \Longrightarrow \hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$$

$$\therefore \nabla w \cdot \hat{n} = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{dy}{ds} - \frac{\partial w}{\partial y}\frac{dx}{ds}$$

$$\therefore \iint_R \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) dx dy = \oint_C \frac{\partial w}{\partial n} ds \quad \square$$

9.5 Surfaces

■ Representation of surface:

$S : z = f(x, y)$

$S : g(x, y, z) = 0$

$S : \vec{r}(u, v) = x(u, v)\hat{i} + y(u, v)\hat{j} + z(u, v)\hat{k}$

Notes:
curve $C : \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

<u>*Ex*</u> :

(1) Circular cylinder:

$$S: x^2 + y^2 = a^2, \qquad -1 \le z \le 1$$

$$S: \vec{r}(u,v) = a\cos u\hat{i} + a\sin u\hat{j} + v\hat{k},$$
$$0 \le u \le 2\pi, \quad -1 \le v \le 1$$

(2) Sphere:

$$S: x^2 + y^2 + z^2 = a^2$$

$$S: \vec{r}(u,v) = a\cos u\cos v\hat{i} + a\sin u\cos v\hat{j} + a\sin v\hat{k},$$
$$0 \le u \le 2\pi, \quad \frac{-\pi}{2} \le v \le \frac{\pi}{2}$$

$$S: \vec{r}(u,v) = a\cos u\sin v\hat{i} + a\sin u\sin v\hat{j} + a\cos v\hat{k},$$
$$0 \le u \le 2\pi, \quad 0 \le v \le \pi$$

(3) Cone:

$$S : z = \sqrt{x^2 + y^2}, \qquad 0 \le z \le h$$

$$S : \vec{r}(u, v) = u \cos v\hat{i} + u \sin v\hat{j} + u\hat{k}$$

$$\begin{cases} 0 \le u \le h \\ 0 \le v \le 2\pi \end{cases}$$

■ Tangent plane and surface normal vector:

• A tangent plane at p contains all the tangent vectors of curves on S passing through p.
• A normal vector of S at p is a vector normal to the tangent plane at p.

To find the normal vector $\vec{N}$ of $S : \vec{r} = \vec{r}(u, v)$, consider two curves on S,

$$C_1 : \vec{r}(u) = \vec{r}(u, v_0),\ \ v_0 = \text{constant}$$

$$C_2 : \vec{r}(v) = \vec{r}(u_0, v),\ \ u_0 = \text{constant}$$

$\therefore$ normal vector $\vec{N} = \dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}$

unit normal vector $\hat{n} = \dfrac{\vec{N}}{|\vec{N}|} = \dfrac{\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}}{\left|\dfrac{\partial \vec{r}}{\partial u} \times \dfrac{\partial \vec{r}}{\partial v}\right|}$

Note: If S is represented by $S : g(x, y, z) = 0$

$\Longrightarrow \vec{N} = \nabla g, \quad \hat{n} = \dfrac{\nabla g}{|\nabla g|}$

$\underline{Ex}$: Surface of a sphere $S : g(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$

$$\nabla g = \frac{\partial g}{\partial x}\hat{i} + \frac{\partial g}{\partial y}\hat{j} + \frac{\partial g}{\partial z}\hat{k} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$|\nabla g| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2a$$

$$\therefore \hat{n} = \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{2a} = \frac{x}{a}\hat{i} + \frac{y}{a}\hat{j} + \frac{z}{a}\hat{k}$$

Or $\quad S : \vec{r}(u, v) = a\cos u\cos v\hat{i} + a\sin u\cos v\hat{j} + a\sin v\hat{k}$

$$\begin{cases} \vec{r_u} &= -a\sin u\cos v\hat{i} + a\cos u\cos v\hat{j} \\ \vec{r_v} &= -a\cos u\sin v\hat{i} - a\sin u\sin v\hat{j} + a\cos v\hat{k} \end{cases}$$

$$\begin{aligned} \vec{r_u} \times \vec{r_v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin u\cos v & a\cos u\cos v & 0 \\ -a\cos u\sin v & -a\sin u\sin v & a\cos v \end{vmatrix} \\ &= \hat{i}(a^2\cos u\cos^2 v) + \hat{j}(a^2\sin u\cos^2 v) \\ &\quad +\hat{k}(a^2\sin^2 u\sin v\cos v + a^2\cos^2 u\sin v\cos v) \\ &= \hat{i}(a^2\cos u\cos^2 v) + \hat{j}(a^2\sin u\cos^2 v) + \hat{k}(a^2\sin v\cos v) \end{aligned}$$

$$\begin{aligned} |\vec{r_u} \times \vec{r_v}| &= a^2\sqrt{\cos^2 u\cos^4 v + \sin^2 u\cos^4 v + \sin^2 v\cos^2 v} \\ &= a^2\sqrt{\cos^4 v + \sin^2 v\cos^2 v} \\ &= a^2\cos v\sqrt{\cos^2 v + \sin^2 v} = a^2\cos v \end{aligned}$$

$$\begin{aligned}
\hat{n} &= \hat{i}\left(\frac{a^2\cos u\cos^2 v}{a^2\cos v}\right) + \hat{j}\left(\frac{a^2\sin u\cos^2 v}{a^2\cos v}\right) + \hat{k}\left(\frac{a^2\sin v\cos v}{a^2\cos v}\right) \\
&= \hat{i}\underbrace{(\cos u\cos v)}_{=\frac{x}{a}} + \hat{j}\underbrace{(\sin u\cos v)}_{=\frac{y}{a}} + \hat{k}\underbrace{(\sin v)}_{=\frac{z}{a}} \quad \square
\end{aligned}$$

9.6 Surface Integral

Consider a surface, $S : \vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$,

with the normal vector: $\vec{N} = \vec{r}_u \times \vec{r}_v, \quad \hat{n} = \frac{\vec{N}}{|\vec{N}|}$.

Surface integral of a vector function $\vec{F}(\vec{r})$ over S is defined as:

$$\iint_S \vec{F}(\vec{r}) \cdot \hat{n} \, dA,$$

where $\vec{F} \cdot \hat{n}$ = component of $\vec{F}$ normal to S.

e.g. :

If $\vec{F} = \rho\vec{v}dA$

$\Longrightarrow \iint_S \vec{F} \cdot \hat{n} = \iint_S \rho\vec{v} \cdot \hat{n}dA$ represents the mass flux across S.

■ **Evaluation of surface integral:** $\iint_S \vec{F}(\vec{r}) \cdot \hat{n} \, dA$

$$\vec{r}(u + \Delta u, v) - \vec{r}(u, v) = \frac{\partial \vec{r}}{\partial u} \Delta u + \ldots$$

$$\vec{r}(u, v + \Delta v) - \vec{r}(u, v) = \frac{\partial \vec{r}}{\partial v} \Delta v + \ldots$$

The area of ΔA is

$$\Delta A \cong \left| \frac{\partial \vec{r}}{\partial u} \Delta u \times \frac{\partial \vec{r}}{\partial v} \Delta v \right| = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| \Delta u \Delta v$$

$\therefore$

$$dA = \lim_{\substack{\Delta u \to 0 \\ \Delta v \to 0}} \Delta A = \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du \, dv$$

$$\therefore \boxed{\iint_S \vec{F}(\vec{r}) \cdot \hat{n} dA} = \iint_R \vec{F}(\vec{r}) \cdot \frac{(\vec{r_u} \times \vec{r_v})}{|\vec{r_u} \times \vec{r_v}|} |\vec{r_u} \times \vec{r_v}| \, dudv$$

$$= \iint_R \vec{F}(\vec{r}) \cdot (\vec{r_u} \times \vec{r_v}) dudv$$

$$= \boxed{\iint_R \vec{F}(\vec{r}) \cdot \vec{N} dudv}$$

i.e. $\hat{n} dA = \vec{N} dudv$

Now represent $\vec{F}$, $\vec{N}$ and $\hat{n}$ by:

$$\vec{F} \equiv F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\vec{N} \equiv N_1 \hat{i} + N_2 \hat{j} + N_3 \hat{k}$$

$$\hat{n} \equiv \cos\alpha \hat{i} + \cos\beta \hat{j} + \cos\gamma \hat{k}$$

where

$\hat{n} \cdot \hat{i} = \cos\alpha$, α is the angle between $\hat{n}$ and x axis
$\hat{n} \cdot \hat{j} = \cos\beta$, β is the angle between $\hat{n}$ and y axis
$\hat{n} \cdot \hat{k} = \cos\gamma$, γ is the angle between $\hat{n}$ and z axis

$$\Longrightarrow \iint_S \vec{F}(\vec{r}) \cdot \hat{n} dA = \iint_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA$$

$$= \iint_R (F_1 N_1 + F_2 N_2 + F_3 N_3) dudv$$

$$dA\cos\gamma = dxdy$$

Similarly,

$$dA\cos\alpha = dydz$$
$$dA\cos\beta = dzdx$$

$$\Longrightarrow \iint_S \vec{F}\cdot\vec{n}dA = \iint_S (F_1dydz + F_2dzdx + F_3dxdy)$$

If $S : z = h(x, y)$

$$\iint_S F_3(x, y, z)\cos\gamma dA = \pm\iint_{R_{xy}} F_3[x, y, h(x, y)]dxdy$$

where R_{xy} is the projection of A on the xy plane.

$$\therefore \iint_S F_3(x, y, z)\cos\gamma dA = \frac{\cos\gamma}{|\cos\gamma|}\iint_{R_{xy}} F_3[x, y, h(x, y)]dxdy$$

Similarly, if $S : y = g(x, z)$

$$\iint_S F_2(x, y, z) \cos\beta dA = \frac{\cos\beta}{|\cos\beta|} \iint_{R_{zx}} F_2[x, g(x, z), z] dz dx$$

If $S : x = f(y, z)$

$$\iint_S F_1(x, y, z) \cos\alpha dA = \frac{\cos\alpha}{|\cos\alpha|} \iint_{R_{yz}} F_1[f(y, z), y, z] dy dz$$

<u>*Ex*</u> : Compute the flux of water through the parabolic cylinder $S : y = x^2, \quad 0 \leq x \leq 2, 0 \leq z \leq 3$ and the velocity vector is $\vec{V} = y\hat{i} + 2\hat{j} + xz\hat{k}$.

$$\text{Flux} = \iint_S \vec{V} \cdot \hat{n} dA = \iint_R \vec{V} \cdot \vec{N} dudv$$

$$S : \vec{r} = u\hat{i} + u^2\hat{j} + v\hat{k}, \quad 0 \leq u \leq 2, \ 0 \leq v \leq 3$$

$$\vec{r_u} = \hat{i} + 2u\hat{j}$$

$$\vec{r_v} = \hat{k}$$

$$\vec{N} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2u\hat{i} - \hat{j}$$

$$\vec{V} = u^2\hat{i} + 2\hat{j} + uv\hat{k}$$

$$\begin{aligned} \Longrightarrow \text{Flux} &= \iint_R \vec{V} \cdot \vec{N} dudv \\ &= \int_0^3 \int_0^2 (u^2\hat{i} + 2\hat{j} + uv\hat{k}) \cdot (2u\hat{i} - \hat{j}) dudv \\ &= \int_0^3 \int_0^2 (2u^3 - 2) dudv = 12 \end{aligned}$$

Or

$S : f(x, y, z) = x^2 - y = 0$

$$\begin{cases} 0 \le x \le 2, & \text{(i.e. } 0 \le y \le 4) \\ 0 \le z \le 3 \end{cases}$$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i} - \hat{j}}{\sqrt{4x^2 + 1}}$$

$\therefore$

$$\cos\alpha = \frac{2x}{\sqrt{4x^2 + 1}} > 0$$

$$\cos\beta = \frac{-1}{\sqrt{4x^2 + 1}} < 0$$

$\therefore$ the flux is

$$\begin{aligned} & \iint_s \vec{V} \cdot \hat{n} dA \\ = & +\iint_{R_{yz}} v_1 dydz - \iint_{R_{zx}} v_2 dzdx \\ = & \int_0^3 \int_0^4 ydydz - \int_0^2 \int_0^3 2dzdx \\ = & 3 \cdot \frac{y^2}{2}\bigg|_0^4 - 2 \cdot 2 \cdot 3 = 24 - 12 \\ = & 12 \quad \square \end{aligned}$$

■ **Integral over oriented surface:** $\iint_S \vec{F} \cdot \hat{n} dA$

We have chosen one of the two possible unit normal vectors ($\hat{n}$ or $-\hat{n}$).
$\therefore$ The integral is over an *oriented surface*.

Smooth surface:
S is smooth if we can choose a normal vector at any point P of S.

The following are not smooth surfaces, but are piecewise smooth.

• *Orientable surface*:
Smooth surface S is orientable if the positive (or negative) normal direction at any point is always in the same direction.

Möbius strip is not oriented surface.

■ Integral over non-oriented surface: $\iint\limits_{s} G(\vec{r})dA$

Recall that $dA = \left|\vec{N}\right| dudv = \left|\vec{r_u} \times \vec{r_v}\right| dudv$

$$\therefore \iint\limits_{s} G(\vec{r})dA = \iint\limits_{R} G[\vec{r}(u,v)] \left|\vec{N}(u,v)\right| dudv$$

If $G(\vec{r}) = 1$

$$\iint\limits_{S} dA = \iint\limits_{R} \left|\vec{r_u} \times \vec{r_v}\right| dudv = \text{area of } S \equiv A(S)$$

<u>*Ex*</u> : Area of a torus surface (doughnut)

$$\vec{r}(u,v) = (a + b\cos v)\cos u\hat{i} + (a + b\cos v)\sin u\hat{j} + b\sin v\hat{k}$$

$$\begin{cases} 0 \le u \le 2\pi \\ 0 \le v \le 2\pi \end{cases}$$

$$\vec{r_u} = -(a + b\cos v)\sin u\hat{i} + (a + b\cos v)\cos u\hat{j}$$

$$\vec{r_v} = -b\sin v\cos u\hat{i} - b\sin v\sin u\hat{j} + b\cos v\hat{k}$$

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -(a + b\cos v)\sin u & (a + b\cos v)\cos u & 0 \\ -b\sin v\cos u & -b\sin v\sin u & b\cos v \end{vmatrix}$$

$$\begin{aligned} &= \hat{i}(a + b\cos v)\cos u\cos v + \hat{j}(a + b\cos v)\sin u\cos v \\ &\quad +\hat{k}[-b\sin v(a + b\cos v)\sin^2 u - b\sin v(a + b\cos v)\cos^2 u] \\ &= b(a + b\cos v)[\cos u\cos v\hat{i} + \sin u\cos v\hat{j} + \sin v\hat{k}] \end{aligned}$$

$\therefore$ $|\vec{r_u} \times \vec{r_v}| = b(a + b\cos v)$

$$
\begin{aligned}
A(S) &= \int_0^{2\pi}\int_0^{2\pi} b(a+b\cos v)dudv \\
&= \int_0^{2\pi}(abu+b^2\sin v)|_{u=0}^{2\pi}dv \\
&= \int_0^{2\pi} 2\pi ab dv = 4\pi^2 ab \quad \square
\end{aligned}
$$

9.7 Triple Integral (Volume Integral) and Divergence Theorem of Gauss

Triple integral (volume integral) of $f(x, y, z)$ over a region T is defined as:

$$\iiint_T f(x,y,z)dxdydz \quad \text{or} \quad \iiint_{\mathrm{T}} \mathrm{f(x,y,z)dV}$$

Theorem: Divergence Theorem (or Gauss's Theorem)
(Transformation between volume integral and surface integral)

Let T be a closed bounded region in space with boundary S which is a piecewise smooth orientable surface.
Vector function $\vec{F}(\vec{r})$ and its first derivatives are continuous.

$$\Longrightarrow \iiint_T \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} dA,$$

where $\hat{n}$ is outward unit normal vector of S □

<u>*Proof*</u> : $\iiint\limits_T \nabla \cdot \vec{F} dV = \iint\limits_S \vec{F} \cdot \hat{n} dA$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$$

i.e. we want to prove

$$\begin{aligned} & \iiint\limits_T (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}) dxdydz \\ = & \iint\limits_S (F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma) dA. \end{aligned}$$

We first prove $\iiint\limits_T \frac{\partial F_3}{\partial z} dxdydz = \iint\limits_s F_3 \cos\gamma dA$

$$\begin{aligned} \iiint\limits_T \frac{\partial F_3}{\partial z} dxdydz &= \iint\limits_R [\int_{g(x,y)}^{h(x,y)} \frac{\partial F_3}{\partial z} dz] dxdy \\ &= \iint\limits_R F_3(x, y, h(x, y)) dxdy - \iint\limits_R F_3(x, y, h(x, y)) dxdy \end{aligned}$$

Recall that $\cos\gamma dA = dxdy$

$$= \iint_S F_3 dxdy = \iint_S F_3 \cos\gamma dA = \iint_S F_3 n_3 dA$$

$$\therefore \iiint_T \frac{\partial F_3}{\partial z} dxdydz = \iint_S F_3 n_3 dA \quad \text{———} \ (1)$$

Similarly

$$\iiint_T \frac{\partial F_1}{\partial x} dxdydz = \iint_S F_1 n_1 dA \quad \text{———} \ (2)$$

$$\iiint_T \frac{\partial F_2}{\partial y} dxdydz = \iint_S F_2 n_2 dA \quad \text{———} \ (3)$$

(1), (2) and (3) $\Longrightarrow \iiint_T \nabla \cdot \vec{F} dV = \iint_s \vec{F} \cdot \hat{n} dA \quad \square$

■ Physical interpretation of divergence theorem

$$\iiint_T \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} ds$$

Divide T into many small boxes:

$$\begin{aligned}
& \text{Flux of } \vec{F} \text{ out of the small box} \\
= \ & (F_1 + \frac{\partial F_1}{\partial x}\Delta x)\Delta y \Delta z - F_1 \Delta y \Delta z \\
& +(F_2 + \frac{\partial F_2}{\partial y}\Delta y)\Delta z \Delta x - F_2 \Delta z \Delta x \\
& +(F_3 + \frac{\partial F_3}{\partial z}\Delta z)\Delta x \Delta y - F_3 \Delta x \Delta y \\
= \ & (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z})\Delta x \Delta y \Delta z \\
= \ & (\nabla \cdot \vec{F})\Delta V
\end{aligned}$$

$$\therefore \iiint_T \nabla \cdot \vec{F} dV = \lim_{\Delta\forall \to 0} \sum_{\Delta V} (\nabla \cdot \vec{F})\Delta V$$

But,

$\therefore$ only the surfaces on S contribute to the flux in Σ

$\therefore \lim\limits_{\Delta V \to 0} \sum\limits_{\Delta V} (\nabla \cdot \vec{F}) \Delta V = \iint\limits_S \vec{F} \cdot \hat{n} ds =$ Flux of $\vec{F}$ out of S $\quad \square$

9.8 Applications of Divergence Theorem

The theory of solutions of Laplace's equation

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is call *potential theory*, and f is called *harmonic function*.

■ Application of divergence theorem in potential theory

$$\iiint_T \nabla \cdot \vec{F} dV = \iint_S \vec{F} \cdot \hat{n} dA$$

(1) $\vec{F} \equiv \nabla f$

$$\vec{F} \cdot \hat{n} = \nabla f \cdot \hat{n} = \frac{\partial f}{\partial n}$$

$$\Longrightarrow \iiint_T \nabla^2 f d\forall = \iint_S \frac{\partial f}{\partial n} dA$$

Theorem:

If f is a harmonic function i.e. $\nabla^2 f = 0$, then $\iint_S \frac{\partial f}{\partial n} dA = 0$

(2) $\vec{F} \equiv f\nabla g$

$$\nabla \cdot \vec{F} = \nabla \cdot (f\nabla g) = \nabla f \cdot \nabla g + f\nabla^2 g$$

$$\vec{F} \cdot \hat{n} = f\nabla g \cdot n = f\frac{\partial g}{\partial n}$$

$$\Longrightarrow \boxed{\iiint_T (f\nabla^2 g + \nabla f \cdot \nabla g)dV = \iint_S f\frac{\partial g}{\partial n}dA} \quad \text{———} \; (*)$$

This is called the Green's first formula (identity).

Interchange f and g i.e. $\vec{F} \equiv g\nabla f$

$$\Longrightarrow \iiint_T (g\nabla^2 f + \nabla f \cdot \nabla g)dV = \iint_S g\frac{\partial f}{\partial n}dA \quad \text{———} \; (**)$$

$(*) - (**)$

$$\Longrightarrow \boxed{\iiint_T (f\nabla^2 g - g\nabla^2 f)dV = \iint_S (f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial n})dA}$$

This is called Green's second formula (identity).

(3) If f satisfies $\nabla^2 f = 0$ in T and $f = 0$ on S,

$$(*) \Longrightarrow \iiint_T (f\nabla^2 f + \nabla f \cdot \nabla f)dV = \iint_S f\frac{\partial f}{\partial n}dA$$

$$\Longrightarrow \iiint_T |\nabla f|^2 \, dV = 0$$

$$\Longrightarrow |\nabla f|^2 = 0$$

$$\Longrightarrow (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2 + (\frac{\partial f}{\partial z})^2 = 0$$

$$\therefore \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

i.e. f = constant in T.
But since f is a continuous function,if $f = 0$ on S, then $f = 0$ in T too

Theorem

If f is a harmonic function in T, and $f = 0$ on S, then $f = 0$ in T. □

If $f_1 = f_2$ on S, i.e. $(f_1 - f_2) = 0$ on S , and f_1 and f_2 are both harmonic functions in T, then

$\nabla^2 f_1 = 0$ and $\nabla^2 f_2 = 0 \quad \Longrightarrow \nabla^2(f_1 - f_2) = 0$ in T

Since $(f_1 - f_2) = 0$ on $S \quad \Longrightarrow (f_1 - f_2) = 0$ in T

i.e. $f_1 = f_2$ in T.

Theorem

If f is a harmonic function in T, then f is uniquely determined in T by its value on S. □

9.9 Stokes' Theorem

Transformation between Surface and Line Integrals

If S is a piecewise smooth oriented surface and C is a piecewise smooth simple closed curve which is the boundary of S,
$\vec{F}(x, y, z)$ and its first derivatives are continuous in S, then

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dA = \oint_C \vec{F} \cdot \vec{r}' ds,$$

where $\vec{r}' = \dfrac{d\vec{r}}{ds} = \hat{t}$.

If S lies on the xy plane (R),

$$(\nabla \times \vec{F}) \cdot \hat{n} = (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\hat{k} \cdot \hat{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$\vec{F} \cdot \frac{d\vec{r}}{ds} ds = \vec{F} \cdot d\vec{r} = F_1 dx + F_2 dy$$

$$\iint_R (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) dxdy = \oint_C (F_1 dx + F_2 dy) \quad \leftarrow \text{Green's theorem in §9.4}$$

<u>*Ex*</u> :

$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$S : z = f(x, y) = 1 - (x^2 + y^2), \quad z \geq 0$

$x = u\cos v$
$y = u\sin v$
$z = 1 - (u^2\cos^2 v + u^2\sin^2 v) = 1 - u^2$

$$S : \vec{r}(u, v) = u\cos v\hat{i} + u\sin v\hat{j} + (1 - u^2)\hat{k}$$
$$0 \leq u \leq 2\pi,\ 0 \leq v \leq 1$$

$$\begin{aligned} \vec{N} &= \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} \\ &= \hat{i}(+2u^2\cos v) + \hat{j}(+2u^2\sin v) + \hat{k}(u) \end{aligned}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\hat{i} - \hat{j} - \hat{k}$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}dA = \int_0^{2\pi}\int_0^1 (-2u^2\cos v - 2u^2\sin v - u)dudv$$

$$= \int_0^{2\pi} (\frac{-2}{3}\cos v - \frac{2}{3}\sin v - \frac{1}{2}) dv = -\pi$$

$$C : \vec{r}(s) = \cos s\hat{\imath} + \sin s\hat{\jmath}, \quad 0 \leq s \leq 2\pi$$

$$\vec{r}'(s) = -\sin s\hat{\imath} + \cos s\hat{\jmath}$$

$$\vec{F}(s) = \sin s\hat{\imath} + \cos s\hat{k}$$

$$\oint_C \vec{F} \cdot \vec{r}' ds = \int_0^{2\pi} (-\sin^2 s) ds = -\pi \quad \square$$

Proof of Stoke's Theorem:

$$\iint_S (\nabla \times \vec{F}) \cdot \vec{n} dA = \oint_C \vec{F} \cdot \vec{r}' ds$$

Recall that $\vec{n} dA = \vec{N} dudv, \quad \vec{N} = [N_1, N_2, N_3] = \vec{r_u} \times \vec{r_v}$

We want to prove

$$\begin{aligned} & \iint_R \left[(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}) N_1 + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}) N_2 + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) N_3 \right] dudv \\ = & \oint_{\bar{C}} (F_1 dx + F_2 dy + F_3 dz) \end{aligned}$$

where R is the region with boundary $\bar{C}$ in the uv-plane.

Proof of $\iint_R (\frac{\partial F_1}{\partial z} N_2 - \frac{\partial F_1}{\partial y} N_3) dudv = \oint_{\bar{C}} F_1 dx$

Consider S which can be represented simultaneously by

$$z = f(x, y), \qquad y = g(x, z), \qquad x = h(y, z)$$

Let $u = x$, $v = y$, then

$$\vec{r}(u, v) = \vec{r}(x, y) = x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

$$\begin{aligned} \therefore \vec{N} &= \vec{r_u} \times \vec{r_v} = \vec{r_x} \times \vec{r_y} \\ &= (\hat{i} + f_x \hat{k}) \times (\hat{j} + f_y \hat{k}) \\ &= -f_x \hat{i} - f_y \hat{j} + \hat{k} \end{aligned}$$

$$
\begin{aligned}
& \iint_R \left(\frac{\partial F_1}{\partial z} N_2 - \frac{\partial F_1}{\partial y} N_3\right) du dv \\
= & \iint_{S^*} \left(\frac{\partial F_1}{\partial z}(-f_y) - \frac{\partial F_1}{\partial y}\right) dx dy \\
= & \iint_{S^*} \frac{-\partial F_1(x, y, f(x, y))}{\partial y} dx dy \qquad \text{(by Green's Theorem)} \\
= & \oint_{C^*} F_1 dx = \oint_{\bar{C}} F_1 dx \quad \text{———} \ (a)
\end{aligned}
$$

Similarly,

$$
\iint_R \left(-\frac{\partial F_2}{\partial z} N_1 + \frac{\partial F_2}{\partial x} N_3\right) du dv = \oint_{\bar{C}} F_2 dy \quad \text{———} \ (b)
$$

$$
\iint_R \left(\frac{\partial F_3}{\partial y} N_1 - \frac{\partial F_3}{\partial x} N_2\right) du dv = \oint_{\bar{C}} F_3 dz \quad \text{———} \ (c)
$$

From (a), (b) and (c)

$$
\begin{aligned}
& \iint_R \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) N_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) N_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) N_3 \right] du dv \\
= & \oint_{\bar{C}} (F_1 dx + F_2 dy + F_3 dz) \quad \square
\end{aligned}
$$

■ Physical meaning of Stoke's theorem

If $\vec{F} \equiv \vec{v}$ (velocity), $\vec{v}(x, y, z) = u\hat{i} + v\hat{j} + w\hat{k}$,

$$\iint_s \underbrace{(\nabla \times \vec{v})}_{\equiv \vec{\omega}} \cdot \hat{n} dA = \underbrace{\oint_c \vec{v} \cdot d\vec{r}}_{\equiv \Gamma}$$

$\vec{\omega}$ is called *vorticity*

Γ is called *circulation*

■ Stoke's theorem applied to independent of path

Recall that in §9.2, Theorem 3:

If a line integral $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of path,

$\Longrightarrow \nabla \times \vec{F} = 0$ in D.

If $\nabla \times \vec{v} = 0$ in D, by Stoke's theorem:

$$\oint_{C_1 + \bar{C}_2} \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} = 0$$

$$\Longrightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = -\int_{\bar{C}_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \qquad (\because\ C_2 = -\bar{C}_2)$$

$\Longrightarrow \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of path. □