# APPLIED MATHEMATICS

## Part 2:

## Vector Differential Calculus

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# Contents

# Chapter 8

# Vector Differential Calculus

## 8.1 Vector Algebra

Vector in Cartesian coordinate system:

vector $\vec{a} = [a_1, a_2, a_3]$

components of $\vec{a} : a_1 = x_2 - x_1,\ a_2 = y_2 - y_1,\ a_3 = z_2 - z_1$

length of $\vec{a}$ : $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

## 8.2 Inner Product (Dot Product)

Definition : Inner product

$$\vec{a} = [a_1, a_2, a_3] \qquad \vec{b} = [b_1, b_2, b_3]$$

$$\Longrightarrow \vec{a} \cdot \vec{b} \equiv |\vec{a}||\vec{b}| \cos \gamma$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3$$

Theorem : Orthogonality

Vectors $\vec{a}$, $\vec{b}$ and $|\vec{a}| \neq 0$, $\left|\vec{b}\right| \neq 0$, $\vec{a} \cdot \vec{b} = 0$

$$\Longleftrightarrow \vec{a} \perp \vec{b}$$

### ■ Properties of Inner Product

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutativity)

$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$ (distributivity)

$$\left.\begin{array}{l} \vec{a} \cdot \vec{a} \geq 0 \\ \vec{a} \cdot \vec{a} = 0 \Longleftrightarrow \vec{a} = 0 \end{array}\right\} \quad \text{(positive-definiteness)}$$

$\left|\vec{a} \cdot \vec{b}\right| \leq |a|\,|b|$ (Schwartz inequality)

Note :

$$\vec{a} \cdot (\vec{b} \cdot \vec{c}) \neq (\vec{a} \cdot \vec{b}) \cdot \vec{c}$$

## ■ Orthonormal Basis

A basis consisting of *orthogonal unit* vectors.

## 8.3 Vector Product (Cross Product)

Definition :

$\vec{a} = [a_1, a_2, a_3] \qquad \vec{b} = [b_1, b_2, b_3]$

$$\begin{aligned} \Longrightarrow \vec{a} \times \vec{b} &= |\vec{a}||\vec{b}| \sin\gamma \hat{e} \\ &= (a_2 b_3 - a_3 b_2)\hat{i} + (a_3 b_1 - a_1 b_3)\hat{j} + (a_1 b_2 - a_2 b_1)\hat{k} \end{aligned}$$

where $\hat{e}$ is an unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{a}$, $\vec{b}$, $\hat{e}$ are in such order that form a "right-handed triple".

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\begin{aligned} |\vec{a} \times \vec{b}| &= |\vec{a}||\vec{b}| \sin\gamma \\ &= \text{area of the parallelogram with } \vec{a} \text{ and } \vec{b} \text{ as two of the sides.} \end{aligned}$$

*Ex* : Vector products of standard basis vectors

$$\begin{array}{lll} \hat{i} \times \hat{j} = \hat{k} & \hat{j} \times \hat{k} = \hat{i} & \hat{k} \times \hat{i} = \hat{j} \\ \hat{j} \times \hat{i} = -\hat{k} & \hat{k} \times \hat{j} = -\hat{i} & \hat{i} \times \hat{k} = -\hat{j} \end{array}$$

## Properties of Cross Product

$$(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$$

$$\left.\begin{aligned} &\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c}) \\ &(\vec{a} + \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) + (\vec{b} \times \vec{c}) \end{aligned}\right\} \quad \text{(distributivity)}$$

$$\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b}) \quad \text{(anti-commutativity)}$$

Note:

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

## Scalar Triple Product

$$\vec{a} = [a_1, a_2, a_3] \quad \vec{b} = [b_1, b_2, b_3] \quad \vec{c} = [c_1, c_2, c_3]$$

$$\begin{aligned} (\vec{a}\;\vec{b}\;\vec{c}) &\equiv \vec{a} \cdot (\vec{b} \times \vec{c}) \\ &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \end{aligned}$$

## Properties of Scalar Triple Product

$$(k\vec{a}\,\vec{b}\,\vec{c}) = k(\vec{a}\,\vec{b}\,\vec{c})$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$$

*Proof* :

$$\begin{aligned}(\vec{a}\times\vec{b})\cdot\vec{c} &= \vec{c}\cdot(\vec{a}\times\vec{b})\\ &= \begin{vmatrix} c_1 & c_2 & c_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} = (-1)\times(-1)\begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}\\ &= \vec{a}\cdot(\vec{b}\times\vec{c})\end{aligned}$$

## Geometric interpretation of $\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right|$

$$\left|\vec{a}\cdot(\vec{b}\times\vec{c})\right| = |\vec{a}|\left|\vec{b}\times\vec{c}\right|\cos\beta$$

$$\left|\vec{b}\times\vec{c}\right| = \text{area of parallelogram}$$

$$|\vec{a}|\cos\beta = \text{height}$$

$$\therefore\ \left|\vec{a}\cdot(\vec{b}\times\vec{c})\right| = \text{volume of parallelepiped}$$

## Linear Independence of Vectors

A set of vectors $\vec{a_{(1)}},\ \vec{a_{(2)}},\ \ldots\ \vec{a_{(m)}}$ are linearly independent if for the vector equation:

$$c_1\vec{a_{(1)}} + c_2\vec{a_{(2)}} + \cdots + c_m\vec{a_{(m)}} = 0$$

the only scalars are $c_1 = 0,\ c_2 = 0\ \ldots\ c_m = 0.$

For $m = 3:\ \ c_1\vec{a_{(1)}} + c_2\vec{a_{(2)}} + c_3\vec{a_{(3)}} = 0$

If $\vec{a_{(1)}}$, $\vec{a_{(2)}}$ and $\vec{a_{(3)}}$ lie on the same plane $\Longrightarrow c_1$, $c_2$ and $c_3 \neq 0$

Theorem :

Three vectors form a linearly independent set if and only if their scalar triple product is not zero.

## 8.4 Vector and Scalar Functions. Field. Derivative

- *vector field* ← *vector function* $\vec{v} = \vec{v}(p) = [v_1(p), v_2(p), v_3(p)]$
- *scalar field* ← *scalar function* $f = f(p)$

where $p$ can be $(x, y, z)$, $(x, y, z, t)$, $(t)$ or other parameters

<u>*Ex*</u> :

- <u>*scalar function*</u>

Distance of any points $p$ from $p_0$:

$$f(p) = f(x, y, z) = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

- <u>*vector field*</u>

Velocity field of rotation:

$$\vec{v}(x, y, z) = \vec{\omega} \times \vec{r} = \vec{\omega} \times (x\hat{i} + y\hat{j} + z\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \omega(-y\hat{i} + x\hat{j})$$

Force (gravitation) field:

$$\vec{G} = -\frac{c}{r^3}\vec{r} \qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

# Vector Calculus

## ■ **Convergence** :

Sequence of vectors $\vec{a}_n$, $n = 1, 2, 3 \ldots$ is said to converge

if $\exists$ $\vec{a}$ such that $\lim\limits_{n\to\infty} |\vec{a}_n - \vec{a}| = 0$.

$\Longrightarrow$ $\vec{a}$ is called the limit vector, i.e. $\lim\limits_{n\to\infty} \vec{a}_n = \vec{a}$ □

## ■ **Limit of vector function** :

If $\vec{v}(t)$ is defined in some neighborhood of $t_0$ and

$$\lim_{t\to t_0} |\vec{v}(t) - \vec{l}| = 0$$

$\Longrightarrow$ $\vec{v}(t)$ has the limit $\vec{l}$ as $t \to t_0$.

i.e. $\lim\limits_{t\to t_0} \vec{v}(t) = \vec{l}$ □

## ■ **Continuity** :

If $\vec{v}(t)$ is defined in some neighborhood of $t$ and $\lim_{t\to t_0} \vec{v}(t) = \vec{v}(t_0)$

$\Longrightarrow$ $v(t)$ is continuous at $t = t_0$ □

## ■ **Derivative** :

$\vec{v}(t)$ is differentiable at point $t$, if

$$\vec{v}'(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t + \Delta t) - \vec{v}(t)}{\Delta t} \quad \text{exists.}$$

$$\Longrightarrow \vec{v}'(t) = [v_1'(t), v_2'(t), v_3'(t)]$$

- *Properties* :

$$(c\vec{v})' = c\vec{v}'$$

$$(\vec{u} + \vec{v})' = \vec{u}' + \vec{v}'$$

$$(\vec{u} \cdot \vec{v})' = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$$

$$(\vec{u} \times \vec{v})' = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$$

$$(\vec{u}\ \vec{v}\ \vec{w})' = (\vec{u}'\ \vec{v}\ \vec{w}) + (\vec{u}\ \vec{v}'\ \vec{w}) + (\vec{u}\ \vec{v}\ \vec{w}')$$

## ■ **Partial derivative** :

$$\vec{v}(t_1, t_2, \ldots t_n), \quad \vec{v} = [v_1, v_2, v_3]$$

$$\frac{\partial \vec{v}}{\partial t_l} = \frac{\partial v_1}{\partial t_l}\hat{i} + \frac{\partial v_2}{\partial t_l}\hat{j} + \frac{\partial v_3}{\partial t_l}\hat{k}$$

$$\frac{\partial^2 \vec{v}}{\partial t_l \partial t_m} = \frac{\partial v_1}{\partial t_l \partial t_m}\hat{i} + \frac{\partial v_2}{\partial t_l \partial t_m}\hat{j} + \frac{\partial v_3}{\partial t_l \partial t_m}\hat{k}$$

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## § **Review of Partial Derivative**

Real function $z = f(x, y)$, $x$ and $y \in R$

- First partial derivative of $f(x, y)$ with respect to $x$ at the point $(x_0, y_0)$ :

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{[f(x_0 + \Delta x, y_0) - f(x_0, y_0)]}{\Delta x}$$

i.e. we keep $y = y_0$ and think of $x$ as a variable.

- Similarly, first partial derivative of $f(x, y)$ with respect to $y$ at $(x_0, y_0)$ :

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{[f(x_0, y_0 + \Delta y) - f(x_0, y_0)]}{\Delta y}$$

<u>*Ex*</u> :

$$f(x, y) = x^2 y + x \sin y$$

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xy + \sin y \\ \\ \dfrac{\partial f}{\partial y} = x^2 + x \cos y \end{cases} \qquad \square$$

• Second partial derivatives :

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = f_{xy}$$

If the $f_x$, $f_y$, $f_{xy}$ and $f_{yx}$ are continuous $\Longrightarrow f_{xy} = f_{yx}$

<u>*Ex*</u> :

$$f(x, y) = x^2 + x \sin y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x + \sin y$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x + \sin y$$

$$\frac{\partial^2 f}{\partial x^2} = 2y$$

$$\frac{\partial^2 f}{\partial y^2} = -x \sin y \quad \square$$

•If $f = f(x, y, z)$, $f_x$, $f_y$ and $f_z$ are defined as for two variables functions.

<u>*Ex*</u> :

$$f(x, y, z) = x^2 + y^2 + z^2 + xye^z$$

$$f_x = 2x + ye^z$$

$$f_z = 2z + xye^z$$

$$f_{yz} = \frac{\partial}{\partial z}(2y + xe^z) = xe^z$$

$$f_{xyz} = \frac{\partial}{\partial z}[\frac{\partial}{\partial y}(2x + ye^z)] = \frac{\partial}{\partial z}[e^z] = e^z \quad \square$$

•Geometric Interpretation of partial derivatives :

$z = f(x, y)$ can be considered as a surface in space:

## 8.5 Curves. Tangents. Arc Length

Differential geometry:
study of curves and surfaces in space by means of calculus.

■ **Cures** :

Curve $C$ : $\vec{r}(t) = [x(t), y(t), z(t)] = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Curve $C$ : $\begin{cases} y = f(x) & \leftarrow \text{projection of } C \text{ into } x - y \text{ plane} \\ z = g(x) & \leftarrow \text{projection of } C \text{ into } x - z \text{ plane} \end{cases}$

Curve $C$ : $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases} \quad \leftarrow \text{intersection of two surfaces}$

<u>*Ex*</u> : Circular helix

$$\vec{r}(t) = a\cos t\hat{i} + a\sin t\hat{j} + ct\hat{k}$$

## ■ **Tangent to a curve** :

Consider a curve $C$: $\vec{r}(t)$

$\Longrightarrow$ Tangent vector of $C$ at $P = \vec{r}'(t) = \lim_{\Delta t \to 0} \dfrac{1}{\Delta t}[\vec{r}(t+\Delta t) - \vec{r}(t)]$

$\Longrightarrow$ Unit tangent vector $\hat{u} = \dfrac{\vec{r}'}{|\vec{r}'|}$

$\Longrightarrow$ Tangent to $C$ at $P$, $T$: $\vec{q}(w) = \vec{r} + w\vec{r}'$ $\square$

<u>*Ex*</u> : Find the tangent to $\dfrac{x^2}{4} + y^2 = 1$ at $P$: $(\sqrt{2}, \dfrac{1}{\sqrt{2}})$.

$$\vec{r}(t) = 2\cos t\hat{i} + \sin t\hat{j}$$

$$\vec{r}'(t) = -2\sin t\hat{i} + \cos t\hat{j}$$

$$P : (\sqrt{2}, \frac{1}{\sqrt{2}}) \quad \Longrightarrow t = \frac{\pi}{4}$$

$$\therefore \vec{q}(w) = (\sqrt{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}) + w(-\sqrt{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}) \quad \square$$

## ■ **Length of a curve** :

$$l = \lim_{m\to\infty} \sum_m \Delta s_m = \int_{s_a}^{s_b} ds$$

$ds$ is the linear element of $c$:

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2 + dz^2} \\ &= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}dt \\ &= \sqrt{\vec{r}' \cdot \vec{r}'}dt \end{aligned}$$

$$\therefore\ l = \int_{t=a}^{b} \sqrt{\vec{r}' \cdot \vec{r}'}dt \quad \square$$

## ■ **Arc length of a curve** :

$b \to t$ (a parameter)

$$\Longrightarrow s(t) = \int_a^t \sqrt{\vec{r}' \cdot \vec{r}'}d\tilde{t} \quad \square$$

*Ex* :

$$\vec{r}(t) = a\cos t\hat{i} + a\sin t\hat{j} + ct\hat{k}$$

$$\vec{r}'(t) = -a\sin t\hat{i} + a\cos t\hat{j} + c\hat{k}$$

$$\vec{r}' \cdot \vec{r}' = a^2 + c^2$$

$$\therefore\ s = \int_0^t \sqrt{a^2 + c^2}d\tilde{t} = t\sqrt{a^2 + c^2}$$

We can change the parameter from $t$ to $s$

i.e. $t = \dfrac{s}{\sqrt{a^2 + c^2}}$

$$\vec{r}(s) = a\cos\frac{s}{\sqrt{a^2 + c^2}}\hat{i} + b\sin\frac{s}{\sqrt{a^2 + c^2}}\hat{j} + \frac{s}{\sqrt{a^2 + c^2}}\hat{k} \quad \square$$

Note that for unit tangent vector $\hat{u}(t)$

$$\hat{u}(t) = \frac{\vec{r}'(t)}{|\vec{r}'|} = \frac{\dfrac{d\vec{r}}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|} = \frac{\dfrac{d\vec{r}}{ds}\dfrac{ds}{dt}}{\left|\dfrac{d\vec{r}}{dt}\right|}$$

$$\because\ ds = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}}\, dt \quad \Longrightarrow \frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$$

$$\therefore\ \hat{u}(s) = \frac{d\vec{r}}{ds}$$

## 8.6 Velocity and Acceleration

Consider a curve (path) $C : \vec{r}(t)$, where $t$ is time.

$$\vec{v}(t) = \vec{r}' = \frac{d\vec{r}}{dt}$$

is the *velocity vector*, which points in the instantaneous direction of motion of a moving point.

$$\vec{a}(t) = \vec{r}'' = \vec{v}'(t)$$

is the *acceleration vector*. □

<u>*Ex*</u> : Centripetal acceleration.

$$C : \vec{r}(t) = R\cos\omega t\hat{i} + R\sin\omega t\hat{j}$$

$$\vec{v}(t) = \vec{r}'(t) = -R\omega\sin\omega t\hat{i} + R\omega\cos\omega t\hat{j} \perp \vec{r}(t)$$

$$\vec{a}(t) = \vec{v}'(t) = -R\omega^2\cos\omega t\hat{i} - R\omega^2\sin\omega t\hat{j} \perp \vec{v}(t)$$

$$\left|v\vec{(t)}\right| = R\omega$$

$$\left|a\vec{(t)}\right| = R\omega^2$$

i.e. $|\vec{v}| =$ constant but $|\vec{a}| \neq 0$. □

# Tangential & Normal Acceleration

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \hat{u}(s)\frac{ds}{dt}$$

$$\begin{aligned}\vec{a}(t) &= \frac{d\vec{v}}{dt} = \frac{d}{dt}(\hat{u}(s)\frac{ds}{dt}) = \frac{d\hat{u}}{dt}\frac{ds}{dt} + \hat{u}(s)\frac{d^2s}{dt^2} \\ &= \underbrace{\frac{d\hat{u}}{ds}(\frac{ds}{dt})^2}_{(*)} + \underbrace{\hat{u}(s)\frac{d^2s}{dt^2}}_{(**)}\end{aligned}$$

Since $\hat{u}(s)$ is "unit" tangent vector.

$\Longrightarrow |\hat{u}(s)| = 1 \qquad \Longrightarrow |\hat{u}(s)|^2 = \hat{u}\cdot\hat{u} = 1$

$\Longrightarrow \dfrac{d(\hat{u}\cdot\hat{u})}{ds} = 0 \qquad \Longrightarrow 2\dfrac{d\hat{u}}{ds}\cdot\hat{u} = 0$

i.e. $\dfrac{d\hat{u}}{ds} \perp \hat{u}$

$\therefore$ $(*) = \dfrac{d\hat{u}}{ds}(\dfrac{ds}{dt})^2$ is the *normal acceleration*,

and $(**) = \hat{u}(s)\dfrac{d^2s}{dt^2}$ is the *tangential acceleration.* □

<u>*Ex*</u> : Coriolis acceleration.

A disk rotating with constant angular velocity $\omega$.

A particle $P$ moves from 0 to the edge of the rotating disk, with constant speed 1.

$$\vec{r}(t) = t\vec{b} = t(\cos wt\hat{i} + \sin wt\hat{j})$$

$$\vec{v}(t) = \vec{r}'(t) = \vec{b} + t\vec{b}'$$

$(\vec{b} + t\vec{b}')$ is tangential to $C : \vec{r}(t) = t\vec{b}$

$$\vec{r}(t) = t\vec{b} = t(\cos wt\hat{i} + \sin wt\hat{j})$$

$$\vec{v}(t) = \vec{r}'(t) = \vec{b} + t\vec{b}'$$

$$\vec{a}(t) = \vec{v}'(t) = \vec{b}' + \vec{b}' + t\vec{b}'' = 2\vec{b}' + t\vec{b}''$$

$$\because \vec{b}'' = -w^2\vec{b}$$

$$\therefore \vec{a}(t) = \underbrace{2\vec{b}'}_{(1)} - \underbrace{tw^2\vec{b}}_{(2)}$$

(1) is the Coriolis acceleration (in direction of rotation)

(2) is the centripetal acceleration (in direction of $\vec{r}$)

<u>*Ex*</u> :

$$
\begin{aligned}
\vec{r}(t) &= R\cos\gamma t\ \hat{b} + R\sin\gamma t\ \hat{k} \\
R &= \text{radius of earth} \\
\gamma &= \text{angular speed of } P \text{ along meridian} \\
\vec{b} &= \cos\omega t\hat{i} + \sin\omega t\hat{j} \\
\omega &= \text{angular speed of earth rotation}
\end{aligned}
$$

$$\vec{v}(t) = \vec{r}' = R\cos\gamma t\ \hat{b}' - \gamma R\sin\gamma t\ \hat{b} + \gamma R\cos\gamma t\ \hat{k}$$

$$\vec{a}(t) = \vec{v}' = R\cos\gamma t\ \hat{b}'' - 2\gamma R\sin\gamma t\ \hat{b}' - \gamma^2 R\cos\gamma t\ \hat{b} - \gamma^2 R\sin\gamma t\ \hat{k}$$

$$
\begin{aligned}
\vec{b}' &= -\omega\sin\omega t\ \hat{i} + \omega\cos\omega t\ \hat{j} \\
\vec{b}'' &= -\omega^2\cos\omega t\ \hat{i} - \omega^2\sin\omega t\ \hat{j} = -\omega^2\ \hat{b}
\end{aligned}
$$

$$\therefore\ \vec{a} = \underbrace{-\omega^2 R\cos\gamma t\ \hat{b}}_{(1)} - \underbrace{2\gamma R\sin\gamma t\ \hat{b}'}_{(2)} - \underbrace{\gamma^2\vec{r}}_{(3)}$$

(1) is centripetal acceleration due to rotation of earth.

(2) is Coriolis acceleration.

(3) is centripetal acceleration due to movement along meridian. □

## 8.7 Curvature and Torsion of a Curve (Optional)

## 8.8 Review from Calculus in Several Variables

### ■ Chain Rules

$w = f(x, y, z)$ is continuous and has first partial derivatives in $xyz$ space

$$\left.\begin{array}{l} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{array}\right\} \text{is continuous and has first partial derivatives in } uv \text{ place}$$

$$\therefore\ w = f(x(u,v), y(u,v), z(u,v))$$

$$\Longrightarrow \begin{cases} \dfrac{\partial w}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial u} \\ \\ \dfrac{\partial w}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial v} \end{cases} \quad \square$$

## ■ Mean Value Theorems

- One-dimensional:

$$f(x_0+h) = f(x_0) + h\,\frac{df}{dx}\bigg|_{x=\xi}, \qquad \text{where } \mathrm{x}_0 \le \xi \le \mathrm{x}_0 + \mathrm{h}$$

$$\text{i.e. } \frac{f(x_0+h)-f(x_0)}{h} = \frac{df}{dx}\bigg|_{x=\xi}$$

- Three-dimensional:

$$f(x_0+h, y_0+k, z_0+l) = f(x_0, y_0, z_0) + \Big(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z}\Big)\bigg|_{(\xi,\eta,\zeta)},$$

where $(\xi, \eta, \zeta)$ in on the straight line segment joining $(x_0, y_0, z_0)$ and $(x_0+h, y_0+k, z_0+l)$. □

## 8.9 Gradient of a Scalar Field. Directional Derivative

$$\mathrm{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

Differential operator: $\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

### ■ <u>Directional Derivative</u>

The rate of change of $f$ at $P$ in the direction $\hat{b} = D_{\hat{b}} f = \frac{df}{ds}$

$$\begin{aligned} C : \vec{r}(s) &= x(s)\hat{i} + y(s)\hat{j} + z(s)\hat{k} \\ &= \vec{P}_0 + s\hat{b} \end{aligned}$$

$$\frac{d\vec{r}}{ds} = \hat{b} = \frac{\partial x}{\partial s}\hat{i} + \frac{\partial y}{\partial s}\hat{j} + \frac{\partial z}{\partial s}\hat{k}$$

$$\begin{aligned} D_{\hat{b}} f &= \frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} \\ &= \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) \cdot \left(\frac{\partial x}{\partial s}\hat{i} + \frac{\partial y}{\partial s}\hat{j} + \frac{\partial z}{\partial s}\hat{k}\right) \\ &= \nabla f \cdot \hat{b} \quad \square \end{aligned}$$

## ■ Gradient Characterizes Maximum Increase of $f(x, y, z)$

Theorem :

$f(x, y, z)$ has continuous first derivatives and $|\nabla f| \neq 0$ at $P$

$\Longrightarrow \nabla f$ is in the direction of maximum increase (or decrease) of $f$ at $P$.

*Proof* :

$|D_{\hat{b}} f| = |\hat{b}| \cdot |\nabla f| \cos\gamma = |\nabla f| \cos\gamma$ is maximum when $\gamma = 0$,

where $\gamma$ is the angle between $\hat{b}$ and $\nabla f$.

$\therefore$ The derivative is maximum, when $\hat{b}$ is in the direction of $\nabla f$. □

## ■ Gradient as surface normal vector

Theorem :

$S : f(x, y, z) = c$ represents a surface, and $|\nabla f| \neq 0$ at $P$

$\Longrightarrow \nabla f$ is a normal vector of $S$ at $P$.

*Proof* :

Let a curve $C : \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

lies on $S : f(x, y, z) = c$,

i.e. $f(x(t), y(t), z(t)) = c$.

$$\Longrightarrow \frac{df}{dt} = 0$$

$$\Longrightarrow \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

$$\Longrightarrow \nabla f \cdot \vec{r}' = 0$$

$\because$ $\vec{r}'$ is the tangent vector on $C$ at $P$

$\therefore$ For all $C$ pass $P$ the tangent vectors form a plane called tangent plane.

$\therefore$ $\nabla f$ is normal to all of these tangent vectors, thus normal to the tangent plane.

$\therefore$ $\nabla f$ is a normal vector of $S$ at $P$. □

## ■ Potential

If a vector field $\vec{v}(p)$ is the gradient of a scalar field $f(p)$, i.e. $\vec{v}(p) = \nabla f(p)$, then the $f(p)$ is called a *potential function* or *potential* of $\vec{v}(p)$.

*Ex* : Gravitational field

$$|F| = \frac{GMm}{r^2}$$

$$\vec{r} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}$$

$$\begin{aligned} \vec{F} &= |F|\Big(-\frac{\vec{r}}{|\vec{r}|}\Big) = -c\frac{\vec{r}}{r^3}, \qquad c = GMm \\ &= -c\Big(\frac{x - x_0}{r^3}\ \hat{i} + \frac{y - y_0}{r^3}\ \hat{j} + \frac{z - z_0}{r^3}\ \hat{k}\Big) \end{aligned}$$

Note that,

$$\frac{\partial}{\partial x}\Big(\frac{1}{r}\Big) = -\frac{2(x - x_0)}{2[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}} = -\frac{x - x_0}{r^3}$$

$$\frac{\partial}{\partial y}\Big(\frac{1}{r}\Big) = -\frac{y - y_0}{r^3}$$

$$\frac{\partial}{\partial z}\Big(\frac{1}{r}\Big) = -\frac{z - z_0}{r^3}$$

$$\Longrightarrow \vec{F} = \nabla\Big(\frac{c}{r}\Big)$$

$\therefore$ $f = \dfrac{c}{r}$ is a potential of gravitational field.

$f = \dfrac{c}{r}$ satisfies *Laplace equation* :

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

$$\left.\begin{aligned} \frac{\partial^2}{\partial x^2}\Big(\frac{1}{r}\Big) &= -\frac{1}{r^3} + \frac{3(x-x_0)^2}{r^5} \\ \frac{\partial^2}{\partial y^2}\Big(\frac{1}{r}\Big) &= -\frac{1}{r^3} + \frac{3(y-y_0)^2}{r^5} \\ \frac{\partial^2}{\partial z^2}\Big(\frac{1}{r}\Big) &= -\frac{1}{r^3} + \frac{3(z-z_0)^2}{r^5} \end{aligned}\right\} \Longrightarrow \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\Big)\Big(\frac{1}{r}\Big) = 0$$

$$\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{is called Laplace operator.} \quad \square$$

## 8.10 Divergence of a Vector Field

$$\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$$

The divergence of the vector $\vec{v}$ is defined as:

$$\begin{aligned} \text{div}\vec{v} = \nabla \cdot \vec{v} &= (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}) \cdot (u\hat{i} + v\hat{j} + w\hat{k}) \\ &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \end{aligned}$$

Theorem : $\nabla \cdot \vec{v}$ does not depend on the choice of the coordinate i.e.

$$\nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*}$$

Note:

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

*Ex* : Meaning of divergence

Consider the motion of a fluid with no points at which fluid is produced or disappears (i.e. no sinks or sources in the fluid). Assume the fluid is compressible i.e. the density $\rho = \rho(x, y, z, t)$

Consider the flow through a fixed small box with volume $\Delta V$, where $\Delta V = \Delta x \Delta y \Delta z$

Conservation of mass:

*(Change of mass in $\Delta V$ within $\Delta t$)*
*=(Mass flux into $\Delta V-$ mass flux out of $\Delta V$ within $\Delta t$)*

- *Change of mass in $\Delta V$ within $\Delta t$*

$$= \frac{\partial(\rho \Delta V)}{\partial t} \Delta t = \frac{\partial \rho}{\partial t} \Delta V \Delta t$$

•*Mass flux (into − out of)* $\Delta V$ *within* $\Delta t$

in $x$-direction

$$= u\rho\Delta y\Delta z\Delta t - \left(u\rho + \frac{\partial(u\rho)}{\partial x}\Delta x\right)\Delta y\Delta z\Delta t$$

$$= -\frac{\partial(u\rho)}{\partial x}(\Delta x\Delta y\Delta z)\Delta t$$

in $y$-direction

$$= v\rho\Delta x\Delta z\Delta t - \left(v\rho + \frac{\partial(v\rho)}{\partial y}\Delta y\right)\Delta x\Delta z\Delta t$$

$$= -\frac{\partial(v\rho)}{\partial x}(\Delta x\Delta y\Delta z)\Delta t$$

in $z$-direction

$$= w\rho\Delta x\Delta y\Delta t - \left(w\rho + \frac{\partial(w\rho)}{\partial z}\Delta z\right)\Delta x\Delta y\Delta t$$

$$= -\frac{\partial(w\rho)}{\partial z}(\Delta x\Delta y\Delta z)\Delta t$$

$\Longrightarrow$ Mass flux (into − out of) $\Delta V$ within $\Delta t$

$$= \left[-\frac{\partial(u\rho)}{\partial x} - \frac{\partial(v\rho)}{\partial y} - \frac{\partial(w\rho)}{\partial z}\right]\Delta x\Delta y\Delta z\Delta t = -\nabla\cdot(\rho\vec{v})\Delta V\Delta t$$

This means that the physical meaning of $\nabla\cdot(\rho\vec{v})$ is the *net* mass flux out of (diverged from) $(x, y, z)$ per unit volume per unit time.

$\therefore$ If the fluid is incompressible

$\Longrightarrow \nabla\cdot\vec{v} = 0$

∴ conservation of mass in $\Delta V$ within $\Delta t$

$$\Longrightarrow \frac{\partial \rho}{\partial t}\Delta V \Delta t = \left[-\frac{\partial(u\rho)}{\partial x} - \frac{\partial(v\rho)}{\partial y} - \frac{\partial(w\rho)}{\partial z}\right]\Delta V \Delta t$$

$$\Longrightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(u\rho)}{\partial x} + \frac{\partial(v\rho)}{\partial y} + \frac{\partial(w\rho)}{\partial z} = 0$$

$$\Longrightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\vec{v}\rho) = 0}$$

This is the condition for the conservation of mass of a compressible flow called continuity equation.

If the flow is steady, i.e. $\dfrac{\partial p}{\partial t} = 0$

$$\Longrightarrow \nabla \cdot (\rho\vec{v}) = 0 \qquad \square$$

## 8.11 Curl of a Vector Field

$$\vec{v} = u\hat{i} + v\hat{j} + w\hat{k}$$

The curl of the vector is defined as

$$\Longrightarrow \text{curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

<u>*Ex*</u> :

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \qquad \vec{\omega} = \xi\hat{i} + \eta\hat{j} + \zeta\hat{k}$$

$$\begin{aligned} \vec{v} &= \vec{w} \times \vec{r} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \xi & \eta & \zeta \\ x & y & z \end{vmatrix} \\ &= (\eta z - \zeta y)\hat{i} + (\zeta x - \xi z)\hat{j} + (\xi y - \eta x)\hat{k} \end{aligned}$$

$$\begin{aligned} \nabla \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (\eta z - \zeta y) & (\zeta x - \xi z) & (\xi y - \eta x)\hat{k} \end{vmatrix} \\ &= \hat{i}(\xi + \xi) + \hat{j}(\eta + \eta) + \hat{k}(\zeta + \zeta) \end{aligned}$$

$$\therefore \nabla \times \vec{v} = 2\vec{w} \qquad \square$$

<u>Note</u> :

$$\nabla \times (\nabla f) = 0$$

$\therefore$ If a vector field is the gradient of a scalar function, then the curl of the vector field is zero, and such flow field is called *irrotational* (or *conservative*).

The scalar function $f$ is called the potential function of the vector field $\vec{v}$, if $\vec{v} = \nabla f$.

## 8.12 Gradient, Divergence and Curl in Curvilinear Coordinate (Optional)

•Cylindrical coordinate :

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

•Spherical coordinate :

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi$$