APPLIED MATHEMATICS

Part 1:

Ordinary Differential Equations

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Contents

Chapter 1

First Order Differential Equations

1.1 Basic Concepts and Ideas

■ Ordinary differential equation (ode):
Equation which contains one or several derivatives of an unknown function of one variable

Ex: Unknown $y = y(x)$

$$y' = \cos x$$

$$x^2 y''' y' + 2e^x y'' = (x^2 + 2) y^2 \quad \square$$

cf: partial differential equation (pde):
Equation which contains one or several partial derivatives of an unknown function of multiple variables

■ [physics modeling] $\Longrightarrow$ [mathematics] $\Longrightarrow$ [solutions]

Ex:

$$m \frac{d^2 y}{dt^2} + ky = 0$$

$$l \frac{d^2 \theta}{dt^2} + g \sin\theta = 0$$

■ Order of an ode is the order of the highest derivative.

Ex:
$F(x, y, y') = 0$ is a first order ode.
$F(x, y, y', y'', \dots y^{(n)}) = 0$ is an n-th order ode. □

■ Solution of ode, $y(x)$, can be
explicit: $y = h(x)$
or
implicit: $H(x, y) = 0$

Ex:

$xy' = 2y$ explicit solution : $y = x^2$

$yy' = -x \; (-1 < x < 1)$ implicit solution : $x^2 + y^2 - 1 = 0 \; (y > 0)$ □

■ General and particular solutions of an ode.

Ex:

$$\frac{dy}{dx} = \cos x$$

$$\Longrightarrow \int \frac{dy}{dx} = \int \cos x dx$$

$$\Longrightarrow y(x) = -\sin x + c \quad \leftarrow \text{general solution}$$

$$y(x) = -\sin x + 1 \text{ or } \pi \text{ or } 0.8 \dots \quad \leftarrow \text{particular solution}$$ □

<u>*Ex*</u>:
$y'^2 = -1$ has no solution.
$|y'| + |y| = 0$ has no general solution $\because$ $y(x) \equiv 0$ □

■ <u>Initial value problems</u> (IVP) (cf: Boundary value problems)
A differential equation with initial conditions.

<u>*Ex*</u>:
Physical system —
Experiments show that a radioactive substance decomposes at a rate proportional to the amount present.

$$\Downarrow$$

Mathematical model —
Let $y(t)$ be the amount of substance at time t then

$$\frac{dy}{dt} = ky, \text{ where } k \text{ is a physical constant}$$

$$\Downarrow$$

General solution —

$$y(t) = ce^{kt}, \text{ where } c \text{ is a constant}$$

$$\Downarrow$$

Particular solution —
If at $t = 0,\ y = 2$ (initial condition)
$\therefore$ $y(0) = c = 2$
$\Longrightarrow y(t) = 2e^{kt}$ (particular solution) □

<u>*Ex*</u>:

$$y' = f(x, y), \qquad y(x_0) = y_0$$

$$F(x, y, y') = 0, \qquad y(x_0) = y_0 \quad \square$$

<u>*Ex*</u>:

Find the curve $y(x)$ through $(1, 1)$ in the x-y plane having at each of its points the slope $-y/x$

<u>Sol</u>:

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\Longrightarrow y(x) = \frac{c}{x} \qquad c = \text{arbitrary constant}$$

i.e. $xy = c = 1 \quad (\because x = 1,\ y = 1)$

$\Longrightarrow$ The curve is $\quad xy = 1 \quad \square$

1.2 Separable Differential Equations

For first order ode with the form:

$$\boxed{g(y)\frac{dy}{dx} = f(x)}$$

$$\Longrightarrow \int g(y)\frac{dy}{dx}dx = \int f(x)dx + c$$

$$\Longrightarrow \int g(y)dy = \int f(x)dx + c$$

<u>*Ex*</u>:

$$9yy' + 4x = 0$$

$$\int 9y\frac{dy}{dx}dx = \int(-4x)dx + c$$

$$\Longrightarrow 9\int ydy = -2x^2 + c$$

$$\Longrightarrow \frac{9}{2}y^2 = -2x^2 + c$$

$$\Longrightarrow \frac{x^2}{9} + \frac{y^2}{4} = c^* \qquad c^* = \frac{c}{18}$$

The solution is in implicit form. □

<u>*Ex*</u>:

$$y' = ky, \quad k = \text{constant}, \quad y(0) = 2$$

$$\Longrightarrow \frac{1}{y}\frac{dy}{dx} = k$$

$$\Longrightarrow \int \frac{1}{y}\frac{dy}{dx}dx = \int k dx + c$$

$$\Longrightarrow \int \frac{dy}{y} = kx + c$$

$$\Longrightarrow \ln|y| = kx + c$$

$$\Longrightarrow |y| = e^{kx+c} = e^c e^{kx}$$

$$\therefore\ y = \pm e^c e^{kx} \equiv c^* e^{kx}$$

Apply the initial condition.

$$y(0) = 2 \Longrightarrow c^* = 2$$

$$\therefore\ y(x) = 2e^{kx} \quad \square$$

1.3 Modeling

1.4 Reduction to Separable Form

Certain first order ode's, which are not separable, can be made separable by changing of variables.

Ex:

$$y' = g(\frac{y}{x})$$

Let $\frac{y}{x} \equiv u$

$$\Longrightarrow y = ux \quad \text{and} \quad y' = u + xu'$$

$$\therefore u + x\frac{du}{dx} = g(u)$$

$$\Longrightarrow x\frac{du}{dx} = g(u) - u$$

$$\Longrightarrow \frac{1}{g(u) - u}\frac{du}{dx} = \frac{1}{x}$$

$$\Longrightarrow \frac{du}{g(u) - u} = \frac{dx}{x} \leftarrow \text{ this is separable} \quad \square$$

<u>*Ex*</u>:

$$2xyy' - y^2 + x^2 = 0 \quad \text{———} \; (*)$$

$$(*) \div x^2$$

$$\Longrightarrow 2\left(\frac{y}{x}\right)y' - \left(\frac{y}{x}\right)^2 + 1 = 0 \qquad \Longrightarrow y' = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)} = g\left(\frac{y}{x}\right)$$

$$\therefore \text{ Let } \quad u \equiv \frac{y}{x}$$

$$\Longrightarrow 2u(u + xu') - u^2 + 1 = 0 \qquad \Longrightarrow 2u^2 + 2ux\frac{du}{dx} - u^2 + 1 = 0$$

$$\Longrightarrow 2ux\frac{du}{dx} = -(u^2 + 1) \qquad \Longrightarrow \frac{2u}{(u^2+1)}\frac{du}{dx} = \frac{1}{x} \quad \text{———} \; (**)$$

$$\int (**) dx$$

$$\Longrightarrow \int \frac{2u}{(u^2+1)}du = -\ln|x| + c^*$$

$$\Longrightarrow \ln(1 + u^2) = -\ln|x| + c^* \equiv \ln\frac{1}{|x|} + \ln c = \ln\frac{c}{|x|}$$

$$\therefore 1 + u^2 = \frac{c}{x} \qquad (x > 0)$$

$$\Longrightarrow 1 + \frac{y^2}{x^2} = \frac{c}{x} \qquad \Longrightarrow x^2 + y^2 = cx \quad \leftarrow \text{general solution} \quad \square$$

1.5 Exact Differential Equation

Function $u(x, y)$ has continuous partial derivatives
$\Longrightarrow$ its total or exact differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$\therefore$ If $u(x, y) = c \quad \Longrightarrow du = 0$

<u>*Ex*</u>:

$$u = x + x^2y^2 = c$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = (1 + 2xy^2)dx + (2x^2y)dy = 0$$

$$\Longrightarrow \frac{dy}{dx} = -\frac{1 + 2xy^2}{2x^2y}$$

differential equation with solution $x + x^2y^2 = c \quad \square$

A first order ode of the form:

$$\boxed{M(x,y)dx + N(x,y)dy = 0} \quad \text{———} \ (*)$$

$$\text{or} \quad M(x,y) + N(x,y)y' = 0$$

is call exact, if $\quad M(x,y)dx + N(x,y)dy = du = 0$

$$\text{i.e.} \quad M(x,y) = \frac{\partial u}{\partial x}, \quad N(x,y) = \frac{\partial u}{\partial y}$$

$\Longrightarrow u(x,y) = c$ is the solution of $(*)$

Since $\dfrac{\partial u}{\partial x} = M, \quad \dfrac{\partial u}{\partial y} = N$

$$\Longrightarrow \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial N}{\partial x}$$

$$\Longrightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \quad \text{if } (*) \text{ is an exact ode}$$

The solution is:

$$\frac{\partial u}{\partial x} = M \quad \Longrightarrow u(x,y) = \int M(x,y)dx + k(y)$$

or

$$\frac{\partial u}{\partial y} = N \quad \Longrightarrow u(x,y) = \int N(x,y)dy + l(x) \quad \square$$

<u>*Ex*</u>: $(x^3 + 3xy^2) + (3x^2y + y^3)y' = 0.$ Solve for $y(x)$.

$$\Longrightarrow (x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$$

<u>*Sol*</u>: Is the differential equation exact? i.e.

$$\frac{\partial u}{\partial x} = x^3 + 3xy^2 \quad \text{———} \quad (1)$$

$$\frac{\partial u}{\partial y} = 3x^2y + y^3 \quad \text{———} \quad (2)$$

$$\Longrightarrow \frac{\partial(1)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = 6xy = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial(2)}{\partial x} \qquad \therefore \text{ exact}$$

$$(1) \Longrightarrow u(x,y) = \int (x^3 + 3xy^2)dx = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + k(y)$$

$$(2) \Longrightarrow \frac{\partial u}{\partial y} = 3x^2y + \frac{dk}{dy} = 3x^2y + y^3$$

$$\therefore k(y) = \frac{1}{4}y^4 + c'$$

$\therefore$ The solution is $\quad u(x,y) = \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 + c' = \text{constant}$

i.e. The implicit form for $y(x)$ is: $\quad \frac{1}{4}x^4 + \frac{3}{2}x^2y^2 + \frac{1}{4}y^4 = c \quad$ ——— $(*)$

Check the solution:

$$\frac{d(*)}{dx} \Longrightarrow x^3 + 3xy^2 + 3x^2y\frac{dy}{dx} + y^3\frac{dy}{dx} = 0$$

$$\Longrightarrow (x^3 + 3xy^2) + (3x^2y + y^3)\frac{dy}{dx} = 0 \quad \square$$

1.6 Integrating Factors

Consider a first order ode:

$$P(x,y)dx + Q(x,y)dy = 0,$$

which is not exact.

i.e. $\quad P \neq \dfrac{\partial u}{\partial x} \quad$ and $\quad Q \neq \dfrac{\partial u}{\partial y}$

or $\quad \dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$

But a suitable function $F(x,y)$ can make

$$F(x,y)P(x,y)dx + F(x,y)Q(x,y)dy = 0$$

exact, i.e. $F(x,y)P(x,u) = \dfrac{\partial u}{\partial x}, \quad F(x,y)Q(x,y) = \dfrac{\partial u}{\partial y}$

$$\Longrightarrow \frac{\partial (FP)}{\partial y} = \frac{\partial (FQ)}{\partial x}$$

$F(x,y)$ is then called the integrating factor. $\square$

■ How to find $F(x,y)$?

i.e. we are looking for $F(x,y)$ that satisfies,

$$\frac{\partial(FP)}{\partial y} = \frac{\partial(FQ)}{\partial x}$$

$$\Longrightarrow \frac{\partial F}{\partial y}P + F\frac{\partial P}{\partial y} = \frac{\partial F}{\partial x}Q + F\frac{\partial Q}{\partial x} \quad \text{———} \; (*)$$

This is a partial differential equation to solve for $F(x,y)$!!
Even more difficult !!!

$\therefore$ Only under certain conditions, can the integrating factor be found systematically.

Let say, if $\boxed{F = F(x)}$ (function of x only)

$$(*) \Longrightarrow F\frac{\partial P}{\partial y} = \frac{dF}{dx}Q + F\frac{\partial Q}{\partial x} \quad \text{———} \; (**)$$

$$\frac{1}{FQ} \times (**) \Longrightarrow \frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) \equiv R$$

$\because \frac{1}{F}\frac{dF}{dx}$ is a function of x only $\Longrightarrow F(x)$ exist only when $R = R(x)$

$$\Longrightarrow \frac{1}{F}dF = Rdx \quad \text{(separable)}$$

$$\Longrightarrow F(x) = e^{\int R(x)dx}$$

If $\boxed{F = F(y)}$ (function y only)

$$(*) \Longrightarrow \frac{dF}{dy}P + F\frac{\partial P}{\partial y} = F\frac{\partial Q}{\partial x} \quad \text{———} \ (***)$$

$$\frac{1}{FP} \times (***) \Longrightarrow \frac{1}{F}\frac{\partial F}{dy} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \equiv \tilde{R}$$

$\because \dfrac{1}{F}\dfrac{\partial F}{dy}$ is a function of y only $\Longrightarrow F(y)$ exist only when $\tilde{R} = \tilde{R}(y)$

$$\Longrightarrow \frac{1}{F}dF = \tilde{R}dy \quad \text{(separable)}$$

$$\Longrightarrow F(y) = e^{\int \tilde{R}(y)dy}$$

If $\boxed{F = F(x, y)} \Longrightarrow F(x, y)$ can only be found by inspection !! $\square$

<u>*Ex*</u>: Consider an initial value problem:

$$2xydx + (4y + 3x^2)dy = 0, \qquad y(0.2) = -1.5.$$

Solve for $y(x)$.

$$\frac{\partial}{\partial y}(2xy) = 2x \neq \frac{\partial}{\partial x}(4y + 3x^2) = 6x$$

$\therefore$ the differential is not exact.

(A) Assume integrating factor $F = F(x)$

$$\frac{\partial}{\partial y}\left(F(x) \times (2xy)\right) = \frac{\partial}{\partial x}\Big(F(x) \times (4y + 3x^2)\Big)$$

$$\Longrightarrow F \times (2x) = \frac{dF}{dx} \times (4y + 3x^2) + F \times 6x$$

$$\Longrightarrow \frac{dF}{dx} \times (4y + 3x^2) = -4x \times F$$

$$\frac{1}{F}\frac{dF}{dx} = \frac{-4x}{4y + 3x^2} \qquad \text{is a function of } x \text{ and } y$$

(B) Assume $F = F(y)$

$$\frac{\partial}{\partial y}(F(y) \times (2xy)) = \frac{\partial}{\partial x}(F(y) \times (4y + 3x^2))$$

$$\Longrightarrow \frac{dF}{dy} \times (2xy) + F \times (2x) = F \times (6x)$$

$$\Longrightarrow \frac{1}{F}\frac{dF}{dy} = \frac{4x}{2xy} = \frac{2}{y}$$

$$\therefore F(y) = y^2$$

$$\Longrightarrow y^2[2xydx + (4y + 3x^2)dy] = 0$$

$$\Longrightarrow 2xy^3dx + (4y^3 + 3x^2y^2)dy = 0$$

$$\Longrightarrow \frac{\partial(2xy^3)}{\partial y} = 6xy^2 = \frac{\partial(4y^3 + 3x^2y^2)}{\partial x} = 6xy^2$$

$\therefore$ exact!!
i.e. the solution is $u(x, y) =$ constant

$$\frac{\partial u}{\partial x} = 2xy^3 \quad \Longrightarrow u(x, y) = x^2y^3 + k(y)$$

$$\frac{\partial u}{\partial y} = 4y^3 + 3x^2y^2 = 3x^2y^2 + \frac{dk}{dy} \quad \Longrightarrow \frac{dk}{dy} = 4y^3 \quad \Longrightarrow k(y) = y^4 + c'$$

$$\therefore u(x, y) = x^2y^3 + y^4 = c \quad \text{is the general solution.}$$

Apply the initial condition $y(0.2) = -1.5$

$$\Longrightarrow (0.2)^2 \times (-1.5)^3 + (-1.5)^4 = c$$

$$\Longrightarrow c = 4.9275$$

$\therefore$ $u(x, y) = x^2y^3 + y^4 = 4.9275$ is the particular solution. $\square$

1.7 Linear Differential Equation

$\boxed{y' + p(x)y = r(x)}$ $\leftarrow$ linear in y and y'

If $r(x) = 0 \Longrightarrow$ homogeneous differential equation

$r(x) \neq 0 \Longrightarrow$ non-homogeneous

■ $\boxed{r(x) = 0}$ (homogeneous)

$$y' + p(x)y = 0$$

$$\Longrightarrow \frac{dy}{y} = -p(x)dx \quad \text{(separable)}$$

$$\Longrightarrow \int \frac{dy}{y} = -\int p(x)dx$$

$$\Longrightarrow \ln|y| = -\int p(x)dx + c^*$$

$$\Longrightarrow |y| = e^{-\int p(x)dx + c^*}$$

$$\Longrightarrow y(x) = ce^{-\int p(x)dx}$$

If $c = 0$, $y(x) = 0$ $\leftarrow$ trivial solution.

■ $\boxed{r(x) \neq 0}$ (non-homogeneous)

$$\frac{dy}{dx} + p(x)y = r(x)$$

$$\Longrightarrow [p(x)y - r(x)]dx + (1)dy = 0 \quad \text{———} \quad (*)$$

To find integrating factor $F(x)$:

$$\frac{\partial}{\partial y}[F(x)(p(x)y - r(x)] = \frac{\partial}{\partial x}[F(x)]$$

$$\Longrightarrow F(x) \cdot p(x) = \frac{dF}{dx}$$

$$\Longrightarrow \int \frac{dF}{F} = \int p(x)dx$$

$$\Longrightarrow \ln|F| = \int p(x)dx$$

$$\Longrightarrow F(x) = e^{\int p(x)dx}$$

$\therefore$ $F(x) \times (*)$

$$\Longrightarrow e^{\int p(x)dx}[p(x)y - r(x)]dx + e^{\int p(x)dx}dy = 0 \quad \text{is exact.}$$

Check:

$$\frac{\partial}{\partial y}\left\{e^{\int p(x)dx}[p(x)y - r(r)]\right\} = e^{\int p(x)dx} \cdot p(x) = \frac{\partial}{\partial x}\left(e^{\int p(x)dx}\right)$$

$\therefore$ the new equation is exact.

$\therefore$ the solution is $u(x,y) =$ constant, where

$$\frac{\partial u}{\partial x} = e^{\int p(x)dx}[p(x)\cdot y - r(x)] \quad \text{———} \ (1)$$

$$\frac{\partial u}{\partial y} = e^{\int p(x)dx} \quad \text{———} \ (2)$$

$$(2) \Longrightarrow u(x,y) = ye^{\int p(x)dx} + k(x)$$

$$(1) \Longrightarrow \frac{\partial u}{\partial x} = yp(x)e^{\int p(x)dx} + \frac{dk}{dx} = e^{\int p(x)dx}[p(x)y - r(x)]$$

$$\Longrightarrow \frac{\partial k}{dx} = -r(x)e^{\int p(x)dx}$$

$$\therefore k(x) = -\int \left(r(x)e^{\int p(x)dx}\right) dx + c^*$$

$\therefore$ The solution is:

$$ye^{\int p(x)dx} - \int \left(r(x)e^{\int p(x)dx}\right) dx + c^* = 0$$

or

$$y(x) = \frac{\int \left(r(x)\cdot e^{\int p(x)dx}\right) dx + c}{e^{\int p(x)dx}} \quad \square$$

<u>*Ex*</u>: $y' - y = e^{2x}$

Separable? No

Exact? No

First order linear ordinary differential equation? Yes!

$$y' + p(x)y = r(x)$$

$$p(x) = -1, \quad r(x) = e^{2x}$$

$$\begin{aligned}
\therefore y(x) &= \frac{\int \left(e^{2x} \cdot e^{\int(-1)dx}\right) dx + c}{e^{\int(-1)dx}} \\
&= \frac{\int \left(e^{2x} \cdot e^{-x}\right) dx + c}{e^{-x}} \\
&= e^x \cdot (e^x + c) \\
&= ce^x + e^{2x} \quad \square
\end{aligned}$$

<u>*Ex*</u>: $y' + (\tan x)y = \sin 2x, \qquad y(0) = 1$

First order linear ordinary differential equation

$p(x) = \tan x, \quad r(x) = \sin 2x$

$$\Longrightarrow y(x) = \frac{\int \left(\sin 2x \cdot e^{\int \tan x dx}\right) dx + c}{e^{\int \tan x dx}}$$

$$= \frac{\int \left(\sin 2x \cdot e^{\ln|\sec x|}\right) dx + c}{e^{\ln|\sec x|}}$$

$$= \frac{\int (\sin 2x \cdot \sec x) \cdot dx + c}{\sec x}$$

$$= \frac{\int (2 \sin x \cdot \cos x \cdot \sec x) dx + c}{\sec x}$$

$$= \cos x(c - 2\cos x)$$

Initial condition $y(0) = 1$:

$$\Longrightarrow y(0) = \cos 0(c - 2\cos 0) = 1(c - 2) = 1$$

$$\Longrightarrow c = 3$$

$$\therefore y(x) = \cos x(3 - 2\cos x) \quad \square$$

Chapter 2

Second-Order Linear Differential Equation

2.1 Homogeneous Linear 2nd-order ODE

$$\boxed{y'' + p(x)y' + q(x)y = 0}$$

where $p(x)$ and $q(x)$ are the coefficients.

Ex: $y'' - y = 0$
$y = e^x, \quad y = e^{-x}, \quad y = -3e^x + 8e^{-x}, \quad y = c_1 e^x + c_2 e^{-x} \ldots\ldots$
all are solutions of the equation. □

Theorem : Superposition or Linearity Principle

If $y = y_1(x)$ and $y = y_2(x)$ are solutions of $y'' + p(x)y' + q(x)y = 0$

$\Longrightarrow y = c_1 y_1 + c_2 y_2$ is also solution.

Proof:

$$\begin{aligned} & y'' + py' + qy \\ &= (c_1y_1 + c_2y_2)'' + p(c_1y_1 + c_2y_2)' + q(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) \\ &= 0 \quad \square \end{aligned}$$

<u>*Note*</u>:
(1) Only the linear and homogeneous differential equations have the superposition property.
(2) Nonlinear differential equations and non-homogeneous differential equations do not have the superposition property.

<u>*Ex*</u>: Nonhomogeneous linear differential equation $y'' + y = 1$.
Both $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions.
But $y = 2(1 + \cos x)$ or $y = (1 + \cos x) + (1 + \sin x)$ are NOT.

<u>*Ex*</u>: Nonlinear differential equation $y''y - xy' = 0$. Both $y = x^2$ and $y = 1$ are solutions.
But $y = -x^2$ or $y = x^2 + 1$ are NOT.

■ Initial Value Problem, Boundary Value Problem

• For *first-order* ordinary differential equations:
General solution has one constant.
Initial condition is used to find the constant $\Longrightarrow$ *particular solution.*

• For *second-order homogeneous linear* ordinary differential equations:
$y'' + p(x)y' + q(x)y = 0$
General solution is $y = c_1 y_1 + c_2 y_2$, c_1 and c_2 are constants.
Need TWO conditions for the constants:
$y(x_0) = k_0, \quad y'(x_0) = k_1 \quad \leftarrow$ initial condition
or $y(x_1) = k_1, \quad y(x_2) = k_2 \quad \leftarrow$ boundary condition
$\Longrightarrow$ *particular solution*

<u>*Ex*</u>: $y'' - y = 0, \qquad y(0) = 5, \quad y'(0) = 3.$

Both e^x and e^{-x} are solutions.

$\therefore$ general solution: $y = c_1 e^x + c_2 e^{-x}$

Apply initial conditions:

$$y(0) = c_1 + c_2 = 5$$
$$y'(0) = c_1 - c_2 = 3$$

$\therefore$ $c_1 = 4$, $c_2 = 1 \Longrightarrow$ particular solution: $y = 4e^x + e^{-x}$

However, if we let the general solution: $y = c_1 e^x + c_2 e^x$

$$y(0) = c_1 + c_2 = 5$$
$$y'(0) = c_1 + c_2 = 3 \qquad \text{oops!!?} \quad \square$$

Basis or Fundamental System

If y_1 and y_2 are solutions of $y'' + p(x)y' + q(x)y = 0$,
and $y_1 \neq ky_2$ or $y_2 \neq ly_1$, i.e. y_1 and y_2 are not *proportional*.

$\Longrightarrow$ y_1 and y_2 are called a **basis** or **fundamental system** of the equation.

$\Longrightarrow$ The general solution is $y = c_1y_1 + c_2y_2$.

$y_1 \neq ky_2$ or $y_2 \neq ly_1$

$\Longrightarrow k_1y_1 + k_2y_2 = 0 \quad \text{iff} \quad k_1 = k_2 = 0$

$\Longrightarrow$ y_1 and y_2 are linearly independent

$\Longrightarrow$ If y_1 and y_2 are basis of *solution*, then y_1 and y_2 are linearly independent.

2.2 Homogeneous Equations with Constant Coefficients

2.3 Homogeneous Equations with Constant Coefficients (continued)

$$\boxed{y'' + ay' + by = 0} \qquad \text{———} \ (1)$$

where a and b are real constants.

Assume the form of the solution is $y = e^{\lambda x}$.
Substitute $y(x)$ into (1):

$$\Longrightarrow \lambda^2 e^{\lambda x} + a\lambda e^{\lambda x} + be^{\lambda x} = 0$$

$$(\lambda^2 + a\lambda + b)e^{\lambda x} = 0$$

Since $e^{\lambda x}$ cannot be zero
$\Longrightarrow \lambda^2 + a\lambda + b = 0 \quad \leftarrow$ **characteristic** or **auxiliary** equation of (1)
Solutions of the characteristic equation:

$$\lambda_{1,2} = \frac{1}{2}(-a \pm \sqrt{a^2 - 4b})$$

$\therefore$ the solutions of (1) is:

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x}$$

■ If $\boxed{a^2 - 4b > 0}$

i.e. λ_1 and λ_2 are two distinct real roots.

$\Longrightarrow y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ is the general solution of (1).

■ If $\boxed{a^2 - 4b = 0}$

i.e. $\lambda_1 = \lambda_2 = -\dfrac{a}{2}$,

$$\Longrightarrow y_1 = e^{-\frac{a}{2}x}$$

To find another basis, try $y_2 = u(x)y_1$

$$y_2'' + ay_2' + by_2 = 0$$

$$\Longrightarrow (u'y_1 + uy_1')' + a(u'y_1 + uy_1') + b(uy_1) = 0$$

$$\Longrightarrow (u''y_1 + 2u'y_1' + uy_1'') + a(u'y_1 + uy_1') + b(uy_1) = 0$$

$$\Longrightarrow u''y_1 + u' \underbrace{(2y_1' + ay_1)}_{\substack{=0 \\ (\because\, y_1 = e^{-\frac{a}{2}x})}} + u \underbrace{(y_1'' + ay_1' + by_1)}_{=0} = 0$$

$$\therefore u''y_1 = 0 \text{ and } y_1 \neq 0$$

$$\Longrightarrow u'' = 0 \qquad \Longrightarrow u = d_1 x + d_2$$

$$\therefore y_2 = xy_1 \quad \text{i.e.,} \quad y = (c_1 + c_2 x)e^{-\frac{a}{2}x}$$

■ If $\boxed{a^2 - 4b < 0}$

$$\lambda_{1,2} = \frac{1}{2}(-a \pm i\sqrt{4b - a^2})$$

$$= -\frac{a}{2} \pm i\sqrt{b - \frac{a^2}{4}} \equiv -\frac{a}{2} \pm iw$$

$$\Longrightarrow y = c_1^* e^{(-\frac{a}{2}+iw)x} + c_2^* e^{(-\frac{a}{2}-iw)x}$$

$$= c_1^* e^{-\frac{a}{2}x} \cdot e^{iwx} + c_2^* e^{-\frac{a}{2}x} \cdot e^{-iwx}$$

Euler formula: $e^{iwx} = \cos wx + i \sin wx$

For real $y(x)$, the bases are $[e^{-\frac{a}{2}x} \cos wx]$ and $[e^{-\frac{a}{2}x} \sin wx]$.

$$\therefore y = c_1 e^{-\frac{a}{2}x} \cos wx + c_2 e^{-\frac{a}{2}x} \sin wx \quad \square$$

Summary :

Case	Roots	Basis	General solution
$b^2 - 4ac > 0$	distinct real roots λ_1, λ_2	$e^{\lambda_1 x}, e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
$b^2 - 4ac = 0$	real double root λ	$e^{\lambda x}, xe^{\lambda x}$	$y = (c_1 + c_2 x)e^{\lambda x}$
$b^2 - 4ac < 0$	complex conjugate $\lambda_1 = \lambda + iw$ $\lambda_2 = \lambda - iw$	$e^{\lambda x} \cos wx$ $e^{\lambda x} \sin wx$	$y = e^{\lambda x}(c_1 \cos wx + c_2 \sin wx)$

<u>*Ex*</u> : $y'' + y' - 2y = 0, \qquad y(0) = 4, \quad y'(0) = -5$

<u>*Sol*</u> :

Characteristic equation: $\lambda^2 + \lambda - 2 = 0$

$$\Longrightarrow \lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1, \quad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2$$

$\therefore$ the general solution is $\quad y(x) = c_1 e^x + c_2 e^{-2x}$

Apply the initial conditions:

$$y(0) = c_1 + c_2 = 4$$
$$y'(0) = c_1 - 2c_2 = -5$$

$$\Longrightarrow c_1 = 1, \quad c_2 = 3$$

$\therefore$ the particular solution is $\quad y(x) = e^x + 3e^{-2x} \quad \square$

<u>*Ex*</u> : $y'' - 4y' + 4y = 0, \qquad y(0) = 3, \quad y'(0) = 1$

<u>*Sol*</u> :

Characteristic equation: $\lambda^2 - 4\lambda + 4 = 0$

$$\Longrightarrow \lambda_1 = \lambda_2 = 2$$

$\therefore$ the general solution is $\quad y(x) = (c_1 + c_2 x)e^{2x}$

Apply the initial conditions:

$$y(0) = c_1 = 3$$
$$y'(0) = c_2 + 2c_1 = 1$$

$$\Longrightarrow c_1 = 3, \quad c_2 = -5$$

$\therefore$ the particular solution is $\quad y(x) = (3 - 5x)e^{2x} \quad \square$

<u>*Ex*</u> : $y'' + y = 0, \qquad y(0) = 3, \quad y'(2\pi) = -3$

<u>*Sol*</u> :

Characteristic equation: $\lambda^2 + 1 = 0$

$$\Longrightarrow \lambda_1 = i, \quad \lambda_2 = -i$$

$\therefore$ the general solution is $\quad y(x) = c_1 \cos x + c_2 \sin x$

Apply the *boundary* conditions:

$$y(0) = c_1 = 3$$

$$y'(2\pi) = c_2 = -3$$

$\therefore$ the particular solution is $\quad y(x) = 3\cos x - 3\sin x \quad \square$

2.4 Differential Operators. *Optional*

2.5 Free Oscillations of Mass-Spring system

■ Undamped System

$$my'' + ky = 0$$

$\Longrightarrow$ characteristic equation:

$$m\lambda^2 + k = 0 \quad \Longrightarrow \quad \lambda = \frac{0 \pm \sqrt{-4mk}}{2m} = \pm i\sqrt{\frac{k}{m}}$$

$$\Longrightarrow y(t) = A\cos\omega_0 t + B\sin\omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

$$\equiv c\cos(\omega_0 t - \delta)$$

where $c = \sqrt{A^2 + B^2} \qquad \delta = \tan^{-1}\frac{B}{A}$

$$\text{period} = \frac{1}{\text{frequency}} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

■ Damped System

$$my'' + cy' + ky = 0$$

where c is the damping constant,
assuming the viscous damping force $\propto y'(t)$ (velocity).

$\Longrightarrow$ characteristic equation:

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

$$= -\alpha \pm \beta$$

(1) $\underline{c^2 > 4mk}$: λ_1, λ_2 are distinct real roots

$$y(t) = c_1 e^{-(\alpha-\beta)t} + c_2 e^{-(\alpha+\beta)t}$$

$$\left.\begin{aligned} \alpha &= \frac{c}{2m} > 0 \\ \beta &= \frac{1}{2m}\sqrt{c^2 - 4mk} > 0 \end{aligned}\right\} \beta^2 = \frac{c^2 - 4mk}{4m^2} = \frac{c^2}{4m^2} - \frac{k}{m} = \alpha^2 - \frac{k}{m}$$

$$\Longrightarrow \alpha > \beta$$

$$\Longrightarrow y(t) \to 0 \quad \text{when} \quad t \to 0$$

$\Longrightarrow$ **overdamping**

(2) $\underline{c^2 = 4mk}$: one real root $\lambda = -\dfrac{c}{2m} \equiv -\alpha$

$$y(t) = \underbrace{(c_1 + c_2 t)}_{=0 \text{ at } t=-\frac{c_1}{c_2}} \underbrace{e^{-\alpha t}}_{\neq 0}$$

$\Longrightarrow$ $y(t)$ may have at most one zero at $t = -\dfrac{c_1}{c_2}$

$\Longrightarrow$ **critical damping**

(3) $\underline{c^2 < 4mk}$: $\lambda_{1,2}$ are complex conjugate roots.

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{i}{2m}\sqrt{4mk - c^2} \equiv -\alpha \pm i\omega^*$$

$$\omega^* = \frac{1}{2m}\sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

$$\begin{aligned} y(t) &= e^{-\alpha t}(c_1 \cos\omega^* t + c_2 \sin\omega^* t) \\ &\equiv ce^{-\alpha t}\cos(\omega^* t - \delta) \\ &\quad (c^2 = c_1^2 + c_2^2, \quad \delta = \tan^{-1}\frac{c_2}{c_1}) \end{aligned}$$

$\Longrightarrow$ **underdamping**

2.6 Euler-Cauchy Equation

$$\boxed{x^2y'' + axy' + by = 0} \qquad a, b \text{ are constants.}$$

We assume the form of the solution is $y = x^m$

$\Longrightarrow x^2m(m-1)x^{m-2} + axmx^{m-1} + bx^m = 0$

$\Longrightarrow m(m-1) + am + b = 0$

$\Longrightarrow m^2 + (a-1)m + b = 0$

(1) $\underline{(a-1)^2 - 4b > 0}$:

Two distinct real $m_{1,2} = \frac{-(a-1)\pm\sqrt{a-1)^2-4b}}{2}$

$\therefore y(x) = c_1x^{m_1} + c_2x^{m_2}$

(2) $\underline{(a-1)^2 - 4b = 0}$:

One real root $m = \frac{1-a}{2}$

$\Longrightarrow y_1 = x^{\frac{(1-a)}{2}}$

Assume the other solution $y_2 = u(x)y_1$, and find $u(x)$.

$$x^2y'' + axy' + by = 0$$

$$\Longrightarrow x^2(u''y_1 + 2u'y_1' + uy_1'') + ax(u'y_1 + uy_1') + buy_1 = 0$$

$$\Longrightarrow u''x^2y_1 + u'x\underbrace{(2xy_1' + ay_1)}_{\equiv (*)} + u\underbrace{(x^2y_1'' + 1xy_1' + by_1)}_{=0} = 0$$

$$(*) = 2x\frac{(1-a)}{2}x^{(\frac{1-a}{2}-1)} + ax^{\frac{(1-a)}{2}} = (1-a)x^{(\frac{1-a}{2})} + ax^{(\frac{1-a}{2})} = x^{\frac{(1-a)}{2}} = y_1$$

$$\Longrightarrow u''x^2y_1 + u'xy_1 = 0$$

or $\quad (u''x^2 + u'x)y_1 = 0$

Since $\quad y_1 \neq 0 \Longrightarrow u''x^2 + u'x = 0$

$$\Longrightarrow \underbrace{(u')'}_{\equiv z'}x + \underbrace{(u')}_{\equiv z} = 0$$

$$\Longrightarrow \frac{dz}{z} = -\frac{dx}{x} \qquad \text{(separable)}$$

$$\Longrightarrow \ln|z| = \ln|u'| = -\ln x \qquad x > 0$$

$$\Longrightarrow u' = \frac{1}{x} \Longrightarrow u = \ln x \qquad x > 0$$

$$\Longrightarrow y_2 = \ln x \cdot x^{(\frac{1-a}{2})}$$

$$\therefore\ y(x) = (c_1 + c_2\ln x)x^{(\frac{1-a}{2})}$$

(3) $\underline{(a-1)^2 - 4b < 0}$:

Two complex roots $m_{1,2} = \dfrac{-(a-1) \pm i\sqrt{4b-(a-1)^2}}{2} \equiv u \pm i\nu$

Since $\begin{cases} x^{i\nu} = e^{i\nu \ln x} = \cos(\nu \ln x) + i \sin(\nu \ln x) \\ x^{-i\nu} = e^{-i\nu \ln x} = \cos(\nu \ln x) - i \sin(\nu \ln x) \end{cases}$

$$\begin{aligned} \Longrightarrow y(x) &= c_1 x^u \cos(\nu \ln x) + c_2 x^u \sin(\nu \ln x) \\ &= x^u (c_1 \cos(\nu \ln x) + c_2 \sin(\nu \ln x)) \quad \Box \end{aligned}$$

2.7 Existence and Uniqueness Theory. Wronskian

$$y'' + p(x)y' + q(x)y = 0 \qquad \text{———— (1)}$$

$p(x), q(x)$ are continuous functions

$\Longrightarrow$ general solution of (1):

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \qquad \text{———— (2)}$$

$y_1(x)$ and $y_2(x)$ form a basis.

i.e. y_1 and y_2 are linear independent

i.e. $k_1 y_1(x) + k_2 y_2(x) = 0$ only when $k_1 = 0$ and $k_2 = 0]$

If y_1 and y_2 are linear dependent

$\Longrightarrow y_1 = ay_2$ or $y_2 = by_1$, where a, b are constants

$\Longrightarrow$ Wronskian (Wronski determinant) W

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1' = 0$$

$$\because W = (ay_2)y_2' - y_2(ay_2') = 0$$

$$\text{or } W = y_1(by_1') - (by_1)y_1' = 0 \quad \square$$

Theorem :

Second order homogeneous, linear ordinary differential equation (1) with continuous $p(x), q(x)$ on some interval I.

$\Longrightarrow$ (1) has a general solution on I, and the solution has the from
$y(x) = c_1 y_1(x) + c_2 y_2(x)$, where y_1 and y_2 are linear independent. □

Initial value problem :

(1) with initial conditions: $y(x_0) = k_0, \quad y'(x_0) = k_1$ ——— (3)

$\Longrightarrow$ c_1 and c_2 of (2) are determined from (3) □

Theorem :

Second order homogeneous, linear ordinary differential equation (1) with continuous $p(x), q(x)$ on some interval I, and initial condition (3) at some x_0 on I

$\Longrightarrow$ The initial value problem (1) and (3) has a *unique* solution on the interval I. □

■ Given $y_1(x)$, how to obtain $y_2(x)$: method of reduction of order

$$y'' + p(x)y' + q(x)y = 0 \quad \text{——— (1)}$$

Given $y_1(x)$

Assume $y_2(x) \equiv u(x)y_1(x)$

$$y_2' = u'y_1 + uy_1'$$

$$y_2'' = u''y_1 + 2u'y_1' + uy_1''$$

$$(1) \Longrightarrow u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1) = 0$$

$$\Longrightarrow u''y_1 + u'(2y_1' + py_1) = 0$$

$$u' \equiv U$$

$$\Longrightarrow U'y_1 + U(2y_1' + py_1) = 0 \quad \leftarrow \text{separable}$$

$$\Longrightarrow \frac{dU}{U} = -(\frac{2y_1' + py_1}{y_1})dx$$

$$\Longrightarrow \ln|U| = -\int \frac{2y_1'}{y_1}dx - \int pdx = -2\ln|y_1| - \int pdx$$

$$\Longrightarrow U = e^{\ln|y_1|^{-2} - \int pdx} = \frac{1}{y_1^2} \cdot e^{-\int pdx}$$

$$\Longrightarrow \frac{du}{dx} = \frac{e^{-\int pdx}}{y_1^2} \qquad \therefore \boxed{u(x) = \int \frac{e^{-\int pdx}}{y_1^2}dx} \quad \square$$

Ex :

$$(x^2-1)y''-2xy'+2y=0$$

$$y_1 = x, \quad \text{find } y_2$$

$$\Longrightarrow y'' - \underbrace{\frac{2x}{x^2-1}}_{=p(x)} y' + \underbrace{\frac{2}{x^2-1}}_{=q(x)} y = 0$$

$$y_2 = u \cdot y_1$$

$$\begin{aligned} u(x) &= \int \frac{e^{-\int p dx}}{y_1^2} dx = \int \frac{e^{\int \frac{2x}{x^2-1} dx}}{x^2} dx \\ &= \int \frac{e^{\ln\left|x^2-1\right|}}{x^2} dx = \int \frac{x^2-1}{x^2} dx = \int (1-\frac{1}{x^2}) dx \\ &= x + \frac{1}{x} \end{aligned}$$

$$\Longrightarrow y_2(x) = \left(x + \frac{1}{x}\right) x = x^2 + 1$$

$$\therefore\ y(x) = c_1 x + c_2(x^2+1) \quad \square$$

2.8 Nonhomogeneous Equations

$$y'' + p(x)y' + q(x)y = r(x), \qquad r(x) \neq 0 \quad \text{———} \; (1)$$

cf: homogeneous equation:

$$y'' + p(x)y' + q(x)y = 0 \quad \text{———} \; (2)$$

The general solution of (1) is $y(x) = y_h(x) + y_p(x)$, where

$y_h(x)$ is the *general* solution of the homogeneous equation (2),

$y_p(x)$ is any *particular* solution of the nonhomogeneous equation (1).

What is the particular solution of (1)?

$$\Longrightarrow y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{= y_h(x)} + y_p(x)$$

where c_1 and c_2 have specific values.

<u>*Ex*</u>:

$$y'' - 4y' + 3y = 10e^{-2x}, \qquad y(0) = 1, y'(0) = -3$$

(1) $y_h(x)$

Characteristic equation:

$$\lambda^2 - 4\lambda + 3 = 0 \quad \Longrightarrow \lambda = 1 \text{ or } 3$$

$$\Longrightarrow y_h(x) = c_1 e^x + c_2 e^{3x}$$

(2) $y_p(x)$

Since $r(x) = 10e^{-2x}$

$$\Longrightarrow y_p(x) = ce^{-2x}$$

$$\Longrightarrow 4ce^{-2x} + 8ce^{-2x} + 3ce^{-2x} = 10e^{-2x}$$

$$\Longrightarrow 15c = 10 \qquad \therefore c = \frac{2}{3}$$

$$\therefore y(x) = c_1 e^x + c_2 e^{3x} + \frac{2}{3} e^{-2x} \quad \square$$

2.9 Solution by Undetermined Coefficients

$$y'' + ay' + by = r(x) \qquad \text{where } a, b \text{ are constants}$$

The general solution is: $y(x) = y_h(x) + y_p(x)$

■ Rules to find $y_p(x)$:

(A) Given the following types of $r(x)$, $y_p(x)$ has the form of:

$r(x)$	$y_p(x)$
ke^{rx}	ce^{rx}
$kx^n \quad (n = 0, 1, 2 \ldots)$	$k_n x^n + k_{n-1} x^{n-1} \ldots + k_1 x + k_0$
$k \cos wx$ $k \sin wx$	$k_1 \cos wx + k_2 \sin wx$
$ke^{\alpha x} \cos wx$ $ke^{\alpha x} \sin wx$	$e^{\alpha x}(k_1 \cos wx + k_2 \sin wx)$

The constants c or k_n are determined by substituting $y_p(x)$ into the nonhomogeneous equation.

(B) If $y_p(x)$ in the above table is a solution of the homogeneous solution.

$\Longrightarrow$ Assume the new $y_p(x) = xy_p(x)$
(or new $y_p(x) = x^2 y_p(x)$ if the homogeneous solution corresponds to a double root of the characteristic equation)

<u>*Ex*</u> : $\quad y'' + 4y = 8x^2$

(1) $\underline{y_h(x)}$:

$$\lambda^2 + 4 = 0 \Longrightarrow \lambda = \pm 2i$$

$$\therefore\ y_h(x) = A\cos 2x + B\sin 2x$$

(2) $\underline{y_p(x)}$:

$$y_p(x) \equiv c_2x^2 + c_1x + c_0$$

$$\Longrightarrow (2c_2) + 4(c_2x^2 + c_1x + c_0) = 8x^2$$

$$\Longrightarrow c_2 = 2, \quad c_1 = 0, \quad c_0 = -1$$

$$\therefore\ y_p(x) = 2x^2 - 1$$

$$\therefore\ y(x) = A\cos 2x + B\sin 2x + 2x^2 - 1 \quad \square$$

<u>*Ex*</u> : $y'' - 3y' + 2y = e^x$

(1) $\underline{y_h(x)}$:

$$\lambda^2 - 3\lambda + 2 = 0 \Longrightarrow \lambda = 2, 1$$

$$\Longrightarrow y_h(x) = c_1 e^x + c_2 e^{2x}$$

(2) $\underline{y_p(x)}$:

$$y_p(x) \equiv cxe^x$$

$$\Longrightarrow c(2 + x)e^x - 3c(1 + x)e^x + 2cxe^x = e^x$$

$$\Longrightarrow c = -1$$

$\therefore\ y(x) = c_1 e^x + c_2 e^{2x} - xe^x \quad \square$

<u>*Ex*</u> : $y'' - 2y' + y = e^x + x \quad y(0) = 1,\, y'(0) = 0$

(1) $\underline{y_h(x)}$:

$$\lambda - 2\lambda + 1 = 0 \Longrightarrow \lambda = 1$$
$$\Longrightarrow y_h(x) = (c_1 + c_2 x)e^x$$

(2) $\underline{y_p(x)}$:

$$y_p(x) \equiv cx^2e^x + k_1x + k_0$$
$$y_p' = c(2xe^x + x^2e^x) + k_1 = ce^x(2x + x^2) + k_1$$
$$y_p'' = ce^x(2x + x^2) + ce^x(2 + 2x) = ce^x(x^2 + 4x + 2)$$

$$\Longrightarrow ce^x(x^2 + 4x + 2) - 2ce^x(2x + x^2) - 2k_1 + cx^2e^x + k_1x + k_0 = e^x + x$$
$$\Longrightarrow 2ce^x + k_1x - 2k_1 + k_0 = e^x + x$$
$$\Longrightarrow c = \frac{1}{2} \quad k_1 = 1 \quad k_0 = 2$$

$\therefore$ the general solution is $y(x) = (c_1 + c_2x)e^x + \frac{1}{2}x^2e^x + x + 2$

Apply the initial conditions:

$$y(0) = 1 \Longrightarrow c_1 + 2 = 1 \Longrightarrow c_1 = -1$$
$$y'(0) = 0 \Longrightarrow c_2 + c_1 + 1 = 0 \Longrightarrow c_2 = 0$$

$\therefore$ the particular solution is $y(x) = -e^x + \frac{1}{2}x^2e^x + x + 2 \quad \square$

2.10 Solution by Variation of Parameters

$$y'' + ay' + by = r(x) \qquad \text{where } a, b \text{ are constants} \quad \text{———} \ (*)$$

The general solution is: $y(x) = y_h(x) + y_p(x)$

We know that the homogeneous solution $y_h(x)$ has the from:

$y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ where y_1 and y_2 from a basis.

■ *Method of variation of "parameters"* :

Replace the parameters c_1 and c_2 by $u(x)$ and $v(x)$

$\Longrightarrow y_p(x) = u(x)y'(x) + v(x)y_2(x)$

Substitute $y_p = uy_1 + vy_2$ into (1), we get one equation for $u(x)$ and $v(x)$.

To get another equation,

$$y_p' = u'y_1 + uy_1' + v'y_2 + vy_2' = \underbrace{u'y_1 + v'y_2}_{\equiv 0} + uy_1' + vy_2'$$

$\Longrightarrow$ the second equation is $u'y_1 + v'y_2 \equiv 0$

$\therefore\ y_p' = uy_1' + vy_2'$

and $y_p'' = u'y_1' + uy_1'' + v'y_2' + vy_2''$

Substitute $y_p = uy_1 + vy_2$ into (*) :

$$\Longrightarrow (u'y_1' + uy_1'' + v'y_2' + vy_2'') + p(uy_1' + vy_2') + q(uy_1 + vy_2) = r$$

$$\Longrightarrow u\underbrace{(y_1'' + py_1' + qy_1)}_{=0} + v\underbrace{(y_2'' + py_2' + qy_2)}_{=0} + u'y_1' + v'y_2' = r$$

$$\Longrightarrow \begin{cases} u'y_1' + v'y_2' = r & \text{———— (1)} \\ u'y_1 + v'y_2 \equiv 0 & \text{———— (2)} \end{cases} \qquad \leftarrow \text{equations for } u', v'$$

$$(2)y_2' - (1)y_2 \;\Longrightarrow\; u'(y_1y_2' - y_2'y_2) = -y_2 r$$

$$(1)y_1 - (2)y_1' \;\Longrightarrow\; v'\underbrace{(y_1y_2' - y_1'y_2)}_{= W(y_1, y_2)} = y_1 r$$

where $W(y_1, y_2)$ is the Wronskian, and $W \neq 0$ i.e. y_1 and y_2 form a basis.

$$\Longrightarrow \begin{cases} u' = -\dfrac{y_2 r}{W} \\ \\ v' = \dfrac{y_1 r}{W} \end{cases}$$

$$\Longrightarrow \begin{cases} u = -\int \dfrac{y_2 r}{W} dx \\ \\ v = \int \dfrac{y_1 r}{W} dx \end{cases}$$

$$\Longrightarrow \boxed{y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx} \qquad \square$$

<u>*Ex*</u> : $y'' + y = \sec x$

(1) $\underline{y_h}$:

$$y_h'' + y_h = 0$$

$$\lambda^2 + 1 = 0 \Longrightarrow \lambda = \pm i$$

$$\Longrightarrow y_h = c_1 \cos x + c_2 \sin x$$

(2) $\underline{y_p}$:

$$y_1 = \cos x \qquad y_2 = \sin x$$

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$$

$$u(x) = \int \frac{\sin x \cdot \sec x}{1} dx = \ln|\cos x|$$

$$v(x) = -\int \frac{\cos x \cdot \sec x}{1} dx = x$$

$$\Longrightarrow y_p = (\ln|\cos x|)\cos x + x \sin x$$

$$\therefore y = y_h + y_p = (c_1 + \ln|\cos x|)\cos x + (c_2 + x)\sin x \quad \square$$