APPLIED MATHEMATICS

Part 5:

Partial Differential Equations

Wu-ting Tsai

Contents

Chapter 11

Partial Differential Equations

11.1 Basic Concepts

Partial differential equation $\Longleftrightarrow$ Ordinary differential equation

$$\begin{cases} \text{order of differential equation} \\ \text{linear} \Longleftrightarrow \text{nonlinear} \\ \text{homogeneous} \Longleftrightarrow \text{nonromogeneous} \end{cases}$$

For examples:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{one-dimensional wave equation}$$

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{one-dimensional heat equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{two-dimensional Laplace equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{two-dimensional Poisson's equation}$$

Note:
$u(x, y) = x^2 - y^2$, $e^x \cos y$ and $\ln(x^2 + y)$ all satisfy the two-dimensional Laplace equation
$\Rightarrow$ "boundary condition" makes the solution unique
(or "initial condition" when t is one of the variable).

Theorem: Superposition of solutions

If u_1 and u_2 are any solutions of a *linear* and *homogeneous* partial differential equation in $\mathcal{R}$
$\Rightarrow u = c_1u_1 + c_2u_2$, where c_1 and c_2 are constant, is also solution of the equation in $\mathcal{R}$ $\square$

11.2 Modeling: Vibrating String. Wave Equation

Consider a vibrating, elastic string with length L and deformation $u(x,t)$.

Assumptions:

1. homogeneous string (i.e. constant density)
2. perfectly elastic and cannot sustain bending
3. neglect gravitation force (i.e. tension $\gg$ gravitational force)
4. small deformation (i.e. du/dx is small)
5. u is in a plane only

Since du/dx (slope) is small
$\Rightarrow$ no horizontal motion
$\Rightarrow$ constant horizontal tension

$$T_1 \cos\alpha = T_2 \cos\beta = T$$

in vertical direction:

$$\left(T_2 \sin\beta - T_1 \sin\alpha = \rho \Delta x \frac{\partial^2 u}{\partial t^2}\right) \div T$$

$$\Rightarrow \frac{T_2 \sin\beta}{T_2 \cos\beta} - \frac{T_1 \sin\alpha}{T_1 \cos\alpha} = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$$\Rightarrow \frac{\left[\left(\frac{\partial u}{\partial x}\right)_x + \Delta x - \left(\frac{\partial u}{\partial x}\right)_x\right]}{\Delta x} = \frac{\rho}{T} \cdot \frac{\partial^2 u}{\partial t^2}$$

$\Delta x \to 0$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}} \qquad \text{where } c^2 \equiv \frac{T}{\rho} \quad \square$$

11.3 Separation of Variables. Use of Fourier Series

One-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{———} \; (1)$$

Boundary conditions:

$$u(0, t) = 0 \quad \text{———} \; (2)$$

$$u(L, t) = 0 \quad \text{———} \; (3)$$

Initial conditions:

$$u(x, 0) = f(x) \quad \text{———} \; (4)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \quad \text{———} \; (5)$$

Method of separation of variables (product method):

$$u(x, t) = F(x)G(t)$$

$$\Rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial t^2} = F \dfrac{d^2 G}{dt^2} \equiv F\ddot{G} \\ \\ \dfrac{\partial^2 u}{\partial x^2} = \dfrac{d^2 F}{dx^2} G \equiv F'' G \end{cases}$$

$$(1) \Rightarrow F\ddot{G} = c^2 F'' G$$

$$\Rightarrow \underbrace{\frac{\ddot{G}}{c^2 G}}_{\text{function of } t} = \underbrace{\frac{F''}{F}}_{\text{function of } x} = k$$

where k is constant to be determined.

$$\Rightarrow \begin{cases} F'' - kF = 0 & \text{———— (6)} \\ \ddot{G} - c^2 kG = 0 & \text{———— (7)} \end{cases}$$

$\boxed{F(x)}$

$$F'' - kF = 0 \qquad F(0) = F(L) = 0$$

- If $k = 0 \quad \Rightarrow F(x) = ax + b$

$$\begin{cases} F(0) = b = 0 \\ F(L) = aL = 0 \end{cases}$$

$$\Rightarrow a = b = 0$$

$$\Rightarrow F(x) = 0 \quad \Rightarrow u(x,t) = 0 \qquad \times$$

- If $k = \mu^2 > 0 \quad \Rightarrow F(x) = Ae^{\mu x} + Be^{-\mu x}$

$$\begin{cases} F(0) = A + B = 0 \\ F(L) = Ae^{\mu L} + Be^{-\mu L} = 0 \end{cases}$$

$$\Rightarrow A = B = 0$$

$$\Rightarrow F(x) = 0 \quad \Rightarrow u(x,t) = 0 \qquad \times$$

$\therefore\ k = -p^2 < 0 \quad \Rightarrow \quad F(x) = A\cos px + B\sin px$

$$\begin{cases} F(0) = A = 0 \\ F(L) = B\sin pL = 0 \end{cases}$$

$$\Rightarrow pL = n\pi \quad \Rightarrow p = \frac{n\pi}{L}, \qquad n = 1, 2, 3\ldots$$

$$\Rightarrow F(x) \equiv F_n(x) = \sin\frac{n\pi}{L}x, \qquad n = 1, 2, 3\ldots$$

$$\text{and } k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$$

$\boxed{G(t)}$

$$\ddot{G} - c^2\left(-\left(\frac{n\pi}{L}\right)^2\right)G = 0$$

$$\Rightarrow \ddot{G} + \lambda_n^2 G = 0, \qquad \lambda_n = \frac{cn\pi}{L}$$

$$G(t) \equiv G_n(t) = B_n\cos\lambda_n t + B_n^*\sin\lambda_n t$$

$$\therefore\ u_n(x,t) = F_n(x)G_n(t)$$

$$= (B_n\cos\lambda_n t + B_n^*\sin\lambda_n t)\sin\frac{n\pi}{L}x \qquad (n = 1, 2, 3\ldots)$$

So far, $u_n(x,t)$ satisfies the differential equation (1), and the boundary conditions (2) and (3).
$u_n(x,t)$ is called *eigenfunction* or *characteristic function*.
λ_n is called *eigenvalue* or *characteristic value*.

A single solution $u_n(x,t)$ does not satisfy initial condition.
Since the differential equation (1) is linear and homogeneous, we therefore assume the solution of (1) is:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

Initial conditions:

$$\begin{cases} u(x,0) = f(x) & \text{——— (4)} \\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = g(x) & \text{——— (5)} \end{cases}$$

$$(4) \Rightarrow u(x,0) = \sum_{n=1} B_n \sin \frac{n\pi}{L} x = f(x) \quad \text{(Fourier sine series of } f(x)\text{)}$$

$$\Rightarrow \boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx} \qquad n = 1, 2, 3 \ldots$$

$$(5) \Rightarrow \left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} (B_n^* \lambda_n \cos \lambda_n t)_{t=0} \sin \frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x) \quad \text{(Fourier sine series of } g(x)\text{)}$$

$$\Rightarrow B_n^* \lambda_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

$$\Rightarrow \boxed{B_n^* = \frac{2}{\lambda_n L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx} \qquad n = 1, 2, 3 \ldots \quad \square$$

■ Discussion of eigenfunctions and eigenvalues:

(1) The eigenfunction

$$u_n(x,t) = (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x$$

represents a harmonic motion having frequency (cycles/unit time) $\dfrac{\lambda_n}{2\pi} = \dfrac{cn}{2L}$. This motion is called the *normal mode*.

(2) The frequency (eigenvalue)

$$w_n = \frac{cn}{2L}, \quad c^2 = \frac{T}{\rho}$$

where T = tension, ρ = density

$\Rightarrow$ increase tension or reduce density will increase the frequency.

■ Discussion of the solution:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} (B_n \cos\lambda_n t + B_n^* \sin\lambda_n t) \sin\frac{n\pi}{L}x \end{aligned}$$

$$\lambda_n = \frac{cn\pi}{L}$$

$$\begin{cases} B_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\sin\frac{n\pi x}{L}dx, & u(x,0) = f(x) \\ B_n^* = \dfrac{2}{\lambda_n L}\displaystyle\int_0^L g(x)\sin\frac{n\pi x}{L}dx, & \left.\dfrac{\partial u}{\partial t}\right|_{t=0} = g(x) \end{cases}$$

Consider the special case in which the string starts from rest, i.e. $g(x) = 0$, and $B_n^* = 0$, then

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n \cos\frac{cn\pi}{L}t \ \sin\frac{n\pi}{L}x \\ &= \frac{1}{2}\sum_{n=1}^{\infty} B_n \left\{\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right\} \\ &= \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x-ct)}_{\text{Fourier sine series of } f^*(x-ct)} + \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}(x+ct)}_{\text{Fourier sine series of } f^*(x+ct)} \\ &= \frac{1}{2}[f^*(x-ct) + f^*(x+ct)] \end{aligned}$$

where $f^*(\bullet)$ is an odd periodic function, and $f(\bullet) = f^*(\bullet)$ for $\bullet \in [0, 2L]$.

Physical meaning of $u(x,t) = \dfrac{1}{2}[f^*(x-ct) + f^*(x+ct)]$:

<u>*Ex*</u> :

$$g(x) = 0$$

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x < \dfrac{L}{2} \\ \\ \dfrac{2k}{L}(L - x), & \dfrac{L}{2} < x < L \end{cases}$$

$$B_n^* = 0$$

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

$$= \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi}{L}x \cdot \cos\frac{(2n-1)\pi c}{L}t\right]$$

11.4 Heat Equation

Consider the heat conduction along a heat-conducting homogeneous rod with temperature distribution $u(x, t)$.

Assumptions:

1. homogeneous rod with uniform cross section A and constant density ρ, specific heat c, thermal conductivity κ

2. insulated laterally so heat flows only in x-direction only

3. temperature is constant at all points of a cross section

4. rate of heat conduction is proportional to $-\partial u/\partial x$

We are going to derive the differential equation governing the temperature distribution $u(x, t)$ along the rod by conservation of energy.

The amount of internal energy within Δx segment is:

$$Q(x, t) = \int_x^{x+\Delta x} \sigma \rho A u(\tilde{x}, t) d\tilde{x}$$

where σ = specific heat (amount of energy required to raise the temperature of a unit mass by one unit).

- Rate of change of internal energy within Δx

$$= \frac{\partial Q}{\partial t} = \int_x^{x+\Delta x} \sigma\rho A \frac{\partial u}{\partial t}(\tilde{x}, t) d\tilde{x}$$

- Rate at which heat flows into Δx

$$= -\kappa A \frac{\partial u}{\partial x}(x, t)$$

- Rate at which heat flows out of Δx

$$= -\kappa A \frac{\partial u}{\partial x}(x + \Delta x, t)$$

$\therefore$ conservation of energy requires that

$$\frac{\partial Q}{\partial t} = \int_x^{x+\Delta x} \sigma\rho A \frac{\partial u}{\partial t}(\tilde{x}, t) d\tilde{x}$$

$$= \sigma\rho A \frac{\partial u}{\partial t}(\xi, t)\Delta x \qquad (x < \xi < x + \Delta x) \quad \text{[by mean value theorem]}$$

$$= -\kappa A \frac{\partial u}{\partial x}(x, t) + \kappa A \frac{\partial u}{\partial x}(x + \Delta x, t)$$

$$\Rightarrow \; \sigma\rho A \frac{\partial u}{\partial t}(\xi, t)\Delta x = \kappa A \left[\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right]$$

$$\Rightarrow \; \frac{\partial u}{\partial t}(\xi, t) - \frac{\kappa}{\sigma\rho} \frac{\left[\frac{\partial u}{\partial x}(x + \Delta x, t) - \frac{\partial u}{\partial x}(x, t)\right]}{\Delta x} = 0$$

As $\Delta x \to 0$

$$\boxed{\frac{\partial u}{\partial t}(x, t) - c^2 \frac{\partial^2 u}{\partial x^2}(x, t) = 0} \qquad c^2 = \frac{k}{\sigma\rho}$$

Initial condition:

$$u(x,0) = f(x) \qquad (0 \le x \le L)$$

Boundary condition:

$$u(0,t) = 0, \quad u(L,t) = 0 \qquad (t \ge 0)$$

Note:

For $3 - D$ heat equation (see §9.8):

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

■ Fourier series solution of the heat equation:

$$\frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad t \geq 0, \quad 0 \leq x \leq L \quad \text{——— (1)}$$

$$\left.\begin{array}{l} u(0,t) = 0 \quad \text{for} \quad t \geq 0, \quad x = 0 \\ u(L,t) = 0 \quad \text{for} \quad t \geq 0, \quad x = L \end{array}\right\} \quad \text{——— (2)}$$

$$u(x,0) = f(x) \quad \text{for} \quad t = 0, \quad 0 \leq x \leq L \quad \text{——— (3)}$$

By method of separation of variables:

$$u(x,t) = F(x) \cdot G(t)$$

$$(1) \Rightarrow F\dot{G} = c^2 F'' G$$

$$\Rightarrow \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = \begin{cases} p^2 & \Rightarrow \dot{G} = c^2 p^2 G \quad \Rightarrow G(t) = Ce^{c^2 p^2 t} \quad \times \\ 0 & \Rightarrow \dot{G} = 0 \quad \Rightarrow G(t) = C \quad \times \\ -p^2 & \end{cases}$$

$$\Rightarrow \begin{cases} F'' + p^2 F = 0 & \text{——— (6)} \\ \dot{G} + c^2 p^2 G = 0 & \text{——— (7)} \end{cases}$$

$\underline{\underline{F(x)}}$:

$(6) \Rightarrow F(x) = A\cos px + B\sin px$

$(2) \Rightarrow F(0)G(t) = 0, \quad F(L)G(t) = 0$

Since $G(t) \neq 0 \quad \Rightarrow F(0) = F(L) = 0$

$\therefore \ A = 0 \quad B\sin pL = 0$

Since $B \neq 0 \quad \Rightarrow \ \sin pL = 0, \qquad$ i.e. $pL = n\pi, \quad n = 1, 2, 3\ldots$

$$\Rightarrow \boxed{p = \frac{n\pi}{L}} \quad n = 1, 2, 3\ldots$$

$$\boxed{F_n(x) = \sin px = \sin\frac{n\pi x}{L}} \quad n = 1, 2, 3\ldots$$

$\underline{\underline{G(t)}}$:

$$(3) \Rightarrow \dot{G} + \lambda_n^2 G = 0 \qquad \lambda_n = \frac{cn\pi}{L}$$

$$\Rightarrow \boxed{G_n(t) = B_n e^{-\lambda_n^2 t}} \qquad n = 1, 2, 3 \ldots$$

$\therefore$ *eigenfunction* $u_n(x,t)$:

$$\boxed{u_n(x,t) = F_n(x) \cdot G_n(t) = B_n \sin \frac{n\pi x}{L} \cdot e^{-\lambda_n^2 t}}$$

with corresponding *eigenvalue* λ_n :

$$\boxed{\lambda_n = \frac{cn\pi}{L}}$$

In order to satisfy initial conduction (3) at $t = 0$:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}$$

$$\Rightarrow u(x,0) = f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \qquad \text{(Fourier sine series of } f(x)\text{)}$$

$$\Rightarrow B_n = \frac{2}{L} \int_0^L f(x) \sin n\pi x dx \quad \square$$

Note:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \cdot e^{-\lambda_n^2 t}$$

$\Rightarrow$ the temperature alway decays.

Ex :

■ Physical meanings of the boundary conditions

$u(0,t) = u(L,t) = \text{constant}$

$\Rightarrow$ Temperatures at $x = 0$ and L remain constant, but heat can still transfer across $x = 0$ and L

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

$\Rightarrow$ No heat transfer across $x = 0$ and L, i.e. these two ends are insulated, but the temperature at these two boundary may vary.

(See Page 654, Problem Set 11.5, Problem 13)

11.5 Laplace Equation in a Rectangular Domain

Consider the steady-state two-dimensional heat flow:

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \quad (\because \text{ steady state})$$

$$\Rightarrow \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \leftarrow \text{two-dimensional Laplace equation}$$

- Boundary value problem (bvp) :

If the boundary conditions are given as

$$\begin{cases} u \text{ given} \quad \Rightarrow \text{Dirichlet problem} \\ \dfrac{\partial u}{\partial n} \text{ given} \quad \Rightarrow \text{Neumann problem} \\ u \text{ given on some boundary }, \dfrac{\partial u}{\partial n} \text{ given on the rest boundary} \\ \qquad\qquad \Rightarrow \text{mixed problem} \end{cases}$$

<u>*Ex*</u> :

Method of separation of variables :

$$u(x, y) = F(x)G(y)$$

$$\Rightarrow \frac{d^2F}{dx^2}G + F\frac{d^2G}{dy^2} = 0$$

$$\Rightarrow -\frac{1}{F}\frac{d^2F}{dx^2} = \frac{1}{G}\frac{d^2G}{dy^2} \equiv k$$

$$\Rightarrow \begin{cases} \dfrac{d^2F}{dx^2} + kF = 0 & \text{———— (1)} \\ \\ \dfrac{d^2G}{dy^2} - kG = 0 & \text{———— (2)} \end{cases}$$

If $\boxed{k < 0}$:

$$(1) \;\Rightarrow\; F(x) = Ae^{\sqrt{k}x} + Be^{-\sqrt{k}x}$$

Boundary conditions: $u(0, y) = 0$ and $u(a, y) = 0$

$$\Rightarrow F(0) = 0, \quad F(a) = 0$$

$$\Rightarrow \begin{cases} A + B = 0 \\ Ae^{\sqrt{k}a} + Be^{-\sqrt{k}a} = 0 \end{cases}$$

$$\Rightarrow A(e^{\sqrt{k}a} - e^{-\sqrt{k}a}) = 0$$

$$\therefore A = 0, \quad B = 0 \qquad \times$$

If $\boxed{k > 0}$:

$$(1) \;\Rightarrow\; F(x) = A\cos\sqrt{k}x + B\sin\sqrt{k}x$$

Boundary conditions:

$$F(0) = 0 \quad \Rightarrow A = 0$$

$$F(a) = 0 \quad \Rightarrow F(a) = B\sin\sqrt{k}a = 0 \quad \Rightarrow \sqrt{k} = \frac{n\pi}{a}$$

$$\Rightarrow \boxed{k_n = \left(\frac{n\pi}{a}\right)^2}$$

and $\boxed{F_n(x) = \sin\frac{n\pi}{a}x}$

$$(2) \;\Rightarrow\; \frac{d^2G}{dy^2} - k_n G = 0$$

$$\Rightarrow G(y) = G_n(y) = A_n e^{\sqrt{k_n}y} + B_n e^{-\sqrt{k_n}y}$$

Boundary condition:

$$u(x,0) = 0 \quad \Rightarrow G(0) = 0$$

$$\Rightarrow A_n + B_n = 0$$

$$\Rightarrow G_n(y) = A_n(e^{\sqrt{k_n}y} - e^{-\sqrt{k_n}y}) = 2A_n \frac{(e^{\sqrt{k_n}y} - e^{-\sqrt{k_n}y})}{2}$$

$$\therefore \boxed{G_n(y) = A_n^* \sinh\sqrt{k_n}y} \qquad A_n^* \equiv 2A_n$$

$$\begin{aligned} \therefore u_n(x,y) &= F_n(x)\cdot G_n(y) \\ &= A_n^* \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \end{aligned}$$

$$\Rightarrow \boxed{u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} A_n^* \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a}}$$

Since $u(x,b) = f(x)$:

$$\Rightarrow u(x,b) = \sum_{n=1}^{\infty} \underbrace{A_n^* \sinh\frac{n\pi b}{a}}_{\equiv\, b_n} \sin\frac{n\pi x}{a} = f(x)$$

$$\Rightarrow b_n = A_n^* \sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}dx$$

$$\therefore \boxed{A_n^* = \frac{2}{a\sinh(\frac{n\pi b}{a})}\int_0^a f(x)\sin\frac{n\pi x}{a}dx} \qquad \square$$

11.6 Two-Dimensional Wave Equation. Use of Double Fourier Series

Consider the motion of a stretched elastic membrane, such as a drumhead.

$\Rightarrow$ Two dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \qquad c^2 = \frac{T}{\rho}$$

$u(x, y, t) =$ vertical deformation
$T =$ tension force per unit length
$\rho =$ mass of the membrane per unit area

Consider the problem of a vibrating rectangular membrane:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \qquad 0 \le x \le a, \quad 0 \le y \le b \quad \text{———} \ (1)$$

Boundary conditions:

$$u(x, y, t) = 0 \quad \text{on the boundary for } t \ge 0 \quad \text{———} \ (2)$$

Initial condition:

$$\begin{cases} u(x, y, 0) = f(x, y) \quad \text{———} \ (3) \\ \dfrac{\partial u}{\partial t}(x, y, 0) = g(x, y) \quad \text{———} \ (4) \end{cases}$$

By method of separation of variables we first determine the solutions of (1) that satisfy the boundary condition (2) :

$$u(x, y, t) = F(x, y) \cdot G(t)$$

$$(1) \ \Rightarrow F\ddot{G} = c^2(F_{xx}G + F_{yy}G) \qquad \div (c^2 FG)$$

$$\Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{1}{F}(F_{xx} + F_{yy}) \equiv -\nu^2 \qquad \text{(cannot be zero or posititice)}$$

$$\Rightarrow \begin{cases} \ddot{G} + \lambda^2 G = 0 \qquad (\lambda \equiv c\nu) \quad \text{———} \ (6) \\ F_{xx} + F_{yy} + \nu^2 F = 0 \quad \text{———} \ (7) \end{cases}$$

$$\begin{cases} \ddot{G} + \lambda^2 G = 0 & \text{———— (6)} \\ F_{xx} + F_{yy} + \nu^2 F = 0 & \text{———— (7)} \end{cases}$$

(7) is called two-dimensional Helmholtz equation, if $\nu = 0$ it become Laplace equation.

• Solution of Helmholtz equation (7) $F_{xx} + F_{yy} + \nu^2 F = 0$

Again, using method of separation of variables :

$$F(x, y) = H(x)\ Q(y)$$

$$(7) \quad \Rightarrow \frac{d^2 H}{dx^2} Q + H \frac{d^2 Q}{dy^2} + \nu^2 H Q = 0 \qquad \div (HQ)$$

$$\Rightarrow \frac{1}{H} \frac{d^2 H}{dx^2} = -\frac{1}{Q} \left(\frac{d^2 Q}{dy^2} + \nu^2 Q \right) \equiv -k^2 \qquad \text{(must be negative)}$$

$$\Rightarrow \begin{cases} \dfrac{d^2 H}{dx^2} + k^2 H = 0 \\[2ex] \dfrac{d^2 Q}{dy^2} + (\nu^2 - k^2) Q = 0 \qquad (\nu^2 - k^2 \equiv p^2) \end{cases}$$

$$\Rightarrow \begin{cases} H(x) = A \cos kx + B \sin kx \\ Q(y) = C \cos py + D \sin py \end{cases}$$

Since $u(x, y, t) = 0$ on $x = 0,\ x = a,\ y = 0,\ y = b$,

$$\Rightarrow \begin{cases} H(0) = 0 & \Rightarrow A = 0 \\ H(a) = 0 & \Rightarrow \sin ka = 0 \quad \Rightarrow k = \dfrac{m\pi}{a},\ m = 1, 2, 3 \ldots \\ Q(0) = 0 & \Rightarrow C = 0 \\ Q(b) = 0 & \Rightarrow \sin pb = 0 \quad \Rightarrow p = \dfrac{n\pi}{b},\ n = 1, 2, 3 \ldots \end{cases}$$

$$\therefore \begin{cases} H(x) = H_m(x) = \sin \dfrac{m\pi x}{a} \\ Q(y) = Q_n(y) = \sin \dfrac{n\pi y}{b} \end{cases}$$

$$\begin{aligned} \Rightarrow F(x,y) &= F_{mn}(x,y) = H_m(x)\ Q_n(y) \\ &= \sin \frac{m\pi x}{a}\ \sin \frac{n\pi y}{b} \qquad (m,n = 1,2,3\ldots) \end{aligned}$$

• Solution of (6) $\ddot{G} + \lambda^2 G = 0$

$$\lambda = c\nu \qquad p^2 = \nu^2 - k^2$$

$$\Rightarrow \lambda = c\sqrt{\nu^2} = c\sqrt{p^2 + k^2} = c\sqrt{(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2}$$

$$\Rightarrow \lambda = \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

(6) $\Rightarrow \ddot{G} + \lambda_{mn}^2 G = 0$

$$\Rightarrow G(t) = G_{mn}(t) = B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t$$

$$\begin{aligned} \therefore u_{mn}(x,y,t) &= F_{mn}(x,y)\cdot G_{mn}(t) \\ &= (\ B_{mn}\cos\lambda_{mn}t + B_{mn}^*\sin\lambda_{mn}t\ )\cdot\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b} \end{aligned}$$

$u_{mn}(x,y,t)$ is the eigenfunction with the corresponding eigenvalue λ_{mn}.

Since every $u_{mn}(x,y,t)$ satisfies (1) and (2), so is the summation:

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (\ B_{mn}\cos\lambda_{mn}t + B_{mn}^{*}\sin\lambda_{mn}t\ ) \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{b}$$

Now apply the initial conditions (3) and (4):

$$\begin{cases} (3)\ \Rightarrow u(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = f(x,y) & \text{———— (8)} \\ (4)\ \Rightarrow \frac{\partial u}{\partial t}(x,y,0) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}^{*}\lambda_{mn}\ \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = g(x,y) & \text{– (9)} \end{cases}$$

(8) and (9) are called double Fourier series of $f(x,y)$ and $g(x,y)$.

We can generalized the Euler formula in one dimension to two dimension:

$$\int_0^b\int_0^a (8)\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}dxdy$$

$$\Rightarrow \quad \int_0^a\int_0^b \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\sin\frac{m\pi x}{a}\cdot\sin\frac{m'\pi x}{a}\cdot\sin\frac{n\pi y}{b}\cdot\sin\frac{n'\pi y}{b}dxdy$$

$$= \int_0^a\int_0^b f(x,y)\sin\frac{m'\pi x}{a}\cdot\sin\frac{n'\pi y}{b}dxdy$$

$$\Rightarrow \quad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\Big[\int_0^a\Big(\sin\frac{m\pi x}{a}\cdot\sin\frac{m'\pi x}{a}\Big)dx \times \int_0^b\Big(\sin\frac{n\pi y}{b}\cdot\sin\frac{n'\pi y}{b}dy\Big)\Big]$$

$$= \int_0^a\int_0^b f(x,y)\sin\frac{m'\pi x}{a}\cdot\sin\frac{n'\pi y}{b}dxdy$$

$$\int_0^a \sin\frac{m\pi x}{a}\cdot\sin\frac{m'\pi x}{a}dx$$

$$=\frac{1}{2}\int_0^a \Big[\cos(m-m')\frac{\pi x}{a} - \underbrace{\cos(m+m')\frac{n\pi x}{a}}_{=0}\Big]dx$$

$$=\frac{1}{2}\int_0^a \cos(m-m')\frac{\pi x}{a}dx$$

$$=\begin{cases} 0 & \text{if} \quad m\neq m' \\ \frac{1}{2}a & \text{if} \quad m=m' \end{cases}$$

Similarly,

$$\int_0^b \sin\frac{n\pi y}{b}\cdot\sin\frac{n'\pi y}{b}dy = \begin{cases} 0 & \text{if} \quad n\neq n' \\ \frac{1}{2}b & \text{if} \quad n=n' \end{cases}$$

$$\Rightarrow B_{mn}\left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = \int_0^a\int_0^b f(x)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dxdy$$

$$\therefore B_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dxdy \qquad (m,n=1,2,3\ldots)$$

Similarly, from (9) we have:

$$B^*_{mn} = \frac{4}{ab\lambda_{mn}}\int_0^a\int_0^b g(x,y)\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}dxdy \quad \square$$

11.7 Heat Equation: Use of Fourier Integral

Consider the heat equation along a heat conducting rod extending to $\pm\infty$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} & (-\infty < x < +\infty) \\ \\ u(x,0) = f(x) & (-\infty < x < +\infty) \end{cases}$$

Again by method of separation of variables:

$$u(x,t) = F(x) \cdot G(t)$$

$$\begin{cases} F'' + p^2 F = 0 & \Rightarrow F(x) = A\cos px + B\sin px \\ \\ \dot{G} + c^2 p^2 G = 0 & \Rightarrow G(t) = e^{-c^2 p^2 t} \end{cases}$$

$$\therefore\ u(x,t;p) = F(x) \cdot G(t) = (A\cos px + B\sin px)e^{-c^2p^2t}$$

Note that there are no boundary conditions to determine p

This means *any* values of p is fine.

$\therefore$ We therefor write

$$\begin{aligned} u(x,t) &= \int_0^p u(x,t;p)dp \\ &= \int_0^p [A\cos px + B\sin px]e^{-c^2p^2t}dp \end{aligned}$$

Recall the Fourier integral for $f(x)$ $\left(\int_{-\infty}^{\infty} |f(x)|\, dx < \infty\right)$

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx] dw$$

$$\begin{cases} A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\cos w\tilde{x} d\tilde{x} \\ \\ B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\sin w\tilde{x} d\tilde{x} \end{cases}$$

Now apply the initial conditions:

$$u(x,0) = \int_0^\infty [A\cos px + B\sin px] dp = f(x)$$

$$\Rightarrow \begin{cases} A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\cos p\tilde{x} d\tilde{x} \\ \\ B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\sin p\tilde{x} d\tilde{x} \end{cases}$$

$$\begin{aligned} \therefore u(x,t) &= \frac{1}{\pi}\int_0^\infty \Big[\Big(\int_{-\infty}^{\infty} f(\tilde{x})\cos p\tilde{x} d\tilde{x} \Big) \cos px \\ &\qquad + \Big(\int_{-\infty}^{\infty} f(\tilde{x})\sin p\tilde{x} d\tilde{x} \Big) \sin px \Big] e^{-c^2p^2t} dp \\ &= \frac{1}{\pi}\int_0^\infty \Big[\int_{-\infty}^{\infty} f(\tilde{x})(\cos p\tilde{x}\cos px + \sin p\tilde{x}\sin px) d\tilde{x} \Big] e^{-c^2p^2t} dp \\ &= \frac{1}{\pi}\int_0^\infty \Big[\int_{-\infty}^{\infty} f(\tilde{x})\cos p(\tilde{x}-x) d\tilde{x} \Big] e^{-c^2p^2t} dp \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x}) \underbrace{\Big[\int_0^\infty \cos p(\tilde{x}-x)\cdot e^{-c^2p^2t} dp \Big]} d\tilde{x} \end{aligned}$$

Let $p \equiv \dfrac{s}{c\sqrt{t}}, \quad dp = \dfrac{ds}{c\sqrt{t}}$

$$
\begin{aligned}
& \int_0^\infty \cos p(\tilde{x}-x) \cdot e^{-c^2p^2t} dp \\
= \quad & \int_0^\infty \cos \frac{s}{c\sqrt{t}}(\tilde{x}-x) \cdot e^{-s^2} \frac{ds}{c\sqrt{t}} \qquad\qquad b \equiv \frac{(\tilde{x}-x)}{2c\sqrt{t}} \\
= \quad & \frac{1}{c\sqrt{t}} \left(\int_0^\infty \cos 2bs \cdot e^{-s^2} ds \right) \\
= \quad & \frac{1}{c\sqrt{t}} \left(\frac{\sqrt{\pi}}{2} e^{-b^2} \right) \\
= \quad & \frac{\sqrt{\pi}}{2c\sqrt{t}} e^{-\frac{(\tilde{x}-x)^2}{4c^2t}}
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ u(x,t) &= \frac{1}{\pi} \int_{-\infty}^{\infty} f(\tilde{x}) \left[\frac{\sqrt{\pi}}{2c\sqrt{t}} e^{-\frac{(\tilde{x}-x)^2}{4c^2t}} \right] d\tilde{x} \\
&= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\tilde{x}) \cdot e^{-\frac{(\tilde{x}-x)^2}{4c^2t}} d\tilde{x}
\end{aligned}
$$

<u>*Ex*</u> :

If $f(x) =$

$$u(x,t) = \frac{u_0}{2c\sqrt{\pi t}} \int_{-1}^{1} e^{-\frac{(\tilde{x}-x)^2}{4c^2 t}} d\tilde{x}$$

If $U_0 = 100\ {}^0\mathrm{C}, \quad c^2 = 1\ \mathrm{cm}^2/\mathrm{sec}.$

11.8 Heat Equation: Use of Fourier Transform

$$\begin{cases} \dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2} & (-\infty < x < \infty) \quad \text{———} \ (1) \\ \\ u(x,0) = f(x) & (-\infty < x < \infty) \quad \text{———} \ (2) \end{cases}$$

Recall that:

$$\hat{u}(w) = \mathcal{F}\{u(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x) e^{-iwx} dx$$

$$\mathcal{F}\{u'(x)\} = iw\mathcal{F}\{u(x)\}$$

$$\mathcal{F}\{u''(x)\} = -w^2\mathcal{F}\{u(x)\}$$

Take Fourier Transform of (1),

$$\Rightarrow \mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = c^2 \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -c^2 w^2 \mathcal{F}\{u\}$$

Also,

$$\mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-iwx} dx = \frac{1}{\sqrt{2\pi}} \frac{\partial}{\partial t} \left[\int_{-\infty}^{\infty} u e^{-iwx} dx\right] = \frac{\partial}{\partial t} \mathcal{F}\{u\}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\mathcal{F}\{u\} = -c^2w^2\mathcal{F}(u)$$

i.e. $$\frac{\partial \hat{u}(w,t)}{\partial t} = -c^2w^2\hat{u}(w,t) \quad \text{———} \; (3)$$

$\therefore$ we have transformed the partial differential equation (1) into an ordinary differential equation (3) !!

Similarly, taking Fourier transform of the initial condition (2), gives:

$$\hat{u}(w,0) = \hat{f}(w) \quad \text{———} \; (4)$$

(3) $\quad \Rightarrow \hat{u}(w,t) = Ce^{-c^2w^2t}$

(4) $\quad \Rightarrow \hat{u}(w,0) = C = \hat{f}(w)$

$$\therefore \hat{u}(w,t) = \hat{f}(w) \cdot e^{-c^2w^2t}$$

Take inverse Fourier transform of $\hat{u}(w,t)$:

$$\begin{cases} \hat{u}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)e^{-iwx}dx \\ \\ u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{u}(w)e^{iwx}dw \end{cases}$$

$$\hat{u}(w,t) = \hat{f}(w)\cdot e^{-c^2w^2t}$$

$$\Rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\cdot e^{-c^2w^2t}\cdot e^{iwx}dw$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} [\, \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\tilde{x})e^{-iw\tilde{x}}d\tilde{x}\,]\, e^{-c^2w^2t}\cdot e^{iwx}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [\, \int_{-\infty}^{\infty} f(\tilde{x})e^{-iw(\tilde{x}-x)}d\tilde{x}\,]\cdot e^{-c^2w^2t}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tilde{x})\, [\, \int_{-\infty}^{\infty} e^{-iw(\tilde{x}-x)}\cdot e^{-c^2w^2t}dw\,]\, d\tilde{x}$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\tilde{x})\, [\, \underbrace{\int_{0}^{\infty} \cos w(\tilde{x}-x)\cdot e^{-c^2w^2t}dw}_{=\frac{\sqrt{\pi}}{2c\sqrt{t}}\, e^{\frac{-(\tilde{x}-x)}{4c^2t}}}\,]\, d\tilde{x}$$

Same as using Fourier integral !! □

11.9 Heat Equation. Use of Fourier Cosine and Sine Transforms

Consider the heat equation along a semi-infinite rod:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (0 \le x < \infty)$$

Recall Fourier cosine and sine transforms of derivatives :

$$\begin{cases} \mathcal{F}_c\{f'\} = w\mathcal{F}_s\{f\} - \sqrt{\frac{2}{\pi}} f(0) \\ \mathcal{F}_s\{f'\} = -w\mathcal{F}_c\{f\} \end{cases}$$

$$\begin{cases} \mathcal{F}_c\{f''\} = -w^2\mathcal{F}_c\{f\} - \sqrt{\frac{2}{\pi}} f'(0) \\ \mathcal{F}_s\{f''\} = -w^2\mathcal{F}_s\{f\} + \sqrt{\frac{2}{\pi}} w f(0) \end{cases}$$

where $\mathcal{F}_c\{f\} = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx dx$

$$\mathcal{F}_s\{f\} = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx dx$$

$\Rightarrow$

If the boundary condition at $x = 0$ is :

$u(0.t) = g(t)$ we use Fourier *sine* transform

$\dfrac{\partial u}{\partial x}(0, t) = g(t)$ we use Fourier *cosine* transform

<u>*Ex*</u> :

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial u}{\partial x^2} \qquad 0 \le x < \infty \quad \text{———} \ (1)$$

$$u(x,0) = f(x) \qquad 0 \le x < \infty \quad \text{———} \ (2)$$

$$u(0,t) = 0 \qquad t \ge 0 \quad \text{———} \ (3)$$

Take Fourier sine transform of (1) :

$$\Rightarrow \mathcal{F}_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial \hat{u}_s}{\partial t} = c^2 \mathcal{F}_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -c^2 w^2 \mathcal{F}_s\{u\} + \sqrt{\frac{2}{\pi}} w u(0,t)$$

$$= -c^2 w^2 \hat{u}_s(w,t)$$

i.e. $\dfrac{\partial \hat{u}_s}{\partial t} = -c^2 w^2 \hat{u}_s$

$$\Rightarrow \hat{u}_s(w,t) = C(w) e^{-c^2 w^2 t}$$

Take Fourier sine transform of the initial condition (2): $u(x,0) = f(x)$

$$\Rightarrow \hat{u}_s(w,0) = \hat{f}_s(w) = C(w)$$

$$\therefore \hat{u}_s(w,t) = \hat{f}_s(w) e^{-c^2 w^2 t} = \left[\sqrt{\frac{2}{\pi}} \int_0^\infty f(\tilde{x}) \sin w\tilde{x} d\tilde{x}\right] e^{-c^2 w^2 t}$$

Take inverse Fourier sine Transform:

$$\therefore u(x,t) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(\tilde{x}) \sin w\tilde{x} \cdot e^{-c^2 w^2 t} \cdot \sin wx \ d\tilde{x} dw \quad \square$$

11.10 Wave Equation. Use of Fourier Transform

Consider wave equation along an infinitely long string :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} \qquad (-\infty < x < \infty) & \text{——— (1)} \\ u(x,0) = f(x) & \text{——— (2)} \\ u_t(x,0) = 0 & \text{——— (3)} \\ u \to 0,\ u_x \to 0 \qquad \text{as } |x| \to \infty & \text{——— (4)} \end{cases}$$

Take Fourier transform of (1):

$$\mathcal{F}\left\{\frac{\partial u}{\partial t^2}\right\} = \frac{\partial^2}{\partial t^2}\mathcal{F}\{u\} = c^2 \mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -c^2 w^2 \mathcal{F}\{u\}$$

$$\Rightarrow\ \hat{u}_{tt} + c^2 w^2 \hat{u} = 0, \qquad \hat{u} = \hat{u}(w,t)$$

$$\therefore\ \hat{u}(w,t) = A(w)\cos cwt + B(w)\sin cwt$$

Take Fourier transform of the initial conditions (2) and (3):

$$\hat{u}(w,0) = \hat{f}(w) = A(w)$$

$$\hat{u}_t(w,0) = 0 = cwB(w)$$

$$\begin{aligned} \Rightarrow \hat{u}(w,t) &= \hat{f}(w)\cos cwt \\ &= \frac{1}{2}\hat{f}(w)\left[e^{icwt} + e^{-icwt}\right] \end{aligned}$$

$$\mathcal{F}\{f(x-a)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{-iwx}dx \qquad x-a \equiv p,\ dx = dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(p)e^{-iw(p+a)}dp$$

$$= \frac{1}{\sqrt{2\pi}}e^{-iwa}\int_{-\infty}^{\infty} f(p)e^{-iwp}dp$$

i.e. $\mathcal{F}\{f(x-a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$

$\therefore\ \mathcal{F}^{-1}\{\mathcal{F}\{f(x-a)\}\} = \mathcal{F}^{-1}\{e^{-iwa}\mathcal{F}\{f(x)\}\} = f(x-a)$

i.e. the inverse Fourier Transform of $e^{-iwa}\mathcal{F}\{f(x)\}$ is $f(x-a)$

Similarly, the inverse Fourier transform of $e^{+iwa}\mathcal{F}\{f(x)\}$ is $f(x+a)$

Since $\hat{u}(w,t) = \frac{1}{2}\hat{f}(w)\left[e^{icwt} + e^{-icwt}\right]$

$$\therefore\ u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] \quad \square$$